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INTERSECTION FORMS OF TORIC HYPERKÄHLER VARIETIES

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Abstract. This note proves combinatorially that the intersection pairing on the middle-dimensional compactly supported cohomology of a toric hyperkähler variety is always definite, providing a large number of non-trivial L^2 harmonic forms for toric hyperkähler metrics on these varieties. This is motivated by a result of Hitchin about the definiteness of the pairing of L^2 harmonic forms on complete hyperkähler manifolds of linear growth.

1. Introduction

A complete hyperkähler manifold M^{4n} has linear growth if for one of the Kähler forms $\omega = d\beta$ with β of linear growth (see [10] for details). In [10, Theorem 4] Hitchin proved that on a complete hyperkähler manifold M^{4n} of linear growth, non-trivial L^2 harmonic forms are of middle dimension and are anti-self-dual (resp. self-dual) if n is odd (resp. n is even). It follows that the intersection form on the Hodge cohomology (the space of L^2 harmonic forms) on such manifolds is always definite (negative definite if n is odd, positive definite if n is even). Moreover, Hodge theory implies that the image of middle-dimensional compactly supported cohomology in ordinary cohomology filters through L^2 harmonic forms. We think of the natural intersection form on the image of compactly supported cohomology in ordinary cohomology as the intersection form, given by cup product and integration, on compactly supported cohomology modulo its null-space. We can thus conjecture that for a complete hyperkähler manifold of linear growth and dimension $4n$ this intersection form is definite; in particular this would imply that the signature of the manifold is non-positive if n is odd and non-negative if n is even. If true this would be a non-trivial topological obstruction for a manifold to carry a complete hyperkähler metric of linear growth.

This is known to hold in all the examples where the intersection form on middle-dimensional compactly supported cohomology of a complete hyperkähler manifold of linear growth has been calculated. These examples include Nakajima's deep result [11, Corollary 11.2] that the intersection form on a quiver variety is always definite, which is deduced from a representation theory result of Kac. In another example, Segal and Selby [13] proved that the intersection form on the moduli space of $SU(2)$ magnetic monopoles on $\mathbb{R}^3$ is definite (fitting very nicely with string theory conjectures of Sen [14] on the Hodge cohomology of such magnetic monopole moduli

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spaces). The final example is [7], where it was shown that the moduli space of rank 2 Higgs bundles of fixed determinant of odd degree, another gauge theoretic example of a complete hyperkähler manifold of linear growth, has a trivial and therefore definite intersection form.

A general theorem, [8, Corollary 7], implies that when the metric of the complete hyperkähler manifold of linear growth M^{4n} is also of fibered boundary (or fibered cusp) type, then the intersection form on middle-dimensional compactly supported cohomology is always semi-definite, and consequently that the intersection form on the image of compactly supported cohomology in ordinary cohomology is definite. The examples of such manifolds include all known gravitational instantons of finite topological type as well as smooth generic ALE toric hyperkähler manifolds (cf. [8, Section 7.2]). In fact, using our main Theorem 1 below, [8, Corollary 11] calculates the full Hodge cohomology of such ALE toric hyperkähler manifolds.

In this paper we study the intersection form of another family of complete hyperkähler manifolds of linear growth. Bielawski and Dancer in [2] construct toric hyperkähler manifolds as hyperkähler quotients of flat quaternionic space $\mathbb{H}^n$ by a hyperkähler action of a torus $T^d \subset T^n$. An account of the algebraic geometry of the underlying varieties from a combinatorial perspective were given in [9]. We will follow the notations and terminology of [9].

The following is the main result of this note.[1]

Theorem 1. *Let $\theta \in \mathbb{N}\mathcal{A}$ be a smooth degree, so that the toric hyperkähler variety $Y(A,\theta)$ of real dimension $4n-4d$ is smooth. Then the intersection form on $H^{2n-2d}_{cpt}(Y(A,\theta),\mathbb{R})$ is always a definite form: it is positive definite if $n-d$ is even, negative definite if $n-d$ is odd.*

When the underlying hyperplane arrangement is co-graphic, the toric hyperkähler variety is also a toric quiver variety (see [9, Section 7]). In these circumstances Nakajima's above-mentioned result already proves the theorem.

In the general case we proceed by considering the bounded complex of the affine hyperplane arrangement $\mathcal{H}^{bd}(\mathcal{B},\psi)$ in $\mathbb{R}^{n-d}$ defined from the data (A,θ) (for details see [9]). Then a basis for $H^{2n-2d}_{cpt}(Y(A,\theta),\mathbb{R})$ is given by the compactly supported cohomology classes η_{X_F} of middle-dimensional projective subvarieties X_F of $Y(A,\theta)$. Each X_F is a toric variety associated to a top-dimensional bounded region F in $\mathcal{H}^{bd}(\mathcal{B},\psi)$. In the next section we will show that in this basis the intersection form is combinatorially given by:

Theorem 2.

$$\int_{Y(A,\theta)} \eta_{X_1}\wedge\eta_{X_2} = (-1)^{\dim \overline{F_1}\cap\overline{F_2}}\left(\textit{number of vertices of } \overline{F_1}\cap\overline{F_2}\right), \tag{1}$$

where F_1 and F_2 are two top-dimensional bounded regions in $\mathcal{H}^{bd}(\mathcal{B},\psi)$, and X_1 and X_2 are the corresponding projective toric varieties in $Y(A,\theta)$. Moreover, the classes η_{X_F}, where F runs through the top (i.e. $n-d$) dimensional bounded regions in $\mathcal{H}^{bd}(\mathcal{B},\psi)$ form a basis for the vector space $H^{2n-2d}_{cpt}(Y(A,\theta),\mathbb{R})$.

[1] After finishing our paper we learned that this theorem could be considered as a special case of Corollary 2.1.13 of [4]. Namely, one always has the proper semismall map $Y(A,\theta)\to Y(A,0)$, and the intersection form on the preimage of the origin under this map is the same as the intersection form on $H^{2n-2d}_{cpt}(Y(A,\theta),\mathbb{R})$ (see [9, Section 6] for details on this). Our proof however is much less involved than [4] and gives a new angle by understanding our main result in the framework of the combinatorics of hyperplane arrangements.

In the last section we then prove that this combinatorial intersection pairing, given purely in terms of the affine hyperplane arrangement $\mathcal{H}(\mathcal{B},\psi)$, is indeed definite. We will in fact construct a natural isomorphism of this pairing with the natural pairing on the $(n-d-1)$ cohomology of the independence complex $\mathcal{N}$ of the matroid $M(\mathcal{B})$.

2. Determining the intersection form

In this section we prove Theorem 2. We will use the terminology and notation of [9]. Let us fix A and a smooth degree θ, so that $\mathcal{B}$ is coloop-free. Thus we are in the situation of [9, Proposition 6.7]. We can assume this, otherwise $H^{2n-2d}(Y,\mathbb{R})=0$ and Theorem 2 holds automatically. For convenience we will write Y for $Y(A,\theta)$ and $\mathcal{H}^{bd}$ for $\mathcal{H}^{bd}(\mathcal{B},\psi)$.

First we note that the subvariety X_F of Y corresponding to a top-dimensional bounded region F in $\mathcal{H}^{bd}$ is a Lagrangian subvariety with respect to the natural holomorphic symplectic structure $\omega_{\mathbb{C}}$ on Y. One way to see this is to use a circle action on Y corresponding to the region F as explained in [6]. By construction the holomorphic symplectic form on $Y(A,\theta)$ is of homogeneity 1 with respect to this circle action, meaning that $\lambda^*(\omega_{\mathbb{C}})=\lambda\omega_{\mathbb{C}}$, where $\lambda\in U(1)$. Moreover, X_F is the minimum of the associated moment map. Now it is clear from [12, Proposition 7.1] that X_F is a Lagrangian subvariety.

Now consider F_1 and F_2 two top-dimensional bounded regions in $\mathcal{H}^{bd}$ and let X_1 and X_2 denote the corresponding projective toric varieties in Y. Then the third equation in [5, Proposition 9.1.1] implies that

$$\int_Y \eta_{X_1}\wedge\eta_{X_2}=\int_{X_{12}} c(N_1)\wedge c(T_{X_2})^{-1}\wedge c(T_{X_{12}}),$$

where X_{12} is the projective toric variety in Y corresponding to the region $\overline{F_1}\cap\overline{F_2}$, and N_1 denotes the normal bundle of X_1 in Y.

Since X_1 and X_2 are Lagrangian subvarieties, we see that $c(N_1)\wedge c(T_{X_2})^{-1}\wedge c(T_{X_{12}})=c(T^*(X_{12}))$ on X_{12}. Therefore,

$$\begin{aligned}\int_Y \eta_{X_1}\wedge\eta_{X_2}&=\int_{X_{12}} c(T^*X_{12})=(-1)^{\dim(X_{12})}\chi(X_{12})\\&=(-1)^{\dim(\overline{F_1}\cap\overline{F_2})}\left(\text{number of vertices of } \overline{F_1}\cap\overline{F_2}\right),\end{aligned}$$

where we used the fact that for the Euler characteristic of a toric variety we have

$$\chi(X_{12})=(\text{number of vertices of } \overline{F_1}\cap\overline{F_2}).$$

To prove the last statement of our theorem list $F_1,F_2,\ldots,F_r$ as the top-dimensional bounded regions of $\mathcal{H}^{bd}$ and the corresponding $X_1,\ldots,X_r$ as middle-dimensional smooth projective subvarieties of Y. Then as in (35) of [9], we can find a basis $\alpha_1,\ldots,\alpha_r$ for $H^{2n-2d}(Y,\mathbb{R})$, which has the property that $\alpha_i\mid_{X_j}\neq 0$ if and only if $i=j$. The Poincaré dual basis for $H^{2n-2d}_{cpt}(Y,\mathbb{R})$ is then clearly $\eta_{X_1},\ldots,\eta_{X_r}$. This completes the proof of Theorem 2.

3. Combinatorial intersection pairing

In this section we prove that the combinatorial intersection pairing given by (1) is $(-1)^{n-d}$ times a positive definite pairing. This will be done by using the classical nerve construction which we now review.

Let Γ be a finite CW-complex and let $\Gamma_1, \ldots, \Gamma_r$ be subcomplexes of Γ such that all non-void intersections are contractible and the union of the Γ_i cover Γ. Let $\mathcal{N}$ be the *nerve* of this cover. That is, $\mathcal{N}$ is the abstract simplicial complex with vertices $v_1, \ldots, v_r$ and whose simplices consist of all $[v_{i_1}, \ldots, v_{i_k}]$ such that $\Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} \neq \emptyset$. Let Z be the subset of $\Gamma \times \mathcal{N}$ defined by

$$\{(x, z) : x \in \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} \text{ and } z \in [v_{i_1}, \ldots, v_{i_k}]\}.$$

Theorem 3 ([3]). *Let $\Gamma, \mathcal{N}$ and Z be as above. The projections π_Γ and $\pi_\mathcal{N}$ of Z onto Γ and $\mathcal{N}$ respectively are homotopy equivalences.*

The bounded complex of any affine hyperplane arrangement is contractible (for a proof see Theorem 3.3 and Theorem 4.7 of [9]; it also follows from [1, Exercise 4.27 (a)]), hence $\mathcal{H}^{bd}(B, \psi)$ is contractible. Let $\Gamma = \mathcal{H}^{bd}(B, \psi) - \bigcup F_j$. Since the F_j are disjoint, Γ is homotopy equivalent to a wedge of r spheres of dimension $n - d - 1$.

Let $\{H_1, \ldots, H_s\}$ be the affine hyperplanes in $\mathcal{H}$. Define $\Gamma_i = \Gamma \cap H_i$ and $\mathcal{U} = \{\Gamma_1, \ldots, \Gamma_s\}$. Now, $\mathcal{U}$ covers Γ and any non-void intersection of members of $\mathcal{U}$ is the bounded complex of an affine hyperplane arrangement, and hence is contractible. Let $\mathcal{N}$ be the nerve of this cover. We denote the vertices of $\mathcal{N}$ by $v_1, \ldots, v_s$. A subset of vertices of $\mathcal{N}$ is a simplex if and only if the corresponding hyperplanes have non-empty intersection, and this holds if and only if the corresponding columns of B are independent. Since this only depends on the matroid $M(B)$, $\mathcal{N}$ is known as the independence (or matroid) complex of the matroid $M(B)$.

Let σ be a k-dimensional cell of Γ and let $(H_{\sigma_1}, \ldots, H_{\sigma_{n-d-k}})$ be the ordered set of hyperplanes containing σ. For each hyperplane H_{σ_i} choose a normal η_{σ_i}. The ordered set of normals $(\eta_{\sigma_1}, \ldots, \eta_{\sigma_n-d-k})$ define an orientation for σ as follows. If $x \in \sigma$ and $W = (w_1, \ldots, w_k)$ is an ordered basis for the tangent space of σ at x, then W is positively oriented if and only if $(\eta_{\sigma_1}, \ldots, \eta_{\sigma_n-d-k}, w_1, \ldots, w_k)$ is a positively oriented basis for the tangent space of $\mathbb{R}^{n-d}$ at x.

Let F be an $(n-d)$-dimensional cell of $\mathcal{H}^{bd}(B, \psi)$. Choose an inward pointing normal for each hyperplane incident to F. Define $[F]$ to be the cycle in $H_\star(\Gamma, \mathbb{Z})$ (CW-homology) given by $\sum[\sigma]$, where the sum is taken over all cells σ of dimension $n-d-1$ on the boundary of F. For each vertex v on the boundary of F let $\Psi(v)$ be the oriented simplex $[v_{i_1}, \ldots, v_{i_{n-d}}]$ in $\mathcal{N}$, where $H_{i_1}, \ldots, H_{i_{n-d}}$ are the hyperplanes which contain v, and the orientation of the simplex is determined by the orientation of the vertex as a zero-cell of Γ. Now define

$$\Psi[F] = \sum_{v \in \partial F} \Psi(v).$$

In order to see that $\Psi[F]$ is a cycle, note that the facets in $\partial\Psi[F]$ correspond to the one-cells in ∂F and that these occur in oppositely oriented pairs, one for each end point of the one-cell. Since $\{[F_1], \ldots, [F_r]\}$ is a basis for $H_\star(\Gamma; \mathbb{Z})$, we can extend Ψ linearly to a map $\Psi_\star : H_\star(\Gamma; \mathbb{Z}) \to H_\star(\mathcal{N}; \mathbb{Z})$.

Proposition 4. *$\Psi_\star$ is an isomorphism.*

Proof. Let $Z = \{(x, z) \in \Gamma \times \mathcal{N} : x \in \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} \text{ and } z \in [v_{i_1}, \ldots, v_{i_k}]\}$. Fix F. By Theorem 3 it is sufficient to find a cycle $[\zeta] \in H_\star(Z; \mathbb{Z})$ such that $(\pi_\Gamma)_\star[\zeta] = [F]$ and $(\pi_\mathcal{N})_\star[\zeta] = \Psi[F]$.

In order to define ζ we introduce the following notation. Let $\Delta = [v_{i_1}, \ldots, v_{i_k}]$ be a (non-empty) simplex of $\mathcal{N}$. Denote by H_Δ the corresponding $(n-d-k)$-cell in Γ with orientation given by $(\eta_{i_1}, \ldots, \eta_{i_k})$. Furthermore, let (H_Δ, v_j) be the cell

$H_\Delta \cap H_j$ with orientation given by $(\eta_{i_1}, \ldots, \eta_{i_k}, \eta_j)$. If $v_j \in \Delta$ or $\Delta \cup v_j$ is not a simplex of $\mathcal{N}$, then (H_Δ, v_j) is empty and the chain $[(H_\Delta, v_j)]$ equals zero. Define a cellular chain ζ in $C_\star(Z)$ by

$$\zeta = \sum_{H_\Delta \subseteq \partial c} (-1)^{|\Delta|}[H_\Delta \times \Delta].$$

Recall that for products of CW-complexes the boundary map is given by

$$\partial(\Omega \times \Psi) = \partial\Omega \times \Psi + (-1)^p \Omega \times \partial\Psi,$$

where Ω is a p-cell and Ψ is a q-cell. Therefore,

$$\begin{aligned} \partial^2(\zeta) &= \sum_j \partial([(H_\Delta, v_j) \times \Delta] + (-1)^{|\Delta|}[H_\Delta \times \partial\Delta]) \\ &= \sum_{j,l}([(H_\Delta, v_j, v_l) \times \Delta] + (-1)^{|\Delta+1|}[(H_\Delta, v_j) \times \partial\Delta] \\ &\qquad + (-1)^{|\Delta|}[(H_\Delta, v_j) \times \partial\Delta] + [H_\Delta \times \partial^2\Delta]) \\ &= 0. \end{aligned}$$

Thus ζ is a cycle. It is easy to see that $(\pi_\Gamma)_\star[\zeta] = [F]$ and $(\pi_{\mathcal{N}})_\star[\zeta] = \Psi[F]$. $\square$

The chain complex $C_\star(\mathcal{N})$ has an inner product structure given by declaring that the set of chains $\{[\Delta] : \Delta \text{ a simplex of } \mathcal{N}\}$ is an orthonormal basis. Since $\mathcal{N}$ is a simplicial complex of dimension $n-d-1$ and is homotopy equivalent to a wedge of $(n-d-1)$-dimensional spheres, $H_{n-d-1}(\mathcal{N})$ is a subspace of $C_{n-d-1}(\mathcal{N})$ and inherits a positive definite inner product.

For convenience we recall the combinatorial pairing introduced in (1) in the present notation. Let $V(B)$ be the vector space whose basis is $F_1, \ldots, F_r$. Set $\sigma_{ij} = \overline{F_i} \cap \overline{F_j}$. Define a pairing $\Phi(F_i, F_j) = (-1)^{\dim \sigma_{ij}}|\{\text{vertices } v : v \in \sigma_{ij}\}|$.

Proposition 5. $\Phi(F_i, F_j) = (-1)^{n-d}\langle \Psi(F_i), \Psi(F_j)\rangle$.

Proof. It is evident that $\langle \Psi(F_i), \Psi(F_j)\rangle$ counts the number of vertices in σ_{ij} weighted with $+1$ if the orientations induced on the vertex by the inward pointing normals of F_i and F_j are the same, -1 if they are different. The orientation of a vertex with respect to the inward pointing normals for F_j is obtained from the orientation of the vertex with respect to F_i by reversing the direction of $n-d-\dim\sigma_{ij}$ of the normals. Hence $\langle \Psi(F_i), \Psi(F_j)\rangle = (-1)^{n-d-\dim\sigma_{ij}}|\{\text{vertices } v : v \in \sigma_{ij}\}|$. $\square$

Remark 1. The map $\Psi_\star$ identifies $H_*(\mathcal{N}, Z)$ with $H_*(\Gamma; Z)$, which in turn could naturally be identified with the middle-dimensional compactly supported cohomology $H^{2n-2d}_{cpt}(Y, \mathbb{Z})$. Moreover, by Theorem 2 and Proposition 5, these identifications also preserve the appropriate inner products on these spaces, so as a byproduct we get Theorem 1.

2. Through the above identifications the map $\Psi_\star$ is defining a flat connection on the middle-dimensional compactly supported cohomology of toric hyperkähler varieties $Y(A, \theta)$ as A is fixed and θ varies. We can think of this as a combinatorial version of the Gauss-Manin connection obtained from the hyperkähler quotient construction.

3. While $\Psi_\star$ makes sense for arbitrary hyperplane arrangements, Proposition 4 may not hold if the arrangement is not generic. In the arrangement pictured in Figure 1

$$\Phi(F_1 + F_2 - F_3 - F_4, F_1 + F_2 - F_3 - F_4) < 0.$$

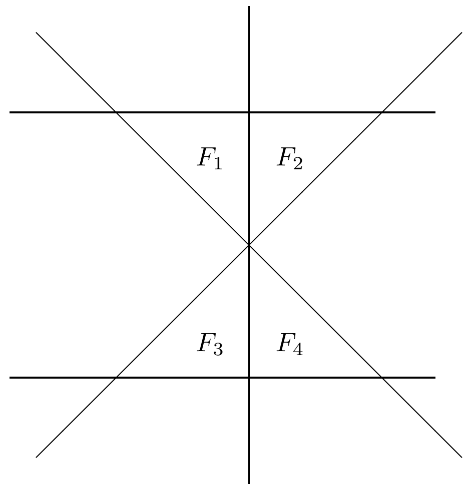

FIGURE 1. A nongeneric arrangement

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