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HOLOMORPHIC APPROXIMATION ON COMPACT, HOLOMORPHICALLY CONVEX, REAL-ANALYTIC VARIETIES

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ABSTRACT. Every continuous function on a compact, holomorphically convex, real-analytic subset of $\mathbb{C}^N$ can be approximated uniformly by functions holomorphic on the set.

1. Introduction

Anderson, Izzo, and Wermer [2] recently proved that if X is a compact, polynomially convex, real-analytic subvariety of $\mathbb{C}^N$, each point of which is a peak point for the algebra $\mathcal{P}(X)$, then each continuous function on X can be approximated uniformly on X by polynomials. It turns out that there is a more general theorem in the same vein:

Theorem 1. *If X is a compact, real-analytic subvariety of $\mathbb{C}^N$ that is holomorphically convex, then every continuous function on X can be approximated uniformly by functions holomorphic on (varying neighborhoods of) X.*

The notion of holomorphic convexity we are considering here is this:

Definition 2. *A compact set X in a complex manifold* is holomorphically convex if every nonzero complex homomorphism of the algebra $\boldsymbol{\mathcal{O}}(X)$ of germs of functions holomorphic on X is of the form $\mathbf{f} \mapsto \mathbf{f}(p)$ for a unique point $p \in X$.

Briefly, X is the spectrum of $\boldsymbol{\mathcal{O}}(X)$ or $\operatorname{spec}\boldsymbol{\mathcal{O}}(X) = X$.

Here there is the customary abuse of notation: The elements of the algebra $\boldsymbol{\mathcal{O}}(X)$ are germs, *i.e.*, equivalence classes of functions, not functions, so, strictly speaking, $\mathbf{f}(p)$ is without meaning. Nonetheless, as usual, we understand by this symbol the value $f(p)$ for any function f in the equivalence class $\mathbf{f}$.

The theory of holomorphically convex sets has been discussed in the paper [8]. Examples in $\mathbb{C}^N$ include the polynomially convex sets, the rationally convex sets, and the sets that are intersections of sequences of domains of holomorphy. The latter examples include all compact totally real submanifolds and, more generally, all compact, totally real subsets of $\mathbb{C}^N$. According to a theorem given by Birtel [3, 4], the compact, connected, holomorphically convex subsets of $\mathbb{C}^N$ are characterized as the compact sets X such that if $\{\Omega_j\}_{j=1,\dots}$ is a sequence of domains in $\mathbb{C}^N$ and if $(\tilde{\Omega}_j, \pi_j)$ is the envelope of holomorphy of Ω_j, then $X = \bigcap_{j=1,\dots} \Omega_j$ implies

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$X = \bigcap_{j=1,\dots} \pi_j(\tilde{\Omega}_j)$. It is necessary to introduce Riemann domains here, because it is not true that a holomorphically convex set is necessarily an intersection of domains of holomorphy. Examples have been given in [5, 14].

With the notation that $\mathcal{C}(X)$ is the algebra of continuous $\mathbb{C}-$valued functions on X, we introduce the notation that $\mathcal{P}(X)$ and $\mathcal{R}(X)$ are, respectively, the subalgebras of $\mathcal{C}(X)$ that consist of the functions that can be approximated uniformly on X by polynomials or by rational functions without poles on X. With this notation, we can state two corollaries of the theorem above.

Corollary 3. *If X is a compact, polynomially convex, real-analytic subvariety of $\mathbb{C}^N$, then $\mathcal{P}(X) = \mathcal{C}(X)$.*

Corollary 4. *If X is a compact, rationally convex, real-analytic subvariety of $\mathbb{C}^N$, then $\mathcal{R}(X) = \mathcal{C}(X)$.*

Alexander Izzo has informed me that he too has obtained the result of the second corollary.

The first corollary contains the result of Anderson, Izzo, and Wermer quoted above.

The second corollary *is* less general than Theorem 1 in that there are examples of compact, holomorphically convex, real-analytic subvarieties of $\mathbb{C}^N$ that are not rationally convex: Duval and Sibony [7] have proved that there is no rationally convex three-sphere in $\mathbb{C}^3$. On the other hand, Ahern and Rudin [1] have shown how to embed a three-sphere in $\mathbb{C}^3$ as a totally real manifold. (The existence of such embeddings is due to Gromov.) Their construction leads to a real-analytic, totally real, three-sphere. Because totally real manifolds have fundamental neighborhood bases that consist of Stein domains [9], it follows that the analytic Ahern-Rudin spheres are holomorphically convex.

It is striking that there is no analogue of Theorem 1 for smooth manifolds: Izzo [11] has given examples of smooth submanifolds X of $\mathcal{C}^N$ that are polynomially convex and for which each point is a peak point for the algebra $\mathcal{P}(X)$ but yet $\mathcal{P}(X) \neq \mathcal{C}(X)$. (In fact, Izzo's example is a smooth manifold with boundary; a small modification of his construction yields an example without boundary.)

2. The structure of compact, holomorphically convex, real-analytic varieties

Fix a compact, holomorphically convex, real-analytic subset X of $\mathbb{C}^N$. For the theory of real-analytic sets, one can consult [12]. Introduce the decomposition $X = X_{reg} \cup X_{sing}$ in which X_{reg} denotes the open subset of X that consists of the points at which X is a real-analytic manifold, and X_{sing} is the set of singular points. Because we are working with real-analytic sets, the set X_{sing} need not be a real-analytic variety.

The set X_{reg} is a bounded, not necessarily connected, closed, real-analytic submanifold of $\mathbb{C}^N \setminus X_{sing}$. It can have components of varying dimensions.

Fix attention on a component Σ of X_{reg}. Put $n = \dim_{\mathbb{R}} \Sigma$. For $j = 0, 1, \dots, [\frac{n}{2}]$, let Σ_j be the set of points $x \in \Sigma$ at which $\dim_{\mathbb{C}} T_x^{\mathbb{C}}(\Sigma) = j$. (In this $T_x^{\mathbb{C}}(\Sigma)$ denotes the largest $\mathbb{C}$-affine subspace contained in the tangent plane $T_x(\Sigma)$.) The number $\dim_{\mathbb{C}} T_x^{\mathbb{C}}(\Sigma)$ is the $CR-$dimension of Σ at x.

Theorem 5. *The set Σ_0 is dense in Σ, and $\Sigma \setminus \Sigma_0$ is a real-analytic subvariety of Σ.*

Thus, the set X is a disjoint union $\mathcal{M} \cup E$ with E a compact subset of X, with $\mathcal{M}$ a bounded, totally real submanifold of $\mathbb{C}^N \setminus E$ that can have components of varying dimensions, and with the dimension of E less than the dimension of $\mathcal{M}$. The set E is not necessarily an analytic set, so *dimension* here should be understood as topological dimension.

The proof of Theorem 5 depends on a simple general fact about $CR-$manifolds, which is probably well known:

Lemma 6. *If $\mathcal{M}$ is a submanifold of class $\mathcal{C}^1$ in the open subset U of $\mathbb{C}^N$ with the property that at each point p, the tangent space to $\mathcal{M}$ is not totally real, then there is an open subset $\mathcal{M}'$ of $\mathcal{M}$ that is a $CR-$submanifold of an open subset U' of $\mathbb{C}^N$.*

The point of the lemma is that, although the $CR-$dimension of $\mathcal{M}$ may vary from point to point, it is constant on a nonempty open subset of $\mathcal{M}$.

Proof. Let the (real) dimension of $\mathcal{M}$ be d. Introduce subsets $\mathcal{M}^k$ for $k = 1, \dots, \left[\frac{d}{2}\right]$ by the condition that $\mathcal{M}^k$ is the set of points $p \in \mathcal{M}$ at which $\dim_{\mathbb{C}} T_p^{\mathbb{C}}(\mathcal{M})$ is at least k. Thus, $\mathcal{M} = \mathcal{M}^o \supset \mathcal{M}^1 \supset \cdots$. Some of these inclusions may be equalities; in any case, $\mathcal{M}^o = \mathcal{M}^1$ by hypothesis. Because the Grassmannian of p-dimensional complex linear subspaces of $\mathbb{C}^N$ is compact, each $\mathcal{M}^k$ is a closed subset of $\mathcal{M}$. Let p be the greatest integer such that $\mathcal{M}^p = \mathcal{M}$. If $p = \left[\frac{d}{2}\right]$, we are done, for $\mathcal{M}$ itself is a $CR-$manifold of this $CR-$dimension. If p is smaller than $\left[\frac{d}{2}\right]$, then we can take $\mathcal{M}' = \mathcal{M} \setminus \mathcal{M}^{p+1}$ and $U' = U \setminus \mathcal{M}^{p+1}$.

3. Proof of Theorem 5

In this proof we shall use the notation that $\mathbb{T}$ and $\mathbb{U}$ are, respectively, the unit circle and the open unit disc in the complex plane.

That $\Sigma \setminus \Sigma_0$ is an analytic subvariety of Σ is so, for, with $\iota : \Sigma \hookrightarrow \mathbb{C}^N$ the inclusion map, this set is the subset of Σ on which the form $\iota^*\vartheta$ vanishes for every holomorphic n-form ϑ on $\mathbb{C}^N$.

Suppose that Σ_0 is not dense in Σ. Thus, for an $m \in \{1, \dots [\frac{n}{2}]\}$, the set Σ_m contains an open subset Ω of Σ. The set Ω is a $CR-$manifold of $CR-$dimension m. Assume without loss of generality that $0 \in \Omega$.

We consider first the *generic case* that Σ is of real codimension $\ell = N - m$. We can suppose that coordinates $z_j = x_j + iy_j$ have been chosen so that

$$T_0\Sigma = \{z \in \mathbb{C}^N : y_1 = \cdots = y_\ell = 0\} = \mathbb{R}^\ell \times \mathbb{C}^{N-\ell}.$$

Then

$$T_0^{\mathbb{C}}\Sigma = \{z \in \mathbb{C}^N : z_1 = \cdots = z_\ell = 0\}.$$

Near the origin, Σ is described by a system of equations

$$y_j = h_j(x_1, \dots, x_\ell, z_{\ell+1}, \dots, z_N), \quad 1 \le j \le \ell,$$

with each h_j a real-analytic real-valued function such that $h_j(0) = 0$ and $dh_j(0) = 0$.

Consider the real-analytic map $\varphi_0 : \mathbb{R} \times \bar{\mathbb{U}} \to T_0^{\mathbb{C}}$ given by

$$\varphi_0(\tau, \zeta) = (\tau, \dots, 0, \tau\zeta, 0, \dots, 0)$$

with $\tau\zeta$ in the $(\ell+1)^{st}$ place. According to Theorem 6.1, p. 347 of [10], for some $\tau_o > 0$, there is an analytic map $\varphi : [-\tau_o, \tau_o] \times \bar{\mathbb{U}} \to \mathbb{C}^N$ that carries $[-\tau_o, \tau_o] \times \bar{\mathbb{U}}$

into Ω, that satisfies $\pi \circ \varphi = \varphi_0$ if π is the real orthogonal projection of $\mathbb{C}^N$ onto $T_0^{\mathbb{C}}\Sigma$, and that has the further property that for each τ, $\varphi(\tau, \cdot)$ is holomorphic on $\mathbb{U}$. Note that because $\varphi_0(0, \cdot)$ is identically zero, $\varphi(0, \cdot)$ is also identically zero.

The *nongeneric case* is that in which

$$T_0\Sigma = \{(z_1, \ldots, z_N) \in \mathbb{C}^N : y_1 = \cdots = y_\ell = z_r = \cdots = z_N = 0\}$$

for an $r \in \{\ell + 2, \ldots, N\}$. In this case, the work of Selvaggi Primicerio and Taiani [15], Theorem 5.2, p. 240, provides a real-analytic map $\varphi : [-\tau_o, \tau_o] \times \bar{U} \to \mathbb{C}^N$ with the properties of the φ of the preceding paragraph. (The result of Selvaggi Primicerio and Taiani is obtained by reducing the nongeneric case to the generic case by means of a suitable projection.)

From here the idea is that the Kontinuitätssatz implies that each function holomorphic on a neighborhood of $\varphi([0, \tau_o] \times b\mathbb{U})$ extends holomorphically to a neighborhood of the set $\varphi([0, \tau_o] \times \bar{\mathbb{U}})$. The holomorphic convexity of X implies that the range of φ is contained in X. As serious monodromy issues arise in connection with the Kontinuitätssatz, we prefer to give the following direct argument to establish the last assertion. For each $t \in [-\tau_o, \tau_o]$ and each $z \in \mathbb{U}$, define a functional $L_{t,z}$ on $\boldsymbol{\mathcal{O}}(X)$ by

$$L_{t,z}(\mathbf{f}) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\mathbf{f}(\varphi(t, \zeta))}{\zeta - z}\, d\zeta.$$

For each $(t, \zeta) \in [-\tau_o, \tau_o] \times \mathbb{T}$, the point $\varphi(t, \zeta)$ lies in Ω, so $\mathbf{f}(\varphi(t, \zeta))$ is defined for every $\mathbf{f} \in \boldsymbol{\mathcal{O}}(X)$. Thus, $L_{t,\zeta}$ is well defined as a $\mathbb{C}-$linear functional on $\boldsymbol{\mathcal{O}}(X)$. It is also multiplicative. To see this, note that the analyticity of φ implies that for a fixed $\mathbf{f} \in \boldsymbol{\mathcal{O}}(X)$ and a fixed $z \in \mathbb{U}$, $L_{t,z}(\mathbf{f})$ depends analytically on t. If the elements $\mathbf{g}$ and $\mathbf{h}$ of $\boldsymbol{\mathcal{O}}(X)$ are represented by the holomorphic functions g and h defined on an open subset W of $\mathbb{C}^N$ that contains X, and if B is a small ball centered at the origin and contained in W, then for all t near 0 the points $\varphi(t, z)$ with $|z| \leq 1$ lie in B, and

$$L_{t,z}(\,\mathbf{gh}) = g(\varphi(t, z))h(\varphi(t, z)) = L_{t,z}(\,\mathbf{g})L_{t,z}(\mathbf{h}).$$

For fixed $\mathbf{f}$, $\mathbf{g}$ and z, both sides of this equation are analytic in their dependence on t, $t \in [-\tau_o, \tau_o]$, so for all $t \in [-\tau_o, \tau_o]$ and $z \in \mathbb{U}$, $L_{t,z}(\mathbf{gh}) = L_{t,z}(\mathbf{g})L_{t,z}(\mathbf{h})$. Thus each $L_{t,z}$ is in $\operatorname{spec}\boldsymbol{\mathcal{O}}(X)$, which, because of the assumed holomorphic convexity of X, implies that the entire set $\varphi([0, \tau_o] \times \bar{\mathbb{U}})$ is contained in X. This, however, implies that the compact, real-analytic variety X contains the germ of an analytic variety, in contradiction to a result of Diederich and Fornæss [6].

The proof of Theorem 5 is complete.

Note, by the way, that this argument implies that the dimension of a compact, holomorphically convex, real-analytic subvariety of $\mathbb{C}^N$ is not more than N. This bound is best possible.

4. Proof of Theorem 1

It will be convenient to use the notation that if Z is a real-analytic set with regular set Z_{reg} and singular set Z_{sing}, then Z^c_{reg} is the set of points at which Z_{reg} is not totally real, a closed, real-analytic subset of Z_{reg}, and Z^* is the set $Z^c_{reg} \cup Z_{sing}$. When Z is compact, the set Z^* is also compact.

Consider now the compact, holomorphically convex, real-analytic variety X of Theorem 1. The approximation theorem of O'Farrell, Preskenis, and Walsh [13], Theorem 2, implies that every continuous function on X that vanishes on X^* can be approximated uniformly on X by functions holomorphic on X. Consequently, if μ is a measure on X that is orthogonal to $\boldsymbol{\mathcal{O}}(X)$, then $\operatorname{supp}\mu \subset X^*$. We shall show, essentially following [2], that μ is the zero measure.

Let d be the dimension of X.

Consider a point $x \in X$. According to [2], there is an $r > 0$ small enough that for a sequence $Y(0), \dots, Y(d)$ of real-analytic subvarieties of $\mathbb{B}_N(x, r)$ we have $Y(0) = X \cap \mathbb{B}_N(x, r)$ and for all choices of k, $\dim Y(k) \leq d - k$ and $Y^*(k-1) \subset Y_k \subset Y(k-1)$. Define $Z(k)$ to be the set $(X \setminus \mathbb{B}_N(x, r)) \cup Y(k)$, a certain compact subset of X.

Lemma 7. *The set $Z(k)$ is holomorphically convex.*

Proof. The set $Z(0) = X$ is holomorphically convex. We assume $Z(k)$ to be holomorphically convex and deduce that $Z(k+1)$ is holomorphically convex. For this, let χ be a character, *i.e.*, a nonzero multiplicative $\mathbb{C}$-valued linear functional, on $\boldsymbol{\mathcal{O}}(Z(k+1))$. We are to show the existence of a point $x \in Z(k+1)$ such that $\chi(\mathbf{f}) = \mathbf{f}(x)$ for all $\mathbf{f} \in \boldsymbol{\mathcal{O}}(Z(k+1))$. The restriction of functions induces a natural homomorphism $\rho : \boldsymbol{\mathcal{O}}(Z(k)) \to \boldsymbol{\mathcal{O}}(Z(k+1))$, and the map $\mathbf{f} \mapsto \chi(\rho(\mathbf{f}))$ is a character on $\boldsymbol{\mathcal{O}}(Z(k))$. Consequently, there is a unique point $p \in Z(k)$ with the property that $\chi(\rho(\mathbf{f})) = \mathbf{f}(p)$ for all $\mathbf{f} \in \boldsymbol{\mathcal{O}}(Z(k))$. We shall show that the point p lies in $Z(k+1)$.

Each character ψ of $\mathcal{O}(Z(k+1))$ satisfies $|\psi(\mathbf{g})| \leq \sup_{Z(k+1)} |\mathbf{g}|$ for all $\mathbf{g} \in \boldsymbol{\mathcal{O}}(Z(k+1))$. An argument parallel to the proof of Lemma 6 shows that $Y^c(k)_{reg}$ is a nowhere dense closed analytic subvariety of $Y(k)_{reg}$. The theorem of O'Farrell, Preskenis, and Walsh used above implies that each $f \in \mathcal{C}(Z(k))$ that vanishes on $(X \setminus \mathbb{B}_N(x, r)) \cup Y(k)^*$ can be approximated uniformly by functions holomorphic on a neighborhood of $Z(k)$. Thus, if $q \in Z(k) \setminus Z(k+1) = Y(k) \setminus Y(k+1)$, there is $\mathbf{h} \in \boldsymbol{\mathcal{O}}(Z(k))$ with $\mathbf{h}(q) = 1 > \sup_{Z(k+1)} |\mathbf{h}|$. It follows that the point p lies in $Z(k+1)$, whence χ is found to be an evaluation at a point of $Z(k+1)$ as claimed.

Proof of Theorem 1*, concluded.* Return to the orthogonal measure μ. We have that it is supported in $X = Z(0)$. Let k be an integer such that $\operatorname{supp}\mu \subset Z(k)$. We claim that then $\operatorname{supp}\mu \subset Z(k+1)$. To see this, we argue as we have already to find that because $Z(k)$ is holomorphically convex and μ is orthogonal to $\boldsymbol{\mathcal{O}}(Z(k))$, the support of μ must be contained in $(X \setminus \mathbb{B}_N(x, r)) \cup Y(k)^*$, which is a subset of $Z(k+1)$. By applying this remark repeatedly, we find finally that $\operatorname{supp}\mu$ is contained in the union of $X \setminus \mathbb{B}_N(x, r)$ and a variety of dimension not more than zero, *i.e.*, the empty set or a discrete subset of $\mathbb{B}_N(x, r)$. A finite number of the balls $\mathbb{B}_N(x, r)$ cover the set X, so the compact set $\operatorname{supp}\mu$ is countable. As μ annihilates all polynomials, it is necessarily the zero measure.

The theorem is proved.

Remark. As noted above, not every compact holomorphically convex set in $\mathbb{C}^N$ is the intersection of a sequence of domains of holomorphy. It is conceivable, though, that every compact, real-analytic subvariety of $\mathbb{C}^N$ that is holomorphically convex is such an intersection. The author knows neither of a proof nor a counterexample to this possibility.

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