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ISOMORPHIC ℓ^p-SUBSPACES IN ORLICZ-LORENTZ SEQUENCE SPACES

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ABSTRACT. Given a decreasing weight w and an Orlicz function φ satisfying the Δ_2-condition at zero, we show that the Orlicz-Lorentz sequence space $d(w,\varphi)$ contains an $(1+\epsilon)$-isomorphic copy of ℓ_p, $1 \le p < \infty$, if and only if the Orlicz sequence space ℓ_φ does, that is, if $p \in [\alpha_\varphi, \beta_\varphi]$, where α_φ and β_φ are the Matuszewska-Orlicz lower and upper indices of φ, respectively. If φ does not satisfy the Δ_2-condition, then a similar result holds true for order continuous subspaces $d_0(w,\varphi)$ and h_φ of $d(w,\varphi)$ and ℓ_φ, respectively.

In the early seventies, Lindenstrauss and Tzafriri studied isomorphic copies of ℓ_p, $1 \le p \le \infty$, in Orlicz sequence spaces. They showed [9] (see also [8]) that ℓ_p (c_0 if $p = \infty$) is isomorphic to a subspace of h_φ, the order continuous part of ℓ_φ, if and only if p belongs to the closed interval determined by Matuszewska-Orlicz indices of the Orlicz function φ. Analogous results were then obtained in function Orlicz spaces by Hernández and Rodriguez-Salinas in [4]. In this paper we extend the result of Lindenstrauss and Tzafriri to Orlicz-Lorentz sequence spaces. Let $d(w,\varphi)$ be an Orlicz-Lorentz space and $d_0(w,\varphi)$ its order continuous subspace. We prove that for every $\epsilon > 0$, $d_0(w,\varphi)$ contains an $(1+\epsilon)$-isomorphic copy of ℓ_p, $1 \le p \le \infty$, where c_0 is considered for $p = \infty$, if and only if h_φ contains an isomorphic copy of ℓ_p. The latter is equivalent to $p \in [\alpha_\varphi, \beta_\varphi]$, where $1 \le \alpha_\varphi \le \beta_\varphi \le \infty$ are Matuszewska-Orlicz indices. If $d_0(w,\varphi)$ coincides with the whole space $d(w,\varphi)$, that is, when φ satisfies condition Δ_2, then the analogous characterization holds for $d(w,\varphi)$, ℓ_φ and ℓ_p, $1 \le p \le \infty$, respectively. The characterization is somewhat unexpected since it does not depend on the weight w, although several other properties in Orlicz-Lorentz spaces do [5, 6, 7, 13].

Throughout the paper we shall use the Banach space theory standard terminology mostly following the monograph [9]. Further, let $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural and real numbers, respectively. Recall that $[x_n]$ denotes the closed linear span of a sequence (x_n) in a Banach space X. The term basis will be strictly reserved for a Schauder basis. We say that two basic sequences in Banach spaces, (x_n) in $(X, \|\cdot\|_X)$ and (y_n) in $(Y, \|\cdot\|_Y)$, are C-equivalent whenever for any real sequence (a_n) we have

$$C^{-1}\Big\|\sum_{n=1}^\infty a_n x_n\Big\|_X \le \Big\|\sum_{n=1}^\infty a_n y_n\Big\|_Y \le C\Big\|\sum_{n=1}^\infty a_n x_n\Big\|_X.$$

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The basic sequences (x_n) and (y_n) are said to be *almost isometrically equivalent* if for all $k \geq 1$ the tails $(x_n)_{n\geq k}$ and $(y_n)_{n\geq k}$ are $(1+\epsilon_k)$-equivalent with $\varepsilon_k \to 0$ when $k \to \infty$. We say that a Banach space Y *contains a C-isomorphic copy* of a Banach space X if there exists a bounded linear operator $T : X \to Y$ with $C^{-1}\|x\| \leq \|Tx\| \leq C\|x\|$ for every $x \in X$.

An *Orlicz function* is a function $\varphi : [0,\infty) \to [0,\infty]$ such that $\varphi(0) = 0$, where φ is convex and not identically zero (for some technical reasons we do not assume that the Orlicz functions φ that we consider are *normalized*, i.e. that $\varphi(1) = 1$). The Orlicz function φ is called *nondegenerate* if it is finite-valued and vanishes only at zero. Throughout the paper we assume that any Orlicz function φ (or ϕ) is nondegenerate. However in the process of studies, degenerate Orlicz functions may also appear. In this case they will be denoted exclusively by the symbol ψ. A *weight sequence* $w = (w(n))$ is a positive decreasing sequence such that $w(1) = 1$, $\lim_{n\to\infty} w(n) = 0$ and $\lim_{n\to\infty} W(n) = \infty$, where $W(n) = \sum_{i=1}^n w(i)$ for every $n \in \mathbb{N}$. The *Orlicz-Lorentz sequence space* $d(w,\varphi)$ consists of all bounded real sequences $\lambda = (\lambda_n)$ such that for some $K > 0$, $I(K\lambda) < \infty$, where

$$I(\lambda) = \sum_{n=1}^{\infty} \varphi(\lambda_n^*)w(n) = \sup\Big\{\sum_{n=1}^{\infty} \varphi(|\lambda_{\pi(n)}|)w(n) : \pi \text{ is an injection } \mathbb{N} \to \mathbb{N}\Big\},$$

and $\lambda^* = (\lambda_n^*)$ is the decreasing rearrangement of $|\lambda| = (|\lambda_n|)$. The space $d(w,\varphi)$, equipped with the norm

$$\|\lambda\| = \inf\{\epsilon : I(\lambda/\epsilon) \leq 1\},$$

is a Banach space. Notice that the assumption $\lim_{n\to\infty} W(n) = \infty$ yields that $d(w,\varphi) \hookrightarrow c_0$. Let $d_0(w,\varphi)$ be the closure of finitely supported sequences in $d(w,\varphi)$. We say that φ satisfies the Δ_2-condition (at zero), if for some $K > 0$ and $t_0 > 0$ it holds for every $0 < t \leq t_0$ that

$$\varphi(2t) \leq K\varphi(t).$$

The symbol e_n, $n \in \mathbb{N}$, will stand for the unit vectors $(0,\dots,0,1_n,0,\dots)$.

If $\varphi(u) = u^p$, $1 \leq p < \infty$, then $d(w,p) := d(w,\varphi)$ is a Lorentz sequence space. If $w(n) = 1$ for every $n \in \mathbb{N}$, then $\ell_\varphi := d(w,\varphi)$ is an Orlicz sequence space, and $h_\varphi = d_0(w,\varphi)$ is its subspace of finite elements. In the case of Orlicz space ℓ_φ, the modular I will be denoted by I_φ, and thus for any sequence $\lambda = (\lambda_n)$,

$$I_\varphi(\lambda) = \sum_{n=1}^{\infty} \varphi(|\lambda_n|).$$

It is well known that the unit vectors e_n form a symmetric basis in h_φ and also that $\ell_\varphi = h_\varphi$ if and only if φ satisfies condition Δ_2 (cf. Propositions 4.a.2, 4.a.4 in [9]). We say that two Orlicz functions φ_1 and φ_2 are equivalent (at zero) whenever there exist $K > 0$ and $t_0 > 0$ such that for all $0 \leq t \leq t_0$,

$$\varphi_1(K^{-1}t) \leq \varphi_2(t) \leq \varphi_1(Kt).$$

Given two Orlicz functions φ_1 and φ_2, the unit vector bases of h_{φ_1} and h_{φ_2} are equivalent if and only if φ_1 and φ_2 are equivalent (cf. Proposition 4.a.5 in [9]).

For an Orlicz function φ, define the lower and upper Matuszewska-Orlicz indices [11, 9] as follows:

$$\alpha_\varphi = \sup\{r : \sup_{0<a,t\le 1} \varphi(at)/\varphi(a)t^r < \infty\},$$
$$\beta_\varphi = \inf\{r : \inf_{0<a,t\le 1} \varphi(at)/\varphi(a)t^r > 0\}.$$

It is well known and easy to show that $\beta_\varphi < \infty$ if and only if φ satisfies condition Δ_2. Recall that $\Phi = (\varphi_n)_{n=1}^\infty = (\varphi_n)$ is called a *Musielak-Orlicz function* if all φ_n are Orlicz functions. Then setting for a real sequence $\lambda = (\lambda_n)$ the modular

$$I_\Phi(\lambda) = \sum_{n=1}^\infty \varphi_n(|\lambda_n|),$$

the *Musielak-Orlicz sequence space* ℓ_Φ is the set of all $\lambda = (\lambda_n)$ such that

$$\|\lambda\|_\Phi = \inf\{\epsilon > 0 : I_\Phi(\lambda/\epsilon) \le 1\} < \infty.$$

The space ℓ_Φ equipped with the norm $\|\cdot\|_\Phi$ is a Banach space. We note that if all functions φ_n coincide with the same Orlicz function φ_0, then ℓ_Φ is the Orlicz space ℓ_{φ_0}.

Given $u = \sum_{i=m+1}^{m+k} a_i e_i$, $m, k \in \mathbb{N}$, $a_i \in \mathbb{R}$, the function

$$\varphi^{(u)}(t) = I(tu) = \sum_{i=1}^k \varphi(t|a_i^*|)w(i), \qquad t \ge 0,$$

will be called the *Orlicz function associated to* u, where $(a_i^*)_{i=1}^k$ is a decreasing rearrangement of $(a_i)_{i=m+1}^{m+k}$. If (u_n) is a block basic sequence in $d(w,\varphi)$, i.e.

$$u_n = \sum_{i=q_n+1}^{q_{n+1}} a_i e_i,$$

where $q_1 < q_2 < \dots$ are integers, then the function $\Phi = (\varphi_n)_{n=1}^\infty = (\varphi_n)$ will be called the *Musielak-Orlicz function associated to* (u_n) whenever $\varphi_n = \varphi^{(u_n)}$ for every $n \in \mathbb{N}$. Then if the sequence (a_i) is decreasing and positive, the modular I_Φ corresponding to Φ has the following form:

$$I_\Phi(\lambda) = \sum_{n=1}^\infty \varphi_n(|\lambda_n|) = \sum_{n=1}^\infty I(\lambda_n u_n) = \sum_{n=1}^\infty \sum_{i=q_n+1}^{q_{n+1}} \varphi(|\lambda_n a_i|)w(i-q_n),$$

where $\lambda = (\lambda_n)$ is an arbitrary sequence in $\mathbb{R}$. Let $C[0,s]$ be the space of all continuous real-valued functions on the interval $[0,s]$ equipped with the usual uniform norm

$$\|f\|_{C[0,s]} = \sup_{t\in[0,s]} |f(t)|.$$

Given an Orlicz function φ and $a \in (0,+\infty)$, let φ_a be the function φ scaled at a, defined by

$$\varphi_a(t) = \frac{\varphi(at)}{\varphi(a)}, \qquad t \ge 0.$$

Let us now define the following sets of functions mapping $[0,+\infty)$ into $[0,+\infty]$. For $0<A<\infty$, let

$$E^0_{\varphi,A}=\{\varphi_a:0<a<A\}\ ;\quad E_{\varphi,A}=\overline{E^0_{\varphi,A}}\ ;\quad C_{\varphi,A}=\overline{\operatorname{conv}E^0_{\varphi,A}}\ ;$$
$$E_\varphi=\bigcap_{A>0}E_{\varphi,A}\ ;\quad C_\varphi=\bigcap_{A>0}C_{\varphi,A}\ .$$

Here $\operatorname{conv}X$ denotes the set of all convex combinations of functions in X, while $\overline{X}$ is the pointwise closure of X (in the space of $[0,+\infty]$-valued functions on $[0,+\infty)$). Note that $0=\varphi_a(0)\le\varphi_a(t)\le\varphi_a(1)=1$ for every $0\le t\le 1$ and $0<a<\infty$, so the functions of $C_{\varphi,A}$ verify the same inequalities (in particular they are real-valued on $[0,1]$): these are possibly degenerate Orlicz functions (they can vanish outside 0 and take the value $+\infty$ at some $t>1$). The sets $E_{\varphi,A},C_{\varphi,A}$ are compact subsets of $[0,+\infty]^{[0,+\infty)}$; hence E_φ,C_φ are nonempty. It is well known that for any $0<s<1$ the restrictions to the interval $[0,s]$ of the sets $E_{\varphi,A},C_{\varphi,A}$ consist of continuous functions and are compact subsets of $C[0,s]$ for the uniform norm (see Lemma 4.a.6 in [9] and the Remark thereafter). It is also well known that if φ satisfies condition Δ_2, then the sets $E_{\varphi,A}$, $C_{\varphi,A}$, E_φ and C_φ consist of nondegenerate Orlicz functions and their restrictions to $[0,1]$ are compact subsets of the space $C[0,1]$.

Note that if $u=\sum_{i=m+1}^{m+k}a_ie_i$ has norm one in $d(w,\varphi)$, then $\varphi^{(u)}$ belongs to $C_{\varphi,A}$, where $A=\|u\|_\infty=\max_{\{i=m+1,\dots,m+k\}}|a_i|$ is the c_0 norm of u. Indeed,

$$\varphi^{(u)}(t)=\sum_{i=1}^k\varphi(ta_i^*)w(i)=\sum_{i=1}^k\varphi(a_i^*)w(i)\varphi_{a_i^*}(t),$$

where $a_i^*\le\|u\|_\infty$ and

$$\varphi^{(u)}(1)=I(u)=\sum_{i=1}^k\varphi(a_i^*)w(i)=1.$$

For more information on Orlicz-Lorentz or Lorentz spaces we refer the reader to [1, 2, 5, 7, 9, 13], on Orlicz spaces to [3, 9, 10], and on Musielak-Orlicz spaces to [12, 14].

The first proposition is a collection of basic properties of $d(w,\varphi)$ and $d_0(w,\varphi)$, which are analogous to the corresponding properties in Orlicz sequence spaces (cf. Propositions 4.a.2 and 4.a.4 in [9]). The proof can be done in a very similar way as in the case of Orlicz spaces, so we skip it. As characteristic features of Orlicz-Lorentz spaces, the following two facts are employed in the proof. The first is that the modular I is orthogonally subadditive, that is, $I(\lambda+\gamma)\le I(\lambda)+I(\gamma)$ for disjoint sequences $\lambda=(\lambda_n)$ and $\gamma=(\gamma_n)$. The second fact is that for any sequence $\lambda^{(m)}=(\lambda_n^{(m)})\subset c_0$ such that $\lambda^{(m)}\downarrow 0$, that is, $\lambda_n^{(m)}\downarrow 0$ as $m\to\infty$ for every $n\in\mathbb{N}$, we also have that $\lambda^{(m)*}\downarrow 0$ as $m\to\infty$.

Proposition 1. I. *The subspace $d_0(w,\varphi)$ coincides with the set of all sequences $\lambda=(\lambda_n)$ such that for every $K>0$, $I(K\lambda)<\infty$. Moreover, the sequence of the unit vectors (e_n) is a symmetric basis in $d_0(w,\varphi)$.*

II. *The following assertions are equivalent:*

(i) *The Orlicz function φ satisfies condition Δ_2.*
(ii) *The unit vectors e_n form a boundedly complete basis in $d_0(w,\varphi)$.*

(iii) $d(w,\varphi) = d_0(w,\varphi)$.
(iv) $d_0(w,\varphi)$ *does not contain a closed subspace isomorphic to* c_0.

Now we state our first main result.

Proposition 2. *Let* φ *be an arbitrary Orlicz function and for the integers* $q_1 < q_2 < \dots$ *and* $n \in \mathbb{N}$ *let*

$$u_n = \sum_{i=q_n+1}^{q_{n+1}} a_i e_i$$

be a normalized block basis in $d(w,\varphi)$. *If* $\lim_{i\to\infty} a_i = 0$, *then there exists a subsequence* $(u_{n_j}) \subset (u_n)$ *which is almost isometrically equivalent to the unit vector basic sequence* (e_j) *in* ℓ_Φ, *where* Φ *is the associated Musielak-Orlicz function to the block basis* (u_{n_j}).

Proof. By symmetry of the basis (e_j) in $d(w,\varphi)$, we can assume that (a_i) is a positive decreasing sequence.

By a standard diagonal argument it will be sufficient to prove that for every $\varepsilon > 0$ there exists a subsequence $(u_{n_j}) \subset (u_n)$ which is $(1+\epsilon)$-equivalent to the unit vector basic sequence (e_j) in ℓ_Φ.

For any sequence (λ_j) of real numbers and any subsequence (u_{n_j}), by orthogonal subbadditivity of the modular I we have

$$I\Big(\sum_{j=1}^{\infty} \lambda_j u_{n_j}\Big) \le \sum_{j=1}^{\infty} I(\lambda_j u_{n_j}). \tag{1}$$

Since $\lim_{i\to\infty} a_i = 0$, it is clear that for any integer $Q \ge 1$

$$\sum_{i=q_n+1}^{q_n+Q} \varphi(a_i) w(i-q_n) \underset{n\to\infty}{\longrightarrow} 0.$$

Put $n_1 = 1$ and $Q_1 = q_2 - q_1$. Then, given $\epsilon \in (0,1)$ there exists $n_2 > n_1$ such that

$$\sum_{i=q_{n_2}+1}^{q_{n_2}+Q_1} \varphi(a_i) w(i-q_{n_2}) < \epsilon/2.$$

Note that since $\varepsilon/2 < I(u_{n_2}) = 1$ we have $q_{n_2+1} - q_{n_2} > Q_1$. Put $Q_2 = Q_1 + q_{n_2+1} - q_{n_2}$.

In a similar manner we will find by induction a sequence $n_1 < n_2 < \dots$ such that for all $j \in \mathbb{N}$,

$$Q_{j-1} := \sum_{i=1}^{j-1} (q_{n_i+1} - q_{n_i}) < q_{n_j+1} - q_{n_j} \quad \text{and} \quad \sum_{i=q_{n_j}+1}^{q_{n_j}+Q_{j-1}} \varphi(a_i) w(i-q_{n_j}) < \epsilon/2^{j-1},$$

where $Q_0 = 0$. Now setting

$$v_{n_j} = \sum_{i=q_{n_j}+1}^{q_{n_j}+Q_{j-1}} a_i e_i,$$

we have for all $j \in \mathbb{N}$,

$$I(v_{n_j}) < \epsilon/2^{j-1}.$$

Now let $\pi : \mathbb{N} \to \mathbb{N}$ be the injective map such that for all $j \in \mathbb{N}$,

$$\pi(i) = q_{n_j} + i \quad \text{for} \quad i = Q_{j-1}+1, \dots, q_{n_j+1} - q_{n_j}, \quad \text{and}$$
$$\pi(i) = 2q_{n_j} - q_{n_j+1} + i \quad \text{for} \quad i = q_{n_j+1} - q_{n_j} + 1, \dots, Q_j.$$

Note that $\pi\{Q_{j-1}+1, \dots, Q_j\} = \{q_{n_j}+1, \dots, q_{n_j+1}\}$ is the support of u_{n_j}. Then for any sequence $(\lambda_j) \subset \mathbb{R}$ we have

$$\begin{aligned} I\Big(\sum_{j=1}^{\infty} \lambda_j u_{n_j}\Big) &\geq \sum_{i=1}^{\infty} \varphi\Big(\Big|\sum_{j=1}^{\infty} \lambda_j u_{n_j}(\pi(i))\Big|\Big) w(i) = \sum_{j=1}^{\infty} \sum_{i=Q_{j-1}+1}^{Q_j} \varphi(|\lambda_j| a_{\pi(i)}) w(i) \\ &\geq \sum_{j=1}^{\infty} \sum_{i=Q_{j-1}+1}^{q_{n_j+1}-q_{n_j}} \varphi(|\lambda_j| a_{\pi(i)}) w(i) \\ &= \sum_{j=1}^{\infty} \sum_{i=q_{n_j}+Q_{j-1}+1}^{q_{n_j+1}} \varphi(|\lambda_j| a_i) w(i - q_{n_j}) \\ &= \sum_{j=1}^{\infty} I(\lambda_j u_{n_j}) - \sum_{j=1}^{\infty} I(\lambda_j v_{n_j}). \end{aligned}$$

Hence

$$\sum_{j=1}^{\infty} I(\lambda_j u_{n_j}) \leq I\Big(\sum_{j=1}^{\infty} \lambda_j u_{n_j}\Big) + \sum_{j=1}^{\infty} I(\lambda_j v_{n_j}). \tag{2}$$

Now, let Φ be the Musielak-Orlicz function associated to (u_{n_j}). Then by (1), for any $\lambda = (\lambda_j)$,

$$I\Big(\sum_{j=1}^{\infty} \lambda_j u_{n_j}\Big) \leq I_\Phi(\lambda).$$

Hence

$$\Big\|\sum_{j=1}^{\infty} \lambda_j u_{n_j}\Big\| \leq \|\lambda\|_\Phi.$$

On the other hand, letting $\|\sum_{j=1}^{\infty} \lambda_j u_{n_j}\| = 1$, we get $I_\Phi(\lambda) \leq 1 + \epsilon$ by (2). Thus by convexity of Φ, $I_\Phi(\lambda/(1+\epsilon)) \leq I_\Phi(\lambda)/(1+\epsilon) \leq 1$, which yields

$$\|\lambda\|_\Phi \leq 1 + \epsilon,$$

and finishes the proof. □

Corollary 3. *Let φ be an Orlicz function and for $n \in \mathbb{N}$ let*

$$u_n = \sum_{i=q_n+1}^{q_{n+1}} a_i e_i$$

be a normalized block basis in $d(w,\varphi)$, where (q_n) is an increasing sequence of natural numbers. Then there exists a normalized block basis (v_n) of (u_n) which is almost isometrically equivalent either to the unit vector basis of c_0 or to the unit vector basis in ℓ_Φ, where Φ is the associated Musielak-Orlicz function to the block basis (v_n). In the second case we may suppose moreover that the sequence (v_n) converges to zero in c_0 norm.

Proof. By Theorem 1.c.10 in [9] if the unconditional basic sequence (u_n) is not boundedly complete, it has a block basis (z_n) of (u_n) equivalent to the unit vector basis in c_0. By the well-known result of James (see 2.e.3 in [9]), for every $\varepsilon > 0$ a further block basis (v_n) is $(1+\varepsilon)$-equivalent to the basis of c_0. By a diagonal argument we obtain a block basis almost isometrically equivalent to the unit vector basis of c_0. On the other hand if (u_n) is boundedly complete, then there exists a sequence $s_1 < s_2 < \dots$ of integers such that the numbers $\alpha_n = \|\sum_{i=s_n+1}^{s_{n+1}} u_i\|$ approach infinity. Then defining

$$z_n = \frac{1}{\alpha_n} \sum_{i=s_n+1}^{s_{n+1}} u_i = \sum_{i=s_n+1}^{s_{n+1}} b_i e_i,$$

(z_n) is a normalized block basis of (e_j) in $d(w,\varphi)$ such that $\lim_{i\to\infty} b_i = 0$. Now applying Proposition 2 we conclude the proof. $\square$

The following result is well known (cf. Theorem 3.6 in [14], under Δ_2-condition), but we state it in a different, more suitable form for our purpose. We also prove it for the sake of completeness.

Lemma 4. *Let $\Phi = (\varphi_n)$ be a Musielak-Orlicz function associated to some normalized sequence (u_n) in $d(w,\varphi)$. Then there exists a subsequence $(n_j) \subset \mathbb{N}$ and a (possibly degenerate) Orlicz function $\psi \in C_{\varphi,1}$ such that (e_j) in h_ψ is almost isometrically equivalent to (e_{n_j}) in ℓ_Φ. If moreover the given sequence (u_n) converges to zero in c_0 norm, then ψ can be found in C_φ.*

Proof. By our assumptions, $\varphi_n \in C_{\varphi,A}$ for every $n \in \mathbb{N}$, where $A = \varphi^{-1}(1)$. By pointwise compactness of $C_{\varphi,A}$, there exists a pointwise limit point ψ of (φ_n) in $C_{\varphi,A}$. By compactness of the restriction of $C_{\varphi,A}$ to each $C[0,s]$, $0 < s < 1$, there exists for each s a subsequence (φ_{n_j}) which converges to ψ in the metric of $C[0,s]$. By a diagonal argument we can find a subsequence of (φ_n) which realizes simultaneously all these convergences, i.e. for every $j \in \mathbb{N}$,

$$d_{1-2^{-j}}(\psi, \bar\varphi_j) = \sup_{t\in[0,1-2^{-j}]} |\psi(t) - \bar\varphi_j(t)| \le 2^{-j-1}, \tag{3}$$

where $\bar\varphi_j = \varphi_{n_j}$. Note that if $\|u_n\|_\infty \to 0$, then ψ belongs to all $C_{\varphi,\rho}$ with $\rho > 0$, hence to C_φ.

Set $\bar\Phi = (\bar\varphi_j)$ and let $\lambda = (\lambda_j)$ be a sequence of reals with $\lambda_j = 0$ for $j < k$. It follows from the preceding that if $I_{\bar\Phi}(\lambda) \le 1$, then $I_\psi((1-2^{-k})\lambda) \le I_{\bar\Phi}(\lambda) + 2^{-k} \le 1 + 2^{-k}$, which yields that $\|\lambda\|_\psi \le \frac{(1+2^{-k})}{(1-2^{-k})}\|\lambda\|_{\bar\Phi} \le (1+2^{-k+2})\|\lambda\|_{\bar\Phi}$ and similarly $\|\lambda\|_{\bar\Phi} \le (1+2^{-k+2})\|\lambda\|_\psi$. Thus $(e_j)_{j\ge k}$ in h_ψ is $(1+2^{-k+2})$-equivalent to $(e_j)_{j\ge k}$ in $\ell_{\bar\Phi}$. $\square$

Corollary 5. *For every closed infinite-dimensional subspace X of $d_0(w,\varphi)$, there exists a closed subspace of X which is almost isometrically equivalent either to c_0 or to some Orlicz space h_ψ associated to a (possibly degenerate) function $\psi \in C_\varphi$.*

Proof. By a well-known result (cf. Proposition 1.a.11 in [9]) there exists a subspace of X with a basis (y_n) which is almost isometrically equivalent to a normalized block basis (u_n) of (e_n) in $d_0(w,\varphi)$. By Corollary 3, there exists a normalized block basis (v_n) of (u_n) which is almost isometrically equivalent to (e_j) either in c_0 or in ℓ_Φ, where Φ is the associated Musielak-Orlicz function to (v_n). In the second case we may suppose moreover that the sequence (v_n) converges to zero in c_0 norm.

Hence by Lemma 4 there exist a subsequence (v_{n_j}) of (v_n) and $\psi \in C_\varphi$ such that (v_{n_j}) in $d_0(w,\varphi)$ is almost isometrically equivalent to (e_j) in h_ψ. Finally, since (e_j) is a basis in h_ψ, h_ψ is $(1+\epsilon_k)$-isomorphic to the subspace $[v_{n_j}]_{j\geq k}$ of X in $d_0(w,\varphi)$, with $\varepsilon_k \to 0$, and the proof is completed. $\square$

We observe that as a direct consequence of Corollary 5 we obtain that every closed infinite-dimensional subspace X of the Lorentz sequence space $d(w,p)$, $1 \leq p < \infty$, contains a subspace which is isomorphic to ℓ^p (cf. [1, 9]).

Proposition 6. *Let φ be an Orlicz function and let $1 \leq p < \infty$. If ℓ_p is isomorphic to a subspace of $d_0(w,\varphi)$, then the function u^p is equivalent to some function in the class C_φ.*

Proof. By Corollary 5, if ℓ_p is isomorphic to a closed subspace X of $d_0(w,\varphi)$, then ℓ_p contains a subspace Y which is isomorphic to some h_ψ, $\psi \in C_\varphi$; the other possibility that Y is isomorphic to c_0 is excluded by the condition $p<\infty$. It follows that $\psi(u)$ is equivalent to some function in $C_{u^p,1}$ by Theorem 4.a.8 in [9], but this last set consists of the only single function $u \mapsto u^p$. $\square$

We say that two sets A, B of functions $[0,+\infty) \to [0,+\infty]$ *coincide on* $[0,s]$ if their restrictions to $[0,s]$ coincide, that is,

$$\{f\,|_{[0,s]} : f \in A\} = \{f\,|_{[0,s]} : f \in B\}.$$

Observe that if two normalized Orlicz functions ψ_1, ψ_2 coincide on $[0,1]$, they define algebraically and isometrically identical Orlicz sequence spaces.

Lemma 7. *For every Orlicz function φ the sets C_φ and $\overline{\operatorname{conv} E_\varphi}$ coincide on $[0,1]$.*

Proof. It is clear that $\overline{\operatorname{conv} E_\varphi} \subset C_\varphi$. Thus we need to prove that for every $0<s<1$ the restriction to $[0,s]$ of any $\psi \in C_\varphi$ can be approximated in the metric d_s of $C[0,s]$ by a sequence of convex combinations of (restrictions of) elements of E_φ. This in turn will be sufficient since all the functions in both sets take the value 1 at the point 1.

Since the restriction of $E_{\varphi,1}$ to $[0,s]$ is compact, for every $\epsilon > 0$ there is a finite covering of $E_{\varphi,1}$ by sets of d_s-diameter less than ϵ. Let $\nu(\epsilon)$ be the minimal cardinal of such a covering $(S_i)_{i\leq\nu(\epsilon)}$. If now F is any subset of $E_{\varphi,1}$, then the sets $S_i \cap F$, $i=1,\dots,\nu(\epsilon)$, also form a covering of F, and have diameter less than ϵ. If we choose a point $\gamma_i \in S_i \cap F$, for each i with $S_i \cap F \neq \emptyset$ and an arbitrary point $\gamma_i \in F$ when $S_i \cap F = \emptyset$, then the family (γ_i) is an ϵ-net in F, that is, the balls $B(\gamma_i,\epsilon) = \{\gamma \in C[0,s] : d_s(\gamma_i,\gamma) < \epsilon\}$ cover F, of cardinality $\nu(\epsilon)$. Given $\xi \in C[0,s]$ and $A \subset C[0,s]$, let $d_s(\xi, A)$ be the distance of ξ to A. Then the set

$$\{\xi \in \operatorname{conv} F : d_s(\xi, \operatorname{conv}(\gamma_i, i=1,\dots,\nu(\epsilon))) \leq \epsilon\}$$

is convex and contains the sets $F \cap B(\gamma_i,\epsilon)$, $i=1,\dots,n$, hence contains F. Thus this set coincides with $\operatorname{conv} F$. In other words for every $\xi \in \operatorname{conv} F$ there is a convex combination $\xi' = \sum_{i=1}^{\nu(\epsilon)} \beta_i \gamma_i$ such that $d_s(\xi,\xi') \leq \epsilon$.

Now fix $\psi \in C_\varphi$ and $\varepsilon > 0$. There exist a sequence $(A_k)_{k\geq 1}$ of positive reals converging to zero and a sequence (ϕ_k) of Orlicz functions in $\operatorname{conv} E^0_{\varphi,A_k}$ such that $d_s(\psi,\phi_k) \longrightarrow 0$ as $k \to \infty$. Applying the preceding to the sets $F_k = E^0_{\varphi,A_k}$, for every $k \geq 1$ we can find a system of $\nu = \nu(\varepsilon)$ elements $\phi_1^{(k)}, \phi_2^{(k)}, \dots, \phi_\nu^{(k)}$ in E^0_{φ,A_k}

and a system $\beta_1^{(k)}, \beta_2^{(k)}, \dots, \beta_\nu^{(k)}$ of nonnegative coefficients with sum 1 such that

$$d_s\big(\phi_k, \sum_{i=1}^{\nu} \beta_i^{(k)} \phi_i^{(k)}\big) \le \epsilon.$$

Up to passing to a subsequence, we may suppose that $\beta_i^{(k)} \longrightarrow \beta_i \in [0,1]$, $\phi_i^{(k)} \longrightarrow \psi_i \in E_\varphi$, $i = 1, \dots, \nu$, as $k \to \infty$. Then

$$d_s\big(\psi, \sum_{i=1}^{\nu} \beta_i \psi_i\big) \le \epsilon,$$

and the proof is finished. □

Theorem 8. *For every $\psi \in C_\varphi$ there exists a basic sequence in $d_0(w,\varphi)$ which is almost isometrically equivalent to the unit vector basis in h_ψ. Thus for every $\epsilon > 0$ there exists an $(1+\epsilon)$-isomorphic copy of h_ψ in $d_0(w,\varphi)$.*

Proof. We shall prove that there exists a sequence of finite blocks u_k of the unit vector basis of $d_0(w,\varphi)$ such that $\|u_k\|_\infty \to 0$, $\|u_k\| \to 1$, and the associated Orlicz functions $\varphi^{(u_k)}$ converge to ψ in all d_s metrics, $0 < s < 1$. Then shifting the u_k to the right we obtain a block basis (z_k) with the same properties. It is easy to see that the normalized block basis $z_k' = z_k/\|z_k\|$ also shares the same properties. In fact we have $\varphi^{(z_k')}(t) = \varphi^{(z_k)}(t/\|z_k\|)$ for all $t > 0$. Hence

$$|\varphi^{(z_k')}(t) - \psi(t)| \le \left|\varphi^{(z_k)}\left(\frac{t}{\|z_k\|}\right) - \psi\left(\frac{t}{\|z_k\|}\right)\right| + \left|\psi\left(\frac{t}{\|z_k\|}\right) - \psi(t)\right|,$$

and the right side converges to zero uniformly in $t \in [0,s]$ since for any $s < s' < 1$, we have $t/\|z_k\| \in [0,s']$ for all $t \in [0,s]$ as soon as $\|z_k\| > s/s'$. By Proposition 2 and Lemma 4 (and its proof) some subsequence of (z_k') is almost isometrically equivalent to the unit vector basis of h_ψ.

For constructing the sequence (u_k) it is sufficient by Lemma 7 to proceed in the case where $\psi \in \operatorname{conv} E_\varphi$. Let $\psi = \sum_{i=1}^{\nu} \beta_i \psi_i$, with $0 < \beta_i \le 1$, $\sum_{i=1}^{\nu} \beta_i = 1$ and $\psi_i \in E_\varphi$. Then for every $k \in \mathbb{N}$ there exist positive reals $b_i^{(k)} > 0$, $i = 1, ..., \nu$, such that the functions

$$\phi_k := \sum_{i=1}^{\nu} \beta_i \varphi_{b_i^{(k)}}$$

converge in all d_s metrics, $0 < s < 1$, to ψ, and moreover $A_k := \sup\limits_{i=1,\dots,\nu} b_i^{(k)} \longrightarrow 0$. In particular we can also assume that $\varphi(A_k) \le \inf\limits_{i=1,\dots,\nu} \beta_i$.

For a moment let k be fixed and let $b_i = b_i^{(k)}$. Up to reordering we may assume that $b_1 \ge b_2 \ge \dots \ge b_\nu$. Since $w(n) \le 1 = w(1)$ for every $n \in \mathbb{N}$ and $W(n) \to \infty$ as $n \to \infty$, we can find by induction the integers $0 = r_0 < r_1 < \dots < r_\nu$ such that

$$W(r_i) - W(r_{i-1}) \le \frac{\beta_i}{\varphi(b_i)} < W(r_i + 1) - W(r_{i-1})$$

for $i = 1, \dots, \nu$. Now, since $w(n)$ is decreasing, $w(r_i+1)/(W(r_i) - W(r_{i-1})) \le 1$, and so for each $i = 1, \dots, \nu$,

$$1 \le \frac{\beta_i}{\varphi(b_i)[W(r_i) - W(r_{i-1})]} \le 1 + \frac{w(r_i + 1)}{W(r_i) - W(r_{i-1})} \le 2.$$

Thus, setting $B_k = \inf_i \beta_i/\varphi(A_k)$ we get

$$W(r_i) - W(r_{i-1}) \geq \frac{1}{2}\frac{\beta_i}{\varphi(b_i)} \geq \frac{1}{2}\inf_{i=1,\dots,\nu}\frac{\beta_i}{\varphi(b_i)} = B_k/2.$$

It follows that

$$1 \leq \frac{\beta_i}{\varphi(b_i)[W(r_i) - W(r_{i-1})]} \leq 1 + 2/B_k.$$

Now, let $1/(1+2/B_k) = 1 - \epsilon_k$, and since $B_k \to \infty$, $0 < \varepsilon_k \longrightarrow 0$. Moreover

$$(1 - \epsilon_k)\beta_i \leq \varphi(b_i)[W(r_i) - W(r_{i-1})] \leq \beta_i, \tag{4}$$

for all $i = 1, \dots, \nu$. Now define the block u_k as follows:

$$u_k = \sum_{i=1}^{\nu} b_i \chi_{S_i},$$

where $S_i = \{r_{i-1} + 1, \dots, r_i\}$. Since (b_i) is decreasing,

$$I(u_k) = \sum_{i=1}^{\nu} \varphi(b_i) \sum_{j=r_{i-1}+1}^{r_i} w(j) = \sum_{i=1}^{\nu} \varphi(b_i)[W(r_i) - W(r_{i-1})],$$

and by (4),

$$1 - \epsilon_k \leq I(u_k) \leq 1,$$

which implies that $1 - \epsilon_k \leq \|u_k\| \leq 1$ and so $\|u_k\| \longrightarrow 1$. The associated Orlicz function to the block u_k is the following:

$$\varphi^{(u_k)}(t) = \sum_{i=1}^{\nu} \varphi(b_i t)[W(r_i) - W(r_{i-1})] = \sum_{i=1}^{\nu} \varphi(b_i)[W(r_i) - W(r_{i-1})]\varphi_{b_i}(t), \quad t \geq 0.$$

Moreover, by the inequality (4) we get

$$(1 - \epsilon_k)\phi_k(t) = (1 - \epsilon_k)\sum_{i=1}^{\nu} \beta_i \varphi_{b_i}(t) \leq \varphi^{(u_k)}(t) \leq \sum_{i=1}^{\nu} \beta_i \varphi_{b_i}(t) = \phi_k(t), \quad t \geq 0. \tag{5}$$

It follows in particular that $\|\varphi^{(u_k)} - \phi_k\|_{C[0,s]} \leq \epsilon_k$ converges to zero for every $0 < s < 1$. Hence $\varphi^{(u_k)} \longrightarrow \psi$ in all d_s metrics, and the proof is completed. □

Now, we are ready to state the main result of this paper.

Theorem 9. *Let φ be an Orlicz function and let $1 \leq p \leq \infty$. Then the following statements are equivalent (where c_0 replaces ℓ_p for $p = \infty$):*

(i) *$d_0(w, \varphi)$ contains an isomorphic copy of ℓ_p.*
(ii) *For every $\epsilon > 0$, $d_0(w, \varphi)$ contains an $(1+\epsilon)$-isomorphic copy of ℓ_p.*
(iii) *h_φ contains an isomorphic copy of ℓ_p.*
(iv) *$p \in [\alpha_\varphi, \beta_\varphi]$.*

Proof. The conditions (iii) and (iv) are equivalent by Theorem 4.a.9 in [9]. Now, if $\psi \in C_{\varphi,1}$, it is easy to show that $[\alpha_\psi, \beta_\psi] \subseteq [\alpha_\varphi, \beta_\varphi]$. Hence if $p \notin [\alpha_\varphi, \beta_\varphi]$, $p < \infty$, then u^p is not equivalent to any function in $C_{\varphi,1}$, and so by Proposition 6, ℓ_p cannot be isomorphic to any subspace of $d_0(w, \varphi)$. If $\beta_\varphi < p = \infty$, then φ satisfies condition Δ_2 and $d_0(w, \varphi)$ does not contain c_0 by Proposition 1. Thus (i) implies (iv). To see that (iv) implies (ii), note that for any $p \in [\alpha_\varphi, \beta_\varphi]$, $p < \infty$, the function u^p belongs to C_φ (see the proof of Theorem 4.a.9 in [9]), and therefore by Theorem 8, ℓ_p has an $(1+\epsilon)$-isomorphic copy in $d_0(w, \varphi)$. If $\beta_\varphi = \infty$, then φ does not satisfy the condition Δ_2, and so $d_0(w, \varphi)$ contains c_0 by Proposition 1. In fact $d_0(w, \varphi)$ contains $(1+\varepsilon)$-isomorphically c_0 by the James Theorem (2.e.3. in [9]). $\square$

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