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EQUIVALENCE OF HARDY-TYPE INEQUALITIES WITH GENERAL MEASURES ON THE CONES OF NON-NEGATIVE RESPECTIVE NON-INCREASING FUNCTIONS

L.-E. PERSSON, V. D. STEPANOV, AND E. P. USHAKOVA

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ABSTRACT. Some Hardy-type integral inequalities in general measure spaces, where the corresponding Hardy operator is replaced by a more general Volterra type integral operator with kernel $k(x, y)$, are considered. The equivalence of such inequalities on the cones of non-negative respective non-increasing functions are established and applied.

1. INTRODUCTION

Let λ and μ be regular Borel measures on $\mathbb{R}_+ : = [0, \infty)$ such that $\lambda[0, x] < \infty$ for all $x \in \mathbb{R}_+$, K is a positive operator, that is, $Kf(x) \geq 0$ for any function $f(y) \geq 0$. A considerable number of works are devoted to the study of inequalities of the type

$$\left(\int_0^\infty (Kf)^q d\mu\right)^{1/q} \leq C \left(\int_0^\infty f^p d\lambda\right)^{1/p}, \tag{1.1}$$

where f runs over a cone of non-negative functions (see, for instance, [1] and the literature given there). Let

$$E : = \{f(x) \geq 0,\ x \in \mathbb{R}_+\},$$
$$E^{\downarrow} : = \{f(x) \geq 0,\ f(x) \text{ is non-increasing for } x \in \mathbb{R}_+\}$$

be two standard cones in the space of all λ-measurable functions. It is well known that inequality (1.1) for all $f \in E$ and the same inequality for all $f \in E^{\downarrow}$ are not equivalent in general [2]. However, as was recently discovered by G. Sinnamon [3], for the Hardy operator of the type $Hf(x) : = \int_0^x f d\lambda$ the equivalence takes place. We generalize this and the other results of [3] to Volterra integral operators

$$\mathbb{K}f(x) : = \int_0^x k(x, y) f(y) d\lambda(y), \tag{1.2}$$

with a kernel $k(x, y) \geq 0$ satisfying some conditions of monotonicity.

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Throughout the paper a relation $A \ll B$ means an inequality $A \leq cB$ with a constant c, depending on the parameters of summation p and q. We note that $p'\colon = p/(p-1)$ for $0 < p < \infty$, $p \neq 1$. If $A \ll B \ll A$ or $A = cB$, we write $A \approx B$. The constants C in the inequalities like (1.1) are supposed to be taken as the least possible. We assume $\int_a^b\colon = \int_{[a,b]}$. Uncertainties of the form $0 \cdot \infty$ are taken to be zeros. The sign $:=$ is used to determine new quantities.

2. Main results

We use the notation

$$L^p(\lambda)\colon = \left\{ f \geq 0,\, \|f\|_{L^p(\lambda)}\colon = \left(\int_0^\infty f^p d\lambda \right)^{1/p} \right\},$$

and we need the following statement.

Proposition 2.1. *If $1 < p < \infty$ and a function $f(x) \geq 0$ is λ-measurable, bounded and $\operatorname{supp} f \subset \mathbb{R}_+$ is compact, then there exists a function $f^\circ \in E^\downarrow$ such that*

(i) $Hf(x) \leq Hf^\circ(x)$ *for all* $x > 0$,

(ii) $\|f^\circ\|_{L^p(\lambda)} \leq \|f\|_{L^p(\lambda)}$.

Remark 2.2. The function f° is called *a level function*. The proof of Proposition 2.1 can be found in [4].

Theorem 2.3. *Let $1 \leq p < \infty$ and $0 < q < \infty$. Let the kernel $k(x,y)\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be non-increasing in $y \in [0,x]$ for every x and let $\mathbb{K}$ be defined by* (1.2). *Then the inequalities*

$$\|\mathbb{K}f\|_{L^q(\mu)} \leq C_1(p,q)\|f\|_{L^p(\lambda)},\ f \in E, \tag{2.1}$$

and

$$\|\mathbb{K}f\|_{L^q(\mu)} \leq C_2(p,q)\|f\|_{L^p(\lambda)},\ f \in E^\downarrow, \tag{2.2}$$

are equivalent and $C_1(p,q) = C_2(p,q)$.

Proof. The implication (2.1)⇒(2.2) and the inequality $C_2(p,q) \leq C_1(p,q)$ are trivial because of $E^\downarrow \subset E$. We show the inverse. If $\operatorname{supp}\mu = \{0\}$, then the assertion of the theorem is obvious. Therefore, let $\operatorname{supp}\mu \neq \{0\}$. By monotonicity of the kernel $k(x,y)$ we have in a sense of Stieltjes' integral that for all $0 \leq y < x < \infty$

$$k(x,y) = k(x,x) + \int_y^x d_z(-k(x,z)).$$

By (i) of Proposition 2.1 it implies for all bounded $f \in E$ with compact support $\operatorname{supp} f \subset \mathbb{R}_+$ that

$$\begin{aligned}\mathbb{K}f(x) &= k(x,x)\int_0^x f d\lambda + \int_0^x \left(\int_y^x d_z(-k(x,z)) \right) f(y) d\lambda(y) \\ &= k(x,x)\int_0^x f d\lambda + \int_0^x \left(\int_0^z f d\lambda \right) d_z(-k(x,z)) \\ &\leq k(x,x)\int_0^x f^\circ d\lambda + \int_0^x \left(\int_0^z f^\circ d\lambda \right) d_z(-k(x,z)) = \mathbb{K}f^\circ(x).\end{aligned}$$

Applying (ii) of Proposition 2.1, we find for $1 < p < \infty$ that

$$\|\mathbb{K}f\|_{L^q(\mu)} \leq \|\mathbb{K}f^\circ\|_{L^q(\mu)} \leq C_2\|f^\circ\|_{L^p(\lambda)} \leq C_2\|f\|_{L^p(\lambda)}.$$

For arbitrary $f \in E$ the required assertion follows by Fatou's theorem. The case $p = 1$ also follows by a limiting process with $p \to 1$. □

Put

$$\Lambda(x) \colon = \int_0^x d\lambda, \qquad P_\lambda f(x) \colon = \frac{1}{\Lambda(x)} Hf(x).$$

The following statement is essentially proved in [3].

Corollary 2.4. *Let $1 \le p < \infty$, $0 < q < \infty$. Then the inequalities*

$$\|P_\lambda f\|_{L^q(\mu)} \le C_1(p,q)\|f\|_{L^p(\lambda)}, \ f \in E, \tag{2.3}$$

and

$$\|P_\lambda f\|_{L^q(\mu)} \le C_2(p,q)\|f\|_{L^p(\lambda)}, \ f \in E^{\downarrow}, \tag{2.4}$$

are equivalent. Moreover, for $p > 1$ (2.3) and (2.4) are equivalent to

$$\|f\|_{L^q(\mu)} \le C_3(p,q)\|f\|_{L^p(\lambda)}, \ f \in E^{\downarrow}, \tag{2.5}$$

and $C_3(p,q) \le C_1(p,q) = C_2(p,q) \le p' C_3(p,q)$.

Proof. The proof follows from Theorem 2.3, the inequality $f(x) \le P_\lambda f(x)$ for any $f \in E^{\downarrow}$ and Hardy's inequality [3]

$$\|P_\lambda f\|_{L^p(\lambda)} \le p'\|f\|_{L^p(\lambda)}, \tag{2.6}$$

where $1 < p < \infty$. □

From this we obtain a sharp form of the well-known Sawyer's theorem [6, Theorem 1].

Corollary 2.5. *Let $1 < p < \infty$. Then*

$$S_p \colon = \sup_{f \in E^{\downarrow}} \frac{\int_0^\infty f d\mu}{\left(\int_0^\infty f^p d\lambda\right)^{1/p}} \in \left[\frac{B_p}{p'}, B_p\right], \tag{2.7}$$

where

$$B_p \colon = \left(\int_0^\infty \left(\int_x^\infty \frac{d\mu}{\Lambda}\right)^{p'} d\lambda(x)\right)^{1/p'}.$$

Moreover, the relation (2.7) is unimprovable in general.

Proof. It is well known [5, Chapter XI, § 1.5, Theorem 4] that for $1 < p < \infty$, $q = 1$ the best constant in the inequality (2.3) has the form $C_1(p,1) = B_p$. From this and Corollary 2.4 it follows the estimate (2.7). Let us show that it is optimal. If $d\lambda(x) = \delta_0(x)\chi_{[0,1]}(x)dx$, $\operatorname{supp}\mu = [0,1]$, where $\delta_0(x)$ denotes the Dirac delta function at the point 0, then it is not difficult to see that

$$S_p \ge \int_0^1 d\mu = B_p$$

in this case. Therefore the inequality $S_p \le B_p$ is sharp. Now, let

$$\alpha \in (0, 1/p'), \qquad d\mu(x) = x^{-\alpha}\chi_{[0,1]}(x)dx, \qquad d\lambda(x) = \chi_{[0,1]}(x)dx,$$

where $\chi_{[0,1]}(x)$ denotes the characteristic function of the interval $[0,1]$. Then

$$S_{p,\alpha} \colon = \sup_{f \in E^{\downarrow}} \frac{\int_0^1 f(x)dx/x^\alpha}{\left(\int_0^1 f^p(x)dx\right)^{1/p}} = (1 - \alpha p')^{-1/p'}.$$

Here the upper bound follows from the Hölder inequality, and the lower bound is achieved for the function $f(x) = x^{-\alpha/(p-1)}$. We also have

$$B_{p,\alpha} := \left(\int_0^1 \left(\int_x^1 \frac{dy}{y^{1+\alpha}}\right)^{p'} dx\right)^{1/p'} = \frac{1}{\alpha}\left(\int_0^1 (1-x^\alpha)^{p'} \frac{dx}{x^{\alpha p'}}\right)^{1/p'}.$$

Applying the elementary inequality

$$(1-x^\alpha)^{p'} \geq 1 - p'x^\alpha,$$

we find

$$B_{p,\alpha} \geq \frac{1}{\alpha}\left(\int_0^1 (1-p'x^\alpha)\frac{dx}{x^{\alpha p'}}\right)^{1/p'} = \frac{1}{\alpha}(1-p'\alpha)^{-1/p'}\left(1 - \frac{p'(1-p'\alpha)}{1-\alpha(p'-1)}\right)^{1/p'}.$$

Hence,

$$\lim_{\alpha\to 1/p'} \frac{S_{p,\alpha}}{B_{p,\alpha}} \leq \frac{1}{p'}$$

and, consequently, the estimate $p'S_p \geq B_p$ is sharp, too. □

Theorem 2.6. *Let* $1 < p < \infty$, $0 < q < \infty$ *and let* m, n *be any non-negative integers. Then the inequalities* (2.1), (2.2)

$$\|(\mathbb{K}P_\lambda^n)f\|_{L^q(\mu)} \leq C_4\|f\|_{L^p(\lambda)},\ f \in E, \tag{2.8}$$

and

$$\|(\mathbb{K}P_\lambda^n)f\|_{L^q(\mu)} \leq C_5\|P_\lambda^m f\|_{L^p(\lambda)},\ f \in E^\downarrow, \tag{2.9}$$

are mutually equivalent.

Proof. If $f \in E^\downarrow$, then $f \leq P_\lambda f \leq \ldots \leq P_\lambda^n f$ and (2.9) implies (2.2) by

$$\|Kf\|_{L^q(\mu)} \leq \|KP_\lambda^n f\|_{L^q(\mu)} \leq C_5\|P_\lambda^m f\|_{L^p(\lambda)} \leq C_5(p')^m\|f\|_{L^p(\lambda)}.$$

Now, (2.2) ⇒ (2.1) follows by Theorem 2.3 and (2.1) ⇒ (2.8) by

$$\|KP_\lambda^n f\|_{L^q(\mu)} \leq C_1\|P_\lambda^n f\|_{L^p(\lambda)} \leq C_1(p')^n\|f\|_{L^p(\lambda)}.$$

Finally, if $f \in E^\downarrow$, then again applying $f \leq P_\lambda f \leq \ldots \leq P_\lambda^m f$ we see that (2.8) ⇒ (2.9) follows by

$$\|KP_\lambda^n f\|_{L^q(\mu)} \leq C_4\|f\|_{L^p(\lambda)} \leq C_4\|P_\lambda^m f\|_{L^p(\lambda)}.$$

□

In the theory of the weighted Hardy-type inequalities the following is well known.

Definition 2.7. A kernel $k(x, y) \geq 0$ of the integral operator of the form (1.2) belongs to *Oinarov's class*, $k \in \mathcal{O}$, if there is a constant $D \geq 1$ such that

$$D^{-1}k(x,y) \leq k(x,z) + k(z,y) \leq Dk(x,y),\ x \geq z \geq y \geq 0. \tag{2.10}$$

Remark 2.8. It is known [7] that if $k \in \mathcal{O}$, then

$$k(x,y) \leq \overline{k}(x,y) \leq Dk(x,y), \tag{2.11}$$

where

$$\overline{k}(x,y) = \sup_{z\in[y,x]} k(x,z).$$

Therefore, Theorem 2.6 is valid for integral operators $\mathbb{K}$ of the form (1.2) with a kernel $k \in \mathcal{O}$, because $\overline{k}(x, y)$ is non-increasing in $y \in [0, x]$.

Put

$$K_\lambda(x) := \int_0^x k(x,y)d\lambda(y),$$

$$P_K f(x) := \frac{1}{K_\lambda(x)} \int_0^x k(x,y) f(y) d\lambda(y). \tag{2.12}$$

Definition 2.9. We write that a kernel $k(x,y) \in \mathcal{D}_\lambda$ if $P_K f \in E^\downarrow$ for any $f \in E^\downarrow$.

Observe that the class D_λ consists of non-negative kernels $k(x,y)$ such that for all $t > 0$

$$\frac{1}{K_\lambda(x)} \int_0^t k(x,y) d\lambda(y)$$

is non-increasing in x when $x \geq t$. Necessity follows from the inequality $P_K f(x) \geq P_K f(z)$ with $x \geq z \geq t$ and $f = \chi_{[0,t]}$. For sufficiency we assume that $f(x) = \int_x^\infty h$, $h \geq 0$. Then for $x \geq t$ we have

$$\begin{aligned} P_K f(x) &= f(x) + \int_t^x h(s) \frac{\int_0^s k(x,y) d\lambda(y)}{K_\lambda(x)} ds \\ &\quad + \int_0^t h(s) \frac{\int_0^s k(x,y) d\lambda(y)}{K_\lambda(x)} ds \\ &\geq \int_t^\infty h + \int_0^t h(s) \frac{\int_0^s k(t,y) d\lambda(y)}{K_\lambda(t)} ds = P_K f(t). \end{aligned}$$

Now we intend to obtain the equivalence like (2.3)⇔(2.5) for the more general inequality (2.1) instead of (2.3). To this end we need an analog of the Hardy inequality (2.6) for the operator P_K (see [8, § 2.3]).

Proposition 2.10. *Let $1 < p < \infty$ and $k(x,y) \in \mathcal{O}$. Then*

$$\|P_K f\|_{L^p(\lambda)} \leq C_6 \|f\|_{L^p(\lambda)}, \ f \in E, \tag{2.13}$$

if and only if

$$\mathbb{A}_0 := \sup_{t \in \mathbb{R}_+} \frac{\left(\int_0^t \left(\frac{1}{K_\lambda(x)} \int_0^x [k(x,y)]^{p'} d\lambda(y) \right)^p d\lambda(x) \right)^{1/p}}{\left(\int_0^t [k(t,z)]^{p'} d\lambda(z) \right)^{1/p}} < \infty$$

and $\mathbb{A}_0 \approx C_6$.

Proof. Necessity. For any $t > 0$ put $f_t(y) = [k(t,y)]^{p'-1} \chi_{[0,t]}(y)$. Then (2.13) with $f = f_t$ gives

$$\begin{aligned} C_6^p \int_0^t [k(t,z)]^{p'} d\lambda(z) &\geq \int_0^t \left(\frac{1}{K_\lambda(x)} \int_0^x [k(t,y)]^{p'-1} k(x,y) d\lambda(y) \right)^p d\lambda(x) \\ &\geq D^{-p'} \int_0^t \left(\frac{1}{K_\lambda(x)} \int_0^x [k(x,y)]^{p'} d\lambda(y) \right)^p d\lambda(x). \end{aligned}$$

Hence, $D^{p'/p} C_6 \geq \mathbb{A}_0$.

For the *sufficiency* we observe that (2.13) is equivalent to

$$\|P_K^* g\|_{L^{p'}(\lambda)} \leq C_6 \|g\|_{L^{p'}(\lambda)}, \ g \in E,$$

where

$$P_K^* g(y) :\, = \int_y^\infty k(x,y) g(x) \frac{d\lambda(x)}{K_\lambda(x)}.$$

Now, applying [8, Lemma 2], whose proof can be duplicated for a general measure, we find that

$$\begin{aligned} \|P_K^* g\|_{L^{p'}(\lambda)}^{p'} &\ll \int_0^\infty g (P_K^* g)^{p'-1} d\lambda \\ &+ \int_0^\infty g(y) \left(\int_y^\infty g(z) \frac{d\lambda(z)}{K_\lambda(z)} \right)^{p'-1} \left(\int_0^y [k(y,s)]^{p'} d\lambda(s) \right) \frac{d\lambda(y)}{K_\lambda(y)} \\ &=: J_1 + J_2. \end{aligned}$$

Assume that g is bounded with compact support and $0 < \|P_K^* g\|_{L^{p'}(\lambda)} < \infty$. By Hölder's inequality we have

$$J_1 \le \|g\|_{L^{p'}(\lambda)} \|P_K^* g\|_{L^{p'}(\lambda)}^{p'/p}.$$

Again by Hölder's inequality we find

$$\begin{aligned} J_2 &\le \|g\|_{L^{p'}(\lambda)} \left(\int_0^\infty \left(\int_y^\infty g(z) \frac{d\lambda(z)}{K_\lambda(z)} \right)^{p'} \left(\frac{1}{K_\lambda(y)} \int_0^y [k(y,s)]^{p'} d\lambda(s) \right)^p d\lambda(y) \right)^{1/p} \\ &=: \|g\|_{L^{p'}(\lambda)} J_{2,1}^{1/p}. \end{aligned}$$

Now, once more applying the measure version of [8, Lemma 2] we obtain

$$\begin{aligned} J_{2,1} &\ll \int_0^\infty g(z) \frac{d\lambda(z)}{K_\lambda(z)} \left(\int_z^\infty g(x) \frac{d\lambda(x)}{K_\lambda(x)} \right)^{p'-1} \int_0^z \left(\frac{1}{K_\lambda(y)} \int_0^y [k(y,s)]^{p'} d\lambda(s) \right)^p d\lambda(y) \\ &\le \mathbb{A}_0^p J_2. \end{aligned}$$

Hence, $J_2 \ll \mathbb{A}_0^{p'} \|g\|_{L^{p'}(\lambda)}^{p'}$. Thus,

$$\|P_K^* g\|_{L^{p'}(\lambda)}^{p'} \ll \|g\|_{L^{p'}(\lambda)} \|P_K^* g\|_{L^{p'}(\lambda)}^{p'/p} + \mathbb{A}_0^{p'} \|g\|_{L^{p'}(\lambda)}^{p'},$$

and the required upper bound follows. □

Theorem 2.11. *Let $1 < p < \infty$, $0 < q < \infty$ and let a kernel $k \in \mathcal{O} \cap \mathcal{D}_\lambda$ be such that $\mathbb{A}_0 < \infty$. Then the inequality* (2.1) *is equivalent to*

$$\|f K_\lambda\|_{L^q(\mu)} \le C_7 \|f\|_{L^p(\lambda)}, \ f \in E^\downarrow, \tag{2.14}$$

and $C_1 \approx C_7$.

Proof. First we show that (2.14)⇒(2.2). Let $f \in E^\downarrow$. Then $P_K f \in E^\downarrow$, because of $k \in \mathcal{D}_\lambda$. Therefore

$$\|Kf\|_{L^q(\mu)} = \|(P_K f) K_\lambda\|_{L^q(\mu)} \le C_7 \|P_K f\|_{L^p(\lambda)},$$

and by the inequality (2.13)

$$\ll C_7 \mathbb{A}_0 \|f\|_{L^p(\lambda)}.$$

In view of Remark 2.8 this implies (2.1) and, moreover, $C_1 \ll C_7 \mathbb{A}_0$. Conversely, if $f \in E^\downarrow$, then by (2.11)

$$K_\lambda(x) \le \overline{K}_\lambda(x) :\, = \int_0^x \overline{k}(x,y) d\lambda(y).$$

Therefore,

$$P_K f(x) \geq \frac{1}{D\overline{K}_\lambda(x)} \int_0^x \overline{k}(x,y) f(y) d\lambda(y) \geq \frac{1}{D} f(x).$$

Thus, if (2.1) is true, then for $f \in E^{\downarrow}$ we have

$$\|fK_\lambda\|_{L^q(\mu)} \leq D\|Kf\|_{L^q(\mu)} \leq DC_1\|f\|_{L^p(\lambda)}.$$

Consequently, $C_7 \leq DC_1$. □

Combining Theorems 2.6 and 2.11, we obtain the following

Corollary 2.12. *Let $1 < p < \infty$, $0 < q < \infty$ and let a kernel $k \in \mathcal{O} \cap \mathcal{D}_\lambda$ be such that $\mathbb{A}_0 < \infty$. Then* (2.14) *is equivalent to* (2.8) *and to* (2.9).

Remark 2.13. The condition $k \in \mathcal{D}_\lambda$ was not used in the proof of (2.1)⇒(2.14), and therefore (2.14) follows either from (2.1), (2.2), (2.8) or (2.9), whenever $k \in \mathcal{O}$ and $\mathbb{A}_0 < \infty$.

Let $\varphi\colon (0,\infty) \to (-\infty,\infty)$ be such a monotone function so that there exists the reverse function φ^{-1} and either

(a) φ is concave and increasing

or

(b) φ is convex and decreasing.

Put

$$\Phi_K f(x)\colon = \varphi^{-1}\left[P_K(\varphi(f))\right](x) = \varphi^{-1}\left[\frac{1}{K_\lambda(x)}\int_0^x k(x,y)[\varphi(f)](y)d\lambda(y)\right].$$

Our following assertion generalizes the result of G. Sinnamon [3, Theorem 4.1], established for $k(x,y) = 1$.

Theorem 2.14. *Let $1 < p < \infty$, $0 < q < \infty$, $k \in \mathcal{O}\cap\mathcal{D}_\lambda$ and $\mathbb{A}_0 < \infty$. Suppose that φ satisfies the condition* (a) *or* (b). *Then the following inequalities are equivalent:*

$$\|f\|_{L^q(\mu)} \leq C_8\|f\|_{L^p(\lambda)}, \; f \in E^{\downarrow}, \tag{2.15}$$

$$\|P_K f\|_{L^q(\mu)} \leq C_9\|f\|_{L^p(\lambda)}, \; f \in E^{\downarrow}, \tag{2.16}$$

$$\|P_K f\|_{L^q(\mu)} \leq C_{10}\|f\|_{L^p(\lambda)}, \; f \in E, \tag{2.17}$$

$$\|\Phi_K f\|_{L^q(\mu)} \leq C_{11}\|f\|_{L^p(\lambda)}, \; f \in E, \tag{2.18}$$

$$\|\Phi_K f\|_{L^q(\mu)} \leq C_{12}\|f\|_{L^p(\lambda)}, \; f \in E^{\downarrow}. \tag{2.19}$$

Moreover, the least possible constants in the inequalities (2.15)–(2.19) *are also pairwise equivalent.*

Proof. (2.15)⇒(2.16)⇒(2.17) follows from Theorems 2.3 and 2.11 and Remark 2.8, and (2.17)⇒(2.18) follows by applying Jensen's inequality $\Phi_K f(x) \leq P_K f(x)$. (2.18)⇒(2.19) is trivial. It is left to show that (2.19)⇒(2.15). Let $f \in E^{\downarrow}$ and φ satisfy condition (a). Then $\varphi(f)$ is non-increasing. This easily implies that $\varphi(f)(x) \leq P_K(\varphi(f))(x)$. Applying φ^{-1} to the both parts of this inequality, we conclude that $f(x) \leq \Phi_K f(x)$ and therefore, (2.19)⇒(2.15); moreover, $C_8 \leq C_{12}$. If φ satisfies condition (b) and $f \in E^{\downarrow}$, then $\varphi(f)$ is non-decreasing. Then $\varphi(f)(x) \geq P_K(\varphi(f))(x)$. Again applying φ^{-1} to the both parts of this inequality, we obtain that $f(x) \leq \Phi_K f(x)$ and therefore (2.19)⇒(2.15) and $C_8 \leq C_{12}$. □

Remark 2.15. Until now we have mainly studied only the equivalence of the inequalities, and it may seem that their own characterization is still open. However, this is not so, because the characterization of the inequality (2.15) is well known (see [9, Proposition 1] for weighted inequalities and [3] for inequalities with general measures). Namely, for $0 < p \le q < \infty$

$$C_8(p,q) = \sup_{f\in E^{\downarrow}} \frac{\left(\int_0^\infty f^q d\mu\right)^{1/q}}{\left(\int_0^\infty f^p d\lambda\right)^{1/p}} = \sup_{t>0} \frac{\left(\int_0^t d\mu\right)^{1/q}}{\left(\int_0^t d\lambda\right)^{1/p}}, \tag{2.20}$$

and for $0 < q < p < \infty$, $1/r\colon = 1/q - 1/p$

$$C_8(p,q) \in \left[\frac{q}{r} B_{p/q}^{1/q},\ B_{p/q}^{1/q}\right], \tag{2.21}$$

where

$$B_{p/q}^{1/q} = \left(\int_0^\infty \left(\int_x^\infty \frac{d\mu}{\Lambda}\right)^{r/q} d\lambda(x)\right)^{1/r}. \tag{2.22}$$

Observe that (2.21) follows by the change $f^q \to f$ in the middle term (2.20) and applying (2.7). Thus, the borders for the constant $C_8(p,q)$ in (2.21) are sharp and the inequalities (2.16)–(2.19) are characterized by finiteness of the right-hand side of (2.20) for $1 < p \le q < \infty$ and by finiteness of the right-hand side of (2.22) for $0 < q < p < \infty$, $p > 1$.

Also note that under some additional restriction on the measure λ there is an alternative two-sided estimate of the constant $C_8(p,q)$ for $0 < q < p < \infty$, also useful in applications. This restriction on the measure λ was studied in [3] and always holds, for example, if the measure λ is non-atomic. For the absolutely continuous measures λ and μ this alternative estimate was pointed out in [10, formula (1.6)].

Finally, we will point out two applications of Theorem 2.14, found for $k(x,y)\equiv 1$ and $f \in E$ in [3].

Proposition 2.16. *Let $0 < p, q < \infty$, $k \in \mathcal{O}\cap\mathcal{D}_\lambda$ and $\mathbb{A}_0 < \infty$. Then the inequality*

$$\left(\int_0^\infty f^{-p} d\lambda\right)^{-1/p} \le C_{13} \left(\int_0^\infty (P_K f)^{-q} d\mu\right)^{-1/q} \tag{2.23}$$

for all $f \in E$ is also equivalent to the same inequality for all $f \in E^{\downarrow}$ and equivalent to the inequality (2.15)*. Moreover, $C_{13} \approx C_8$.*

Proof. If $0 < q < \infty, p > 1$, then applying Theorem 2.14 with $\varphi(x) = 1/x$, we obtain the required assertion and $C_{13} \approx C_8$. If $0 < p \le 1$, then by the change $f^p = g^2$ in the inequality (2.15) it is equivalently reproduced in the form

$$\left(\int_0^\infty g^{2q/p} d\mu\right)^{p/2q} \le C_8^{p/2} \left(\int_0^\infty g^2 d\lambda\right)^{1/2},\ g \in E^{\downarrow}. \tag{2.24}$$

Moreover, by Theorem 2.14 with $\varphi(x) = x^{-2/p}$, the inequality (2.24) is equivalent to the inequality

$$\left(\int_0^\infty (P_K g^{-2/p})^{-q} d\mu\right)^{p/2q} \le C_8^{p/2} \left(\int_0^\infty g^2 d\lambda\right)^{1/2} \tag{2.25}$$

for all $f \in E$ or for all $f \in E^{\downarrow}$. Performing in (2.25) the reverse change $g^{-2/p} = f$, we arrive at the inequality (2.23). □

By a similar method we obtain the following.

Proposition 2.17. *Let* $0 < p, q < \infty$, $k \in \mathcal{O} \cap \mathcal{D}_\lambda$ *and* $\mathbb{A}_0 < \infty$. *Then the inequalities*

$$\left(\int_0^\infty (\exp[P_K(\log f)])^q \, d\mu\right)^{1/q} \leq C_{14} \left(\int_0^\infty f^p d\lambda\right)^{1/p} \tag{2.26}$$

and

$$\left(\int_0^\infty f^{-p} d\lambda\right)^{-1/p} \leq C_{15} \left(\int_0^\infty (\exp[P_K(\log f)])^{-q} \, d\mu\right)^{-1/q} \tag{2.27}$$

either for all $f \in E$ *or for all* $f \in E^{\downarrow}$ *are equivalent to each other and also equivalent to the inequality* (2.15). *Moreover,* $C_{14} \approx C_{15} \approx C_8$.

Proof. The inequalities (2.26) and (2.27) for $0 < q < \infty$, $p > 1$ follow from Theorem 2.14 with $\varphi(x) = \log x$, and $\varphi(x) = \log x^{-1}$, respectively. For $0 < p \leq 1$ we use the arguments from the proof of Proposition 2.16 with the functions $\varphi(x) = \log x^{2/p}$ and $\varphi(x) = \log x^{-2/p}$. □

Remark 2.18. Proposition 2.16 supplements the results of the Prokhorov recent paper [11], and Proposition 2.17 supplements the results of the papers [12], [13].

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Department of Mathematics, Lulea University of Technology, SE-97187 Lulea, Sweden
E-mail address: larserik@sm.luth.se

Department of Mathematical Analysis, Peoples Friendship University of Russia, Miklukho-Maklay 6, Moscow, 117198, Russia
E-mail address: vstepanov@sci.pfu.edu.ru

Computer Center of FEB RAS, Tikhookeanskaya 153, Khabarovsk, 680042, Russia
E-mail address: ushakova@as.khb.ru