PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 9, September 2006, Pages 2767–2770
S 0002-9939(06)08257-8
Article electronically published on March 21, 2006

BOUNDARY STRUCTURE OF HYPERBOLIC 3-MANIFOLDS ADMITTING ANNULAR FILLINGS AT LARGE DISTANCE

SANGYOP LEE

(Communicated by Ronald A. Fintushel)

ABSTRACT. We show that if a hyperbolic 3-manifold M with ∂M a union of tori admits two annular Dehn fillings at distance $\Delta \geq 3$, then M is bounded by at most three tori.

An annulus or torus embedded in a 3-manifold is *essential* if it is incompressible, boundary-incompressible and is not boundary-parallel. A 3-manifold is *annular* (resp. *toroidal*) if it contains an essential annulus (resp. torus). Otherwise, it is *anannular* (resp. *atoroidal*). Also, a 3-manifold is *irreducible* if any embedded 2-sphere bounds an embedded 3-ball, and is *boundary-irreducible* if its boundary is incompressible. Thurston [9] has shown that a compact, orientable 3-manifold M with non-empty boundary is irreducible, boundary-irreducible, atoroidal and anannular if and only if it is *hyperbolic*, in the sense that M with its boundary tori removed admits complete finite volume hyperbolic structure with totally geodesic boundary.

Let M be a compact, connected, orientable 3-manifold with torus boundary component T_0. The Dehn filling of M with slope γ is the manifold $M(\gamma)$ obtained by attaching a solid torus V_γ to M along their boundary so that a meridian of V_γ is identified with a curve of slope γ on T_0. For two slopes γ_1, γ_2 on T_0, $\Delta(\gamma_1, \gamma_2)$ denotes their minimal geometric intersection number and is called a *distance* between the slopes.

Suppose that given a hyperbolic 3-manifold M, there are two slopes γ_1, γ_2 on T_0 such that both $M(\gamma_1)$ and $M(\gamma_2)$ are annular. Gordon [2] showed that $\Delta(\gamma_1, \gamma_2) \leq 5$, and furthermore, together with Wu [5], showed that there are only three specific manifolds M realizing $\Delta(\gamma_1, \gamma_2) = 4$ and 5. These manifolds are the exteriors of the Whitehead link, the Whitehead sister link and the 2-bridge link corresponding to 3/10 in the 3-sphere S^3. Also, he [3, Theorem 5.3] constructed a hyperbolic k-component link in $S^2 \times S^1$ for any $k \geq 4$ whose exterior realizes $\Delta(\gamma_1, \gamma_2) = 2$, and he asked [3, Question 5.3] what is the maximal value for $\Delta(\gamma_1, \gamma_2)$ if M is bounded by at least four tori. In this paper, we give an answer to this question.

Theorem 1. *Let M be a hyperbolic 3-manifold with ∂M a union of tori. Suppose that there are two slopes γ_1 and γ_2 on a specified boundary torus of M such that*

Received by the editors January 27, 2005 and, in revised form, March 21, 2005.
2000 *Mathematics Subject Classification.* Primary 57M25.
Key words and phrases. Dehn filling, annular manifold.

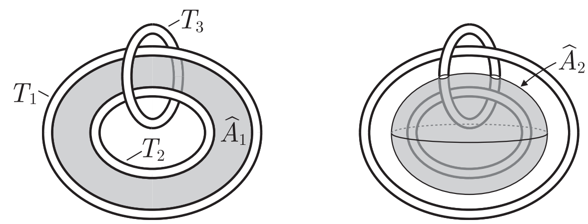

FIGURE 1.

both $M(\gamma_1)$ and $M(\gamma_2)$ are annular. If $\Delta(\gamma_1, \gamma_2) \geq 3$, then ∂M is a union of at most three tori.

We remark that there is a hyperbolic 3-manifold with three boundary tori, called the *magic manifold*, which has two annular Dehn fillings at distance 3. See [4, Lemma 7.6].

We now go on to the proof of Theorem 1. As mentioned before, if $\Delta(\gamma_1, \gamma_2) \geq 4$, then M is bounded by only two tori. So, we assume $\Delta = 3$ and assume for contradiction that ∂M is a union of at least four tori.

Lemma 2. *$M(\gamma_i)$ is homeomorphic to $P \times S^1$, where P is a pair of pants.*

Proof. Put $X = M(\gamma_i)$. Then X is irreducible by [8, 10], and atoroidal by [7]. Since ∂M is a union of at least four tori, X has at least three torus boundary components T_1, T_2, T_3.

By assumption X is annular. If A is an essential annulus with boundary components on T_1 and T_2, then the frontier of $N(A \cup T_1 \cup T_2)$ is a torus T. Since X is connected, irreducible and atoroidal, T must be parallel to T_3, hence $X = N(A \cup T_1 \cup T_2)$ and the result follows. The case that both boundary components of A are on the same torus is similar. We omit the details. $\square$

Let T_1, T_2, T_3 be the three torus boundary components of $M(\gamma_i)$. Note that $P \times S^1$ contains an essential annulus connecting two distinct boundary tori and an essential annulus meeting only one boundary torus.

Let $\widehat{A}_1$ be an essential annulus in $M(\gamma_1)$ which has one boundary circle in each of T_1 and T_2. Let $\widehat{A}_2$ be an essential annulus in $M(\gamma_2)$ whose boundary is contained in T_3. Note that $\partial\widehat{A}_1 \cap \partial\widehat{A}_2 = \emptyset$. Also, notice that $\widehat{A}_1$ is non-separating, while $\widehat{A}_2$ is separating. See Figure 1.

We may assume that $\widehat{A}_1$ meets the core of V_{γ_1} transversely. Then $\widehat{A}_1 \cap V_{\gamma_1}$ is a disjoint union of meridian disks $u_1, u_2, \ldots, u_{n_1}$ of V_{γ_1}, numbered successively along V_{γ_1}, and n_1 is chosen to be minimal among all essential annuli. Similarly, $\widehat{A}_2$ intersects V_{γ_2} in meridian disks $v_1, v_2, \ldots, v_{n_2}$ of V_{γ_2} (n_2 is even). Let $A_i = \widehat{A}_i \cap M (i = 1, 2)$ and assume that A_1 and A_2 intersect transversely and minimally.

In the usual way [1, 2], the arc components of $A_1 \cap A_2$ define labelled graphs G_1 on $\widehat{A}_1$ and G_2 on $\widehat{A}_2$. The vertices of G_i are $u_1, u_2, \ldots, u_{n_1}$ (or $v_1, v_2, \ldots, v_{n_2}$), each having a sign according to whether the core of the attached solid torus passes $\widehat{A}_i$ from the positive side or the negative side at the vertex. An edge of G_i is an arc component of $A_1 \cap A_2$. Since $\partial\widehat{A}_1 \cap \partial\widehat{A}_2 = \emptyset$, each edge of G_i connects two (possibly the same) vertices of G_i. If an edge endpoint lies in $\partial u_x \cap \partial v_y$, then the point is labelled y at u_x and labelled x at v_y. An edge is called an *x-edge* if one endpoint is labelled x.

An edge in G_i is *positive* if it connects the vertices of the same sign. Otherwise, it is *negative*. The parity rule [1] says that an edge is positive in one graph if and only if it is negative in the other graph.

Assume that G_i has at least two labels. For a label x, $G_i^+(x)$ denotes a subgraph of G_i consisting of all vertices and all positive x-edges of G_i. A disk face in $G_i^+(x)$ is called an *x-face*. A cycle σ in G_i consisting of positive edges is a *Scharlemann cycle* if it bounds a disk face of G_i, and all the edges in σ have the same pair of labels $\{x, x+1\}$ at their endpoints. It is a well-known fact that any x-face contains a Scharlemann cycle [6, Proposition 5.1].

Lemma 3. *Neither G_1 nor G_2 contains a Scharlemann cycle.*

Proof. By [10, Lemma 5.4(1)], G_2 cannot contain a Scharlemann cycle.

Assume that G_1 contains a Scharlemann cycle with label pair $\{x, x+1\}$ and let f be the disk face bounded by the Scharlemann cycle. Let H be the annulus in ∂V_{γ_2} cobounded by ∂v_x and ∂v_{x+1} such that $H \cap \partial f \neq \emptyset$. Then surgering $(\widehat{A}_2 - \mathrm{Int}(v_x \cup v_{x+1})) \cup H$ along f gives an annulus A in $M(\gamma_2)$ which intersects the core of V_{γ_2} less than n_2 times. Since $\partial A = \partial \widehat{A}_2$, A is incompressible. Thus A is boundary-parallel in $M(\gamma_2)$. Then one side of $\widehat{A}_2$ cannot contain a component of ∂M. This is a contradiction. See Figure 1. $\square$

If $n_1 = 1$, then the edges of G_1 are mutually parallel. Then G_1 would contain Scharlemann cycles. Thus $n_1 \geq 2$. Since $\widehat{A}_2$ is separating, $n_2 \geq 2$.

Lemma 4. *Let $\{i, j\} = \{1, 2\}$. Then any vertex of G_i has at most n_j negative edge endpoints.*

Proof. Assume, for example, that a vertex u_x has more than n_2 negative edge endpoints. Suppose that $G_2^+(x)$ has V vertices, E edges, and F disk faces. Then $V = n_2, E > n_2$ and $F \geq E - V + \chi(\widehat{A}_2) = E - V > 0$. Thus G_2 contains an x-face and hence it contains a Scharlemann cycle, contradicting Lemma 3. $\square$

Let K_i be the number of negative edge endpoints in G_i. Then by Lemma 4 we have $K_i \leq n_1 n_2$. Since there is a total of $3n_1n_2$ edge endpoints on G_1, by the parity rule we have $K_1 = 3n_1n_2 - K_2$. Therefore

$$K_1 = 3n_1n_2 - K_2 \geq 3n_1n_2 - n_1n_2 = 2n_1n_2.$$

This gives a contradiction and completes the proof of Theorem 1.

I thank the referee for helpful comments.

References

1. M. Culler, C. McA. Gordon, J. Luecke, and P.B. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237–300. MR0881270 (88a:57026)
2. C. McA. Gordon, *Boundary slopes on punctured tori in* 3*-manifolds*, Trans. Amer. Math. Soc. **350** (1998), 1713–1790. MR1390037 (98h:57032)
3. C. McA. Gordon, *Small surfaces and Dehn filling*, Proceedings of the Kirbyfest (Berkeley, CA, 1999), Geom. Topol. Monogr. **2**, 177–199. MR1734408 (2000j:57036)
4. C. McA. Gordon and Y.-Q. Wu, *Toroidal and annular Dehn fillings*, Proc. London Math. Soc. **78** (1999), 662-700. MR1674841 (2000b:57029)
5. C. McA. Gordon and Y.-Q. Wu, *Annular Dehn fillings*, Comment. Math. Helv. **75** (2000), 430–456. MR1793797 (2001j:57024)

6. C. Hayashi and K. Motegi, *Only single twists on unknots can produce composite knots*, Trans. Amer. Math. Soc. **349** (1997), 4465–4479. MR1355073 (98b:57010b)
7. S. Lee, and M. Teragaito, *Boundary structure of hyperbolic* 3*-manifolds admitting annular and toroidal fillings at large distance*, to appear in Canad. J. Math.
8. R. Qiu, *Reducible Dehn surgery and annular Dehn surgery*, Pacific J. Math. **192** (2000), 357–368. MR1744575 (2001b:57036)
9. W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982) 357–381. MR0648524 (83h:57019)
10. Y.-Q. Wu, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, J. Diff. Geom. **48** (1998), 407–437. MR1638025 (99h:57043)

School of Mathematics, Korea Institute for Advanced Study, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-722, Korea

E-mail address: slee@kias.re.kr