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# ON OPERATORS WHICH COMMUTE WITH ANALYTIC TOEPLITZ OPERATORS MODULO THE FINITE RANK OPERATORS

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Abstract. It is shown that an operator $S$ on the Hardy space $H^2(\mathbb{D}^n)$ (or $H^2(\mathbb{B}_n)$) commutes with all analytic Toeplitz operators modulo the finite rank operators if and only if $S = T_g + F$. Here $F$ is a finite rank operator, and in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function.

## 1. Introduction

Davidson [Da] studied when an operator $S$ on the classical Hardy space $H^2(\mathbb{D})$ essentially commutes with all analytic Toeplitz operators. He proved that commutators $[S, T_f] = ST_f - T_f S$ are compact for all $f \in H^\infty(\mathbb{D})$ if and only if $S = T_g + K$, where $g \in H^\infty(\mathbb{D}) + C(\mathbb{T})$ and $K$ is compact. Guo [Guo] generalized this result to the Hardy space over the unit ball $\mathbb{B}_n$.

A natural problem is to characterize operators $S$ such that the commutators $[S, T_f]$ belong to the Schatten-von Neumann class $\mathcal{L}^p$ for all $f \in H^\infty$. Note the inclusion

$$\mathcal{L}^0 \subset \cdots \subset \mathcal{L}^1 \subset \cdots \subset \mathcal{L}^\infty,$$

where $\mathcal{L}^0$ is the class of all finite rank operators, and $\mathcal{L}^\infty$ is the class of all compact operators. Therefore, Davidson's work completed the case of $\mathcal{L}^\infty$ of the classical Hardy space in the above sequence. Furthermore, on the classical Hardy space $H^2(\mathbb{D})$, Gu [Gu1] showed that for each $f \in H^\infty(\mathbb{D})$, $[S, T_f] = ST_f - T_f S \in \mathcal{L}^0$ if and only if $S = T_g + F$, where $g$ is a sum of a bounded analytic function and a rational function, and $F$ is a finite rank operator. However, the proof in [Gu1] is considerably technical because Gu solved a more general problem from which the above-mentioned result is a consequence. In this note, we will generalize Gu's result to the case of higher dimension, and in the dimension $n = 1$, our proof is different from the proof in [Gu1]. The following is the main result in this note.

**Theorem 1.1.** *Let $S$ be a bounded linear operator on $H^2(\mathbb{D}^n)$ (or $H^2(\mathbb{B}_n)$). Then $[S, T_f]$ is of finite rank for all $f \in H^\infty$ if and only if $S = T_g + F$. Here $F$ is a finite rank operator, and in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function.*

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## 2. Preliminaries

Let $\mathbb{D}$ be the open unit disk of the complex plane and $\mathbb{T}$ the unit circle. For $n \geq 1$, let $\mathbb{D}^n$ and $\mathbb{T}^n$ be the unit polydisk and $n$-torus, respectively. The Hardy space $H^2(\mathbb{D}^n)$ is the closure of all polynomials in $L^2(\mathbb{T}^n)$ (with respect to the measure $d\theta_1 \cdots d\theta_n/(2\pi)^n$ on $\mathbb{T}^n$). Let $P$ be the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. The Toeplitz operator $T_f : H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n)$ with symbol $f \in L^\infty(\mathbb{T}^n)$ is defined by $T_f(h) = P(fh)$ for all $h \in H^2(\mathbb{D}^n)$. The Hankel operator $H_f$ with symbol $f$ is defined as $H_f h = (I - P)(fh)$ for all $h \in H^2(\mathbb{D}^n)$. For $f, g \in L^\infty(\mathbb{T}^n)$, Toeplitz and Hankel operators are connected by the following formula:

$$T_{fg} - T_f T_g = H^*_{\bar{f}} H_g.$$

Let us recall the Hardy space $H^2(\mathbb{B}_n)$ over the unit ball. Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$, and $\partial\mathbb{B}_n$ its boundary. The Hardy space $H^2(\mathbb{B}_n)$ is the closure of all polynomials in $L^2(\partial\mathbb{B}_n)$ (with respect to the unique rotation-invariant probability measure $d\sigma$ on $\partial\mathbb{B}_n$). In the same way, one can define Toeplitz and Hankel operators on the Hardy space $H^2(\mathbb{B}_n)$.

It is well known that there exist a lot of inner functions both in the unit ball $\mathbb{B}^n$ and the unit polydisk $\mathbb{D}^n$ [R1, R2]. Inner functions will play an important role in this note. In what follows we will use $H^2$ to denote the Hardy space $H^2(\mathbb{B}_n)$ or $H^2(\mathbb{D}_n)$.

**Lemma 2.1.** *Let $\eta$ be a nonconstant inner function. Then $T_{\bar{\eta}^n} \to 0$ $(SOT)$ on the Hardy space $H^2$. Furthermore, for each compact operator $K$, we have $T_{\bar{\eta}^n} K \overset{\|\cdot\|}{\to} 0$ as $n \to \infty$.*

*Proof.* Let $K_\lambda$ be the reproducing kernel of $H^2$. Take $f = \sum a_i K_{\lambda_i}$ to be a finite linear combination of reproducing kernels. Note that

$$T_{\bar{\eta}^n}(\sum a_i K_{\lambda_i}) = \sum a_i \overline{\eta(\lambda_i)}^n K_{\lambda_i}.$$

Hence

$$\|T_{\bar{\eta}^n}(\sum a_i K_{\lambda_i})\| \leq \sum |a_i|\, |\eta(\lambda_i)|^n\, \|K_{\lambda_i}\| \to 0,$$

as $n \to \infty$. Because the above linear combinations are dense in $H^2$, $T_{\bar{\eta}^n} \to 0$ $(SOT)$. For a rank one operator $f \otimes g$ (here $f \otimes g(h) = \langle h, g\rangle f$), we have

$$T_{\bar{\eta}^n} f \otimes g = (T_{\bar{\eta}^n} f) \otimes g.$$

Note that the set of all finite rank operators is dense in all compact operators, the desired result follows. $\square$

The following lemma may be known by many people. We note it here.

**Lemma 2.2.** *Let $\{T_\alpha\}$ be a net of operators on $H^2$ such that $T_\alpha \to T$ (WOT). If there exists a natural number $M$ such that $\operatorname{rank} T_\alpha \leq M$, then*

$$\operatorname{rank} T \leq M.$$

*Proof.* If $\operatorname{rank} T \geq M + 1$, then there exist $\{f_i\}_{i=1}^{M+1}$ and $\{g_i\}_{i=1}^{M+1}$ such that

$$\det[\langle T f_i, g_j\rangle]_{1 \leqslant i,j \leqslant M+1} \neq 0.$$

Set

$$d_\alpha = [\langle T_\alpha f_i, g_j\rangle]_{1 \leqslant i,j \leqslant M+1}.$$

Then $d_\alpha = 0$ since rank$T_\alpha \leq M$. This leads to a contradiction since

$$d_\alpha \to \det[\langle Tf_i, g_j\rangle]_{1\leqslant i,j\leqslant M+1} \neq 0.$$

$\square$

**Lemma 2.3.** *Let $S$ be an operator on $H^2$. If $[S, T_f]$ is of finite rank for all $f \in H^\infty$, then there exists a natural number $M$ such that*

$$\operatorname{rank}[S, T_f] \leq M.$$

*Proof.* Let $\Gamma_n = \{f : \operatorname{rank}[S, T_f] \leq n\}$. By Lemma 2.2, $\Gamma_n$ is a norm closed subset of $H^\infty$. Since $H^\infty$ is a Banach space and $H^\infty = \bigcup_n \Gamma_n$, the Baire Category Theorem implies that there exists a natural number $N$ such that $\Gamma_N$ contains an open subset of $H^\infty$. This means that the set $\{f - g : f, g \in \Gamma_N\}$ is a neighborhood of the function $f = 0$. Thus for each $h \in H^\infty$, there exists a real number $\gamma$ and two functions $f, g \in \Gamma_N$ such that $h = \gamma(f - g)$ and hence

$$\operatorname{rank}[S, T_h] \leq \operatorname{rank}[S, T_f] + \operatorname{rank}[S, T_g] \leqq 2N.$$

$\square$

The next proposition says that Toeplitz operators on the Hardy space $H^2$ can be completely characterized by algebraic equations.

**Proposition 2.1.** *For a bounded linear operator $T$ on $H^2$, then $T$ is a Toeplitz operator if and only if $T^*_\eta T T_\eta = T$ for each inner function $\eta$.*

*Proof.* We will prove the proposition in the case of the unit ball, and the same reasoning is valid in the unit polydisk. Set

$$\mathcal{A} = \{\bar\eta\, h : \eta \text{ are inner functions, } h \in H^2\}.$$

Then $\mathcal{A}$ is a dense linear subspace of $L^2(\partial\mathbb{B}_n)$ by [R1, Theorem 11.4]. Define a map

$$\Phi : \mathcal{A} \to \mathbb{C}$$

by $\Phi(\bar\eta h) = \langle h, T\eta\rangle$. Then $\Phi$ is well defined and linear. In fact, if $\bar\eta_1 h_1 = \bar\eta_2 h_2$, then we have

$$\begin{aligned}
\Phi(\bar\eta_1 h_1) &= \langle h_1, T\eta_1\rangle = \langle h_1, T^*_{\eta_2} T T_{\eta_2}\eta_1\rangle \\
&= \langle \eta_2 h_1, T\eta_2\eta_1\rangle = \langle \eta_1 h_2, T\eta_1\eta_2\rangle \\
&= \langle h_2, T^*_{\eta_1} T T_{\eta_1}\eta_2\rangle = \langle h_2, T\eta_2\rangle \\
&= \Phi(\bar\eta_2 h_2).
\end{aligned}$$

So, $\Phi$ is well defined. The same reasoning shows that $\Phi$ is linear. From the definition of $\Phi$,

$$|\Phi(\bar\eta\, h)| \leq \|T\|\|h\| = \|T\|\|\bar\eta\, h\|.$$

So, $\Phi$ is a bounded linear functional on $\mathcal{A}$. Since $\mathcal{A}$ is dense in $L^2(\partial\mathbb{B}_n)$, there exists a unique $\phi \in L^2(\partial\mathbb{B}_n)$ such that

$$\Phi(\bar\eta\, h) = \langle \bar\eta h, \phi\rangle.$$

Since $\mathcal{A}$ is dense in $L^2(\partial\mathbb{B}_n)$ and

$$|\langle \bar\eta h, \phi\rangle| = |\int_{\partial\mathbb{B}_n} (\bar\eta h)\bar\phi\, d\sigma| \leq \|T\|\|\bar\eta\, h\|,$$

we see $\phi \in L^\infty$. From the equalities

$$\langle h, T\eta\rangle = \Phi(\bar\eta h) = \langle \bar\eta h, \phi\rangle = \langle h, \phi\eta\rangle = \langle h, T_\phi\eta\rangle,$$

and note that the set of all finite linear combinations of inner functions is dense in $H^2$ [R1, Theorem 11.1], we obtain $T = T_\phi$. $\square$

By Kronecker's result about a finite rank Hankel operator, $H_g$ on $H^2(\mathbb{D})$ is of finite rank if and only if $g$ is the sum of a rational function and a bounded analytic function over the disk $\mathbb{D}$ [P]. For $n \geq 2$, if a Hankel operator $H_g$ on $H^2(\mathbb{D}^n)$ has finite rank, then it must be zero (see [Gu2, GuZ]). In the case of unit ball, the following proposition may be known by many people.

**Proposition 2.2.** *Assume $n \geq 2$ and a Hankel operator $H_g$ is of finite rank. Then $H_g = 0$, that is, $g \in H^\infty$.*

*Proof.* Setting $M = \ker H_g$, then $M$ is a finitely codimensional multiplier invariant subspace. As a finitely codimensional multiplier invariant subspace, $M$ is generated by finitely many polynomials $P_1, \dots, P_m$, and the set $\bigcap_{i=1}^m Z(P_i)$ is finite [CG, Corollary 2.2.6], where $Z(P)$ denotes the set of zero points of a polynomial $P$. From $H_g P_i = 0$, we have $Q_i = gP_i \in H^\infty, \quad i = 1, \dots, m$. On $\partial\mathbb{B}_n$, noting

$$g = \frac{Q_1}{P_1} = \cdots = \frac{Q_m}{P_m},$$

this implies

$$\frac{Q_1(z)}{P_1(z)} = \cdots = \frac{Q_m(z)}{P_m(z)}, \quad z \in \mathbb{B}_n \setminus \bigcup_{i=1}^m Z(P_i).$$

Putting $g(z) = \frac{Q_1(z)}{P_1(z)}$, then from the above equalities, $g$ can be analytically extended to $\mathbb{B}_n \setminus \bigcap_{i=1}^m Z(P_i)$. Since the set $\bigcap_{i=1}^m Z(P_i)$ is finite, $g$ has an extension on $\mathbb{B}_n$ by [Kr]. Furthermore, we have $g \in H^\infty$. The reasoning is as follows: since the set $\bigcap_{i=1}^m Z(P_i)$ is finite, there exist $0 < r < 1$ and $\epsilon > 0$ such that $\sum_{i=1}^m |P_i(z)|^2 > \epsilon$ if $|z| > r$. By

$$|g(z)|^2 = \frac{|Q_1(z)|^2}{|P_1(z)|^2} = \dots = \frac{|Q_m(z)|^2}{|P_m(z)|^2} = \frac{\sum_{i=1}^m |Q_i(z)|^2}{\sum_{i=1}^m |P_i(z)|^2},$$

we see that $|g(z)|$ is bounded on $r < |z| < 1$. This implies $g \in H^\infty$. $\square$

## 3. The proof of the main theorem

Before proving the main theorem we need the following lemma.

**Lemma 3.1.** *On the Hardy space $H^2$, if $[T_g, T_f]$ is of finite rank for all $f \in H^\infty$, then in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function.*

*Proof.* By Lemma 2.3 there exists a natural number $M$ such that

$$\operatorname{rank}[T_g, T_f] \leq M$$

for all $f \in H^\infty$. Below we will give the proof of Lemma 3.1 in the case of unit ball, and the same reasoning is valid in the case of the unit polydisk. For any $h, f \in H^\infty$,

$$\operatorname{rank}(T_{\bar{h}fg} - T_{\bar{h}f}T_g) = \operatorname{rank}(T_{\bar{h}}(T_gT_f - T_fT_g)) \leq \operatorname{rank}(T_gT_f - T_fT_g) \leq M.$$

Note that $\{\bar{\eta}f : \eta$ are inner functions, $f \in H^\infty\}$ is $W^*$ dense in $L^\infty(\partial\mathbb{B}_n)$ [R1, Theorem 11.2]. By Lemma 2.2, $T_{\phi g} - T_\phi T_g$ is of finite rank for all $\phi \in L^\infty(\partial\mathbb{B}_n)$. It follows that $T_{\overline{g}g} - T_{\overline{g}}T_g = H_g^* H_g$ is of finite rank, that is, $H_g$ is of finite rank. In the case $n = 1$, by Kronecker's result about a finite rank Hankel operator, $H_g$ on

$H^2(\mathbb{D})$ is of finite rank if and only if $g$ is a sum of a rational function and a bounded analytic function over the disk $\mathbb{D}$ [P]. In the case $n \geq 2$, applying Proposition 2.2 gives $g \in H^\infty$. $\square$

**Theorem 3.1.** *Given a bounded linear operator $S$ on $H^2$, then $[S, T_f]$ is of finite rank for all $f \in H^\infty$ if and only if $S = T_g + F$. Here $F$ is a finite rank operator, and in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function.*

*Remark.* In the case $n = 1$, Gu obtained the above result [Gu1]. However, the present proof is different from Gu's proof.

*Proof.* First we claim that $S$ has the form $S = T_g + F$, where $g \in L^\infty$, and $F$ is a finite rank operator. For this claim, pick a nonconstant inner function $\eta$, and consider the sequence $\{T^*_{\eta^n} S T_{\eta^n}\}$. Note that $\{T^*_{\eta^n} S T_{\eta^n}\}$ is a bounded set, and hence without a loss of generality we may assume that $\{T^*_{\eta^n} S T_{\eta^n}\}$ converges to a operator $A$ in the weak operator topology (if not, we can choose a subnet). It follows that

$$S - A = (\mathrm{WOT}) \lim (S - T^*_{\eta^n} S T_{\eta^n}).$$

Applying Lemma 2.3 shows that there exists a natural number $M$ such that $\operatorname{rank} [S, T_f] \leq M$ for all $f \in H^\infty$ and hence

$$\operatorname{rank}(S - T^*_{\eta^n} S T_{\eta^n}) = \operatorname{rank} T^*_{\eta^n}(T_{\eta^n} S - S T_{\eta^n}) \leq M.$$

By Lemma 2.2, $\operatorname{rank}(S - A) \leq M$, that is, $F = S - A$ is a finite rank operator. Now we prove that $A$ is a Toeplitz operator. For each inner function $\zeta$

$$\begin{aligned} T^*_\zeta A T_\zeta &= (\mathrm{WOT}) \lim T^*_\zeta T^*_{\eta^n} S T_{\eta^n} T_\zeta = (\mathrm{WOT}) \lim T^*_{\eta^n} T^*_\zeta S T_\zeta T_{\eta^n} \\ &= (\mathrm{WOT}) \lim T^*_{\eta^n} T^*_\zeta T_\zeta S T_{\eta^n} + (\mathrm{WOT}) \lim T^*_\zeta T^*_{\eta^n} (S T_\zeta - T_\zeta S) T_{\eta^n}. \end{aligned}$$

Since $S T_\zeta - T_\zeta S$ is of finite rank, the latter is zero by Lemma 2.1. It follows that

$$T^*_\zeta A T_\zeta = (\mathrm{WOT}) \lim T^*_{\eta^n} T^*_\zeta T_\zeta S T_{\eta^n} = A.$$

Proposition 2.1 says that $A$ is a Toeplitz operator, that is, there is a $g \in L^\infty$ such that $A = T_g$. The claim follows. Using Lemma 3.1, we see that in the case $n = 1$, $g$ is a sum of a rational function and a bounded analytic function, and in the case $n \geq 2$, $g$ is a bounded analytic function. The opposite direction is easily proved. $\square$

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