THE BANACH ALGEBRA GENERATED BY A CONTRACTION

H. S. MUSTAFAYEV

(Communicated by Joseph A. Ball)

Abstract. Let $T$ be a contraction on a Banach space and $A_T$ the Banach algebra generated by $T$. Let $\sigma_u(T)$ be the unitary spectrum (i.e., the intersection of $\sigma(T)$ with the unit circle) of $T$. We prove the following theorem of Katznelson-Tzafriri type: If $\sigma_u(T)$ is at most countable, then the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u(T)$ if and only if $\lim_{n \to \infty} \|T^nR\| = 0$. Let $X$ be a complex Banach space $B(X)$, the algebra of all bounded linear operators on $X$, and let $I$ be the identity operator on $X$. $\sigma(T)$ will denote the spectrum of an operator $T \in B(X)$, and $R_z(T) = (z - T)^{-1}$ will denote the resolvent of $T$. If $A$ is a uniformly closed subalgebra of $B(X)$ with identity $I$, then $\sigma_A(T)$ will denote the spectrum of $T \in A$ with respect to $A$. If $T \in B(X)$, by $A_T$ we will denote the uniformly closed subalgebra of $B(X)$ generated by $T$ and $I$. $A_T$ is a commutative unital Banach algebra. As is well known, the maximal ideal space of $A_T$ can be identified with $\sigma_{A_T}(T)$. $\hat{R}$ will denote the Gelfand transform of any $R \in A_T$.

Let $T$ be a contraction (i.e., a linear operator of norm $\leq 1$) on a Banach space $X$. Then for every $x \in X$ the limit $\lim_{n \to \infty} \|T^n x\|$ exists and is equal to $\inf_{n \in \mathbb{N}} \|T^n x\|$. Note also that $\sigma(T) \subset \sigma_{A_T}(T) \subset \hat{D}$; $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $\Gamma$ be the unit circle. $\sigma_u(T) = \sigma(T) \cap \Gamma$ is called the unitary spectrum of $T$. It is easy to see that if $\sigma_u(T) = \emptyset$, then $\lim_{n \to \infty} \|T^n\| = 0$.

It follows from the Y. Katznelson and L. Tzafriri Theorem [7, Theorem 5] that if $\sigma_u(T)$ is at most countable and $f \equiv 0$ on $\sigma_u(T)$, then $\lim_{n \to \infty} \|T^n f(T)\| = 0$, and that $f(z)$ is a function analytic in $D$, which has absolutely convergent Taylor series. In this note we obtain the following extension of this result.

Theorem 1. Let $T$ be a contraction on a Banach space such that the unitary spectrum $\sigma_u(T)$ of $T$ is at most countable. Then the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u(T)$ if and only if $\lim_{n \to \infty} \|T^n R\| = 0$.

For the proof we need some preliminary results.

The proof of the following lemma is similar to that of [7] Lemma 2.1.

Lemma 1. Let $T$ be a contraction on a Banach space $X$ such that $\sigma(T) \neq \hat{D}$ and $\inf_{x \in X} \|T^n x\| > 0$ for some $x \in X \setminus \{0\}$. Then there exist a Banach space $Y \neq \{0\}$, a bounded linear operator $J : X \to Y$ with dense range and a surjective isometry $S$.
on $Y$ such that:
(i) $\|Jx\| = \lim_{n \to -\infty} \|T^n x\|$;
(ii) $SJ = JT$;
(iii) $\sigma(S) \subset \sigma(T)$.

Proof. On $X$ we define the semi-norm $p$ by $p(x) = \lim_{n \to -\infty} \|T^n x\|$. Put $E = \ker p$. Then $E$ is a closed invariant subspace of $T$ and $E \neq X$. Let $J : X \to X/E$ be the quotient mapping. Then the semi-norm $p$ induces a norm $\tilde{p}$ on $Y_0 = X/E$ by $\tilde{p}(Jx) = p(x)$, and we have $\|Jx\| = \lim_{n \to -\infty} \|T^n x\|$. Let $Y$ be the completion of $Y_0$ with respect to the norm $\tilde{p}$. Define $S_0 : Y_0 \to Y_0$ by $S_0 J = JT$. Since $\|S_0 J x\| = \|Jx\|$, $S_0$ extends to an isometry $S$ on $Y$. Then we have $SJ = JT$, where $J : X \to Y$ has dense range.

Let $z \notin \sigma(T)$. From the obvious inequality $p(R_z(T)x) \leq \|R_z(T)x\| (x \in X)$, it follows that $\sigma(S) \subset \sigma(T)$. If $S$ is a non-surjective isometry, then $\sigma(S) = \tilde{D}$ [p. 27], and therefore $\sigma(T) \neq \tilde{D}$. Hence, $S$ is a surjective isometry. The proof is complete. \hfill \Box

If $T$ is a surjective isometry on a Banach space $X$, then $\sigma(T) \subset \Gamma$ and

$$R_z(T) = \begin{cases} \sum_{n=0}^{\infty} z^{-n-1} T^n, & |z| > 1, \\ -\sum_{n=1}^{\infty} z^{n-1} T^{-n}, & |z| < 1. \end{cases}$$

It follows that $\|R_z(T)\| \leq |z| - 1^{-1} (|z| \neq 1)$. Now let $f \in L^1(Z)$ and

$$\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} f(n) \xi^n \quad (\xi \in \Gamma),$$

the Fourier transform of $f$. We can define $\hat{f}(T) \in B(X)$ by

$$\hat{f}(T) = \sum_{n \in \mathbb{Z}} f(n) T^{-n}.$$ 

Lemma 2. Let $T$ be a surjective isometry on a Banach space and let $f \in L^1(Z)$. If $\hat{f}(\xi) = 0$ in a neighborhood of $\sigma(T)$, then $\hat{f}(T) = 0$.

Proof. Let $U$ be an open set in $\Gamma$ that contains $\sigma(T)$. Assume that $\hat{f}$ vanishes on $U$. Then we have

$$\hat{f}(T) = \lim_{r \to 1^-} \sum_{n \in \mathbb{Z}} r^{|n|} f(n) T^{-n} = \lim_{r \to 1^-} \int \hat{f}(\xi) \left( \sum_{n \in \mathbb{Z}} r^{|n|} \xi^n T^{-n} \right) d\xi$$

$$= \lim_{r \to 1^-} \int_{\Gamma \setminus U} \hat{f}(\xi) \left( TR_{r^{-1}} T(T) - TR_\xi(T) \right) d\xi$$

$$+ \lim_{r \to 1^-} \int_{\Gamma \setminus U} \hat{f}(\xi) \left( TR_\xi(T) - TR_{r \xi}(T) \right) d\xi = 0.$$

We will also need the following notation.

Recall that $\varphi = \{ \varphi(n) \} \in L^\infty(Z)$ is almost periodic on $Z$ if $\{ \varphi_m : m \in \mathbb{Z} \}$ is relatively compact in the norm topology of $L^\infty(Z)$, where $\varphi_m(n) = \varphi(n + m)$. We denote by $AP(Z)$ the set of all almost periodic functions on $Z$. $AP(Z)$ is a closed
subalgebra of \(L^\infty (\mathbb{Z})\). As is well known, there exists a unique \(\Phi \in AP (\mathbb{Z})^*\) (which is called \emph{invariant mean on} \(AP (\mathbb{Z})\)) such that:

(i) \(\Phi (1) = 1\), where \(1 (n) \equiv 1\);
(ii) \(\Phi (\varphi) \geq 0\) for all \(\varphi \geq 0\);
(iii) \(\Phi (\varphi_m) = \Phi (\varphi)\) for all \(\varphi \in L^\infty (\mathbb{Z})\) and \(m \in \mathbb{Z}\).

The \emph{hull} of any ideal \(I \subset L^1 (\mathbb{Z})\) is \(h (I) = \{ \xi \in \Gamma : \hat{f} (\xi) = 0, f \in I \}\). For a closed subset \(K \subset \Gamma\), let \(I_K = \{ f \in L^1 (\mathbb{Z}) : \hat{f} (K) = \{ 0 \}\}\) and \(J_K = \{ f \in L^1 (\mathbb{Z}) : \text{supp} \hat{f} \cap K = \emptyset \}\). \(K\) is a set of \emph{synthesis} if \(I_K = J_K^0\). For example, closed countable sets are sets of synthesis. As is well known (Malliavin’s theorem), there exists a unique \(\Phi \in L^\infty (\mathbb{Z})\) of any ideal \(\varphi \in L^\infty (\mathbb{Z})\) is at most countable. Then for every \(\varphi \in L^\infty (\mathbb{Z})\) and \(\varphi * f\) will denote the convolution of \(\varphi\) and \(f\). Recall that the \(w^*\)-spectrum \(\sigma_*(\varphi)\) of \(\varphi \in L^\infty (\mathbb{Z})\) is defined as the hull of the closed ideal \(I_\varphi = \{ f \in L^1 (\mathbb{Z}) : \varphi * f = 0 \}\). The well-known theorem of Loomis [6] states that if the \(w^*\)-spectrum of \(\varphi \in L^\infty (\mathbb{Z})\) is at most countable, then \(\varphi \in AP (\mathbb{Z})\).

**Lemma 3.** Let \(S\) be a surjective isometry on a Banach space \(Y\) such that \(\sigma (S)\) is at most countable. Then for every \(\phi \in Y^*\), there exists a Hilbert space \(H_\phi\), a bounded linear operator \(J_\phi : Y \to H_\phi\) with dense range and a unitary operator \(U_\phi\) on \(H_\phi\) such that:

(i) \(U_\phi J_\phi = J_\phi S\);
(ii) \(\sigma (U_\phi^*) \subset \sigma (S)\).

**Proof.** (i) Let \(\phi \in Y^*\). For given \(y \in Y\), define the function \(\bar{y}_\phi\) on \(\mathbb{Z}\) by \(\bar{y}_\phi (n) = \phi (S^n y)\). Since \(\|\bar{y}_\phi\|_\infty \leq \|\phi\| \|y\|\), \(\bar{y}_\phi\) is a bounded function. We claim that \(\sigma_*(\bar{y}_\phi) \subset \sigma (S)\). Assume that there exists a \(\xi_0 \in \sigma_*(\bar{y}_\phi)\), but \(\xi_0 \notin \sigma (S)\). Then there exists an \(f \in L^1 (\mathbb{Z})\) such that \(\hat{f} (\xi_0) \neq 0\) and \(\hat{f} (\xi) = 0\) on some neighborhood of \(\sigma (S)\).

By Lemma 2, \(\hat{f} (S) = 0\) and consequently,
\[
0 = \phi \left( S^n \hat{f} (S) y \right) = (\bar{y}_\phi * f) (n), \quad \text{for all } n \in \mathbb{Z}.
\]

Since \(\xi_0 \in \sigma_*(\bar{y}_\phi)\) it follows that \(\hat{f} (\xi_0) = 0\). This contradiction proves the claim.

Hence, \(\sigma_*(\bar{y}_\phi) \) is at most countable. By the Loomis Theorem [6], \(\bar{y}_\phi \in AP (\mathbb{Z})\).

Let \(\bar{H}_\phi\) denote the linear set \(\{ \bar{y}_\phi : y \in Y\}\) with the inner product defined by
\[
\langle \bar{y}_\phi, \bar{z}_\phi \rangle = \Phi \left( \{ \bar{y}_\phi (n) \bar{z}_\phi (n) \}_{n \in \mathbb{Z}} \right), \quad z \in Y,
\]
where \(\Phi\) is the invariant mean on \(AP (\mathbb{Z})\). Let \(H_\phi\) be the completion of \(\bar{H}_\phi\) with respect to the norm
\[
\|\bar{y}_\phi\|_2^2 = \Phi \left( \{ |\bar{y}_\phi (n)|^2 \}_{n \in \mathbb{Z}} \right).
\]

Then \(H_\phi\) is a Hilbert space. Note also that \(\|\bar{y}_\phi\|_2 \leq \|\bar{y}_\phi\|_\infty \leq \|\phi\| \|y\|\). It follows that the map \(J_\phi : Y \to H_\phi\), defined by \(J_\phi y = \bar{y}_\phi\), is a bounded linear operator with dense range. Now define the map \(U_\phi : H_\phi \to H_\phi\), by \(U_\phi \bar{y}_\phi = (S\bar{y})\phi\). It is easy to see that \(U_\phi\) is a unitary operator and \(U_\phi J_\phi = J_\phi S\). We have proved (i).

Next we prove (ii). We have
\[
(1) \quad S^* J_\phi^* = J_\phi^* U_\phi^*.
\]
Assume that there exists $\xi \in \sigma \left( U_\phi \right)$, but $\xi \notin \sigma \left( S \right) = \sigma \left( S^* \right)$. Put $\delta = \| (S^* - \xi)^{-1} \|^{-1}$. Choose $\varepsilon > 0$ such that $\varepsilon < \delta$. Let $\Gamma_\varepsilon = \{ z \in \mathbb{C} : |z - \xi| < \varepsilon \} \cap \Gamma$ and let $E(\cdot)$ be the spectral measure for $U_\phi^*$. Since $\sigma \left( U_\phi^* \right) \cap \Gamma_\varepsilon \neq \emptyset$, we have $E(\Delta_\varepsilon) \neq 0$. Let $h \in E(\Delta_\varepsilon) H_\phi$ be such that $\| h \| = 1$. From the identity

\[
(U_\phi^* - \xi)^n h = \int_{\Gamma_\varepsilon} (t - \xi)^n dE(t) h, \quad n \in \mathbb{N},
\]

we have

\[
\| (U_\phi^* - \xi)^n h \| \leq \varepsilon^n.
\]

On the other hand from (1) we can write

\[
(S^* - \xi)^n J^*_\phi h = J^*_\phi (U_\phi^* - \xi)^n h.
\]

It follows that

\[
\| (S^* - \xi)^n J^*_\phi h \| \leq \varepsilon^n \| J^*_\phi \|.
\]

Consequently, we have

\[
\| J^*_\phi h \| \leq \| (S^* - \xi)^{-1} \| \| (S^* - \xi)^n J^*_\phi h \| \leq (\frac{\varepsilon}{\delta})^n \| J^*_\phi \| \to 0, \quad \text{as} \ n \to \infty.
\]

Hence, $J^*_\phi h = 0$. Since $J^*_\phi$ has zero kernel, we obtain $h = 0$. This is a contradiction. The proof is complete.

**Proof of Theorem 1.** Let $R \in A_T$. Assume that $\lim_{n \to \infty} \| T^n R \| = 0$. Then for any $\xi \in \sigma_u (T)$, $\| T^n (\xi) \hat{R} (\xi) \| \to 0$, as $n \to \infty$. Since $|\hat{T}(\xi)| = 1$, it follows that $\hat{R}(\xi) = 0$. Now assume that $\sigma_u (T)$ is at most countable and $\hat{R}(\xi) \equiv 0$ on $\sigma_u (T)$. It is enough to prove that $\lim_{n \to \infty} \| T^n R x \| = 0$ for all $x \in X$. Indeed, suppose that this is proved. For $C \in B(X)$, let $L_C$ be the left multiplication operator on $B(X)$ defined by $L_C F = C F$. Then $L_T$ is a contraction. Moreover, the maximal ideal spaces of $A_{L_T}$ and $A_T$ are the same, and $\sigma (L_T) = \sigma (T)$. Note also that $L_R \in A_{L_T}$ and the Gelfand transform of $L_R$ vanishes on $\sigma_u (L_T)$. Therefore,

\[
\lim_{n \to \infty} \| L_T^n L_R F \| = 0,
\]

for all $F \in B(X)$. Taking $F = I$, we get the desired conclusion.

If $\lim_{n \to \infty} \| T^n x \| = 0$ for all $x \in X$, then there is nothing to prove. Hence, we may assume that $\lim_{n \to \infty} \| T^n x \| > 0$ for some $x \neq 0$. On the other hand, since $\sigma_u (T)$ is at most countable, $\sigma (T) \neq \hat{D}$. In view of Lemma 1 there exists a Banach space $Y \neq \{0\}$, a bounded linear operator $J : X \to Y$ with dense range and a surjective isometry $S$ on $Y$ such that:

(i) $\| J x \| = \lim_{n \to \infty} \| T^n x \|;

(ii) $S J = JT$;

(iii) $\sigma (S) \subset \sigma (T)$.

It follows from (iii) that $\sigma (S) \subset \sigma_u (T)$ and therefore, $\sigma (S)$ is at most countable.

Now let $\phi \in Y^*$ be given. By Lemma 3, there exists a Hilbert space $H_\phi$, a bounded linear operator $J_\phi : Y \to H_\phi$ with dense range and a unitary operator $U_\phi$ on $H_\phi$ such that

\[
U_\phi J_\phi = J_\phi S
\]
and $\sigma \left( U^*_\phi \right) \subset \sigma (S) \subset \sigma_u (T)$. Moreover, from (ii) and (2) we obtain

$$U^*_\phi J = J^* J \phi T.$$  

(3)

Further, since $R \in A_T$, there exists a sequence of polynomials $\{P_n (z)\}_{n \in \mathbb{N}}$ such that $\|P_n (T) - R\| \to 0$. Also since the Gelfand transform of $R$ vanishes on $\sigma_u (T)$, the sequence $\{P_n (z)\}_{n \in \mathbb{N}}$ uniformly converges to zero on $\sigma (T)$. Hence, the sequence $\{P_n (z)\}_{n \in \mathbb{N}}$ uniformly converges to zero on $\sigma (U^*_\phi)$. It follows that $\|P_n \left( U^*_\phi \right)\| \to 0$.

On the other hand, from the identity (3) we can write

$$J^* J^*_\phi P_n \left( U^*_\phi \right) = P_n \left( T^* \right) J^* J^*_\phi.$$  

This clearly implies that $R^* J^* J^*_\phi = 0$ and so $J_\phi J_R = 0$. Hence, $\phi (JRx) = 0$ for all $\phi \in Y^*$ and $x \in X$. Thus, we obtain that

$$0 = \|JRx\| = \lim_{n \to \infty} \|T^nx\|,$$

for all $x \in X$. This completes the proof. □

We do not know whether Theorem 1 true if $\sigma_u (T)$ is a synthesis set.

**Remark 1.** Theorem 1 remains valid if $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$. Indeed, in this case $\|x\| = \sup_{n \in \mathbb{N}} \|T^nx\|$ is an equivalent norm on $X$ with respect to which $T$ becomes a contraction.

In contrast with the unitary operator on a Hilbert space, there exists a surjective isometry on a Banach space that generates a non-semisimple algebra (see [3]). But surjective isometry on a Banach space with countable spectrum generated a semisimple algebra [3]. The following example shows that even on a Hilbert space there exists a contraction with countable unitary spectrum that generates a non-semisimple algebra: Let $V$ be the Volterra operator on $\ell^2 \left[ 0, 1 \right]$ defined by $(Vf) (t) = \int_0^t f (s) ds$ and let $T = (I + V)^{-1}$. Then $\|T^n\| = 1$ for all $n \in \mathbb{N}$ and $\sigma (T) = \{1\}$. But $T \neq I$.

Recall that a contraction $T$ on a Banach space $X$ is said to be a $C_1$-contraction if $\inf_{n \in \mathbb{N}} \|T^nx\| > 0$ for all $x \in X \setminus \{0\}$ [1] p. 250.

**Corollary 1.** Let $T$ be a $C_1$-contraction on a Banach space such that the unitary spectrum $\sigma_u (T)$ of $T$ is at most countable. If the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u (T)$, then $R = 0$. In particular, $A_T$ is semisimple.

For contractions on a Hilbert space, the Katznelson-Tzafriri theorem can be improved as follows [2]: If $T$ is a contraction on a Hilbert space and $f \in A(D)$ vanishes on $\sigma_u (T)$, then

$$\lim_{n \to \infty} \|T^nf (T)\| = 0;$$

$A(D)$ is the disc algebra. Now let $R \in A_T$ be such that $\hat{R} (\xi) \equiv 0$ on $\sigma_u (T)$. Is it then true that $\lim_{n \to \infty} \|T^nR\| = 0$? We do not know the answer to this question.

However, we prove the following.

**Theorem 2.** Let $H$ be a Hilbert space and let $T$ be a contraction on $H$. If the Gelfand transform of $R \in A_T$ vanishes on $\sigma_u (T)$, then

$$\lim_{n \to \infty} \|T^nRx\| = 0,$$

for all $x \in H$. 

---

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. Let $R \in \mathcal{A}_T$ be such that $\hat{R}(\xi) \equiv 0$ on $\sigma_u(T)$. We define a new inner product on $H$ by the formula

$$[x, y] = \lim_{n \to \infty} \langle T^n x, T^n y \rangle$$

(it is easy to see that the limit on the right-hand side exists). This induced a semi-norm on $H$ defined by

$$p(x) = \left( \lim_{n \to \infty} \|T^n x\|^2 \right)^{1/2}.$$ 

Let $E = \ker p$. It is clear that $E$ is a closed invariant subspace of $T$. If $E = H$, then there is nothing to prove. Hence, we may assume that $E \neq H$. Let $J : H \to H/E$ be the quotient mapping. Then the semi-norm $p$ induces a norm $\bar{p}$ on $K_0 = H/E$ by $\bar{p}(Jx) = p(x)$, and we have

$$\|Jx\| = \left( \lim_{n \to \infty} \|T^n x\|^2 \right)^{1/2}.$$ 

Let $K$ be the completion of $K_0$ with respect to the norm $\bar{p}$. Define $U_0 : K_0 \to K_0$ by $U_0 J = JT$. Since $\|U_0 Jx\| \leq \|T\| \|Jx\|$, $U_0$ extends to a bounded operator $U$ on $K$. Then we have $UJ = JT$, where $J : H \to K$ has dense range. Also since

$$[Up(x), Up(y)] = [p(x), p(y)], \quad x, y \in H,$$

$U$ is an isometry on $K$. As in the proof of Lemma 1 we can see that $\sigma(U) \subset \sigma(T)$.

Now assume that $U$ is a non-surjective isometry. Then $\sigma(U) = \hat{D}$ and consequently, $\sigma(T) = \hat{D}$. Hence, $\sigma_u(T) = \Gamma$. Since $R \in \mathcal{A}_T$, there exists a sequence $\{P_n(z)\}_{n \in \mathbb{N}}$ of polynomials such that $\|P_n(T) - R\| \to 0$. It follows that $P_n(z) \to 0$ uniformly on $\Gamma$. By the von Neumann inequality,

$$\|P_n(T)\| \leq \sup_{\xi \in \Gamma} |P_n(\xi)| \to 0,$$

and so $R = 0$. Hence, we may assume that $U$ is a unitary operator. As above, there exists a sequence $\{P_n(z)\}_{n \in \mathbb{N}}$ of polynomials such that $\|P_n(T) - R\| \to 0$. It follows that $P_n(z) \to 0$ uniformly on $\sigma_u(T)$. Since $\sigma(U) \subset \sigma_u(T)$, we have $\|P_n(U)\| \to 0$. Now from the identity $P_n(U) J = JP_n(T)$ we obtain that $JR = 0$. Hence we have that $\lim_{n \to \infty} \|T^n Rx\| = 0$ for all $x \in H$. The proof is complete. \(\square\)

A similar result holds for the power-bounded operators.

**Theorem 3.** Let $T$ be a power-bounded operator on a Hilbert space $H$. If the Gelfand transform of $R \in \mathcal{A}_T$ vanishes on $\sigma_u(T)$, then for all $x \in H$,

$$\text{l.i.m.} \|T^n Rx\| = 0,$$

where l.i.m. is a Banach limit on $\mathbb{N}$.

**Corollary 2.** Let $T$ be a contraction on a Hilbert space. If $R \in \mathcal{A}_T$ is a compact operator and $\hat{R}(\xi) \equiv 0$ on $\sigma_u(T)$, then

$$\lim_{n \to \infty} \|T^n R\| = 0.$$
References


Department of Mathematics, Faculty of Arts and Sciences, Yüzyücü Yıl University, 65080, Van, Turkey

E-mail address: hsmustafayev@yahoo.com