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THE PROOF OF TCHAKALOFF'S THEOREM

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Abstract. We provide a simple proof of Tchakaloff's Theorem on the existence of cubature formulas of degree m for Borel measures with moments up to order m. The result improves known results for non-compact support, since we do not need conditions on $(m+1)$st moments. In fact, we reduce the classical assertion of Tchakaloff's Theorem to a well-known statement going back to F. Riesz.

We consider the question of existence of cubature formulas of degree m for *Borel measures* μ, i.e. a measure defined on the Borel σ-algebra, where moments up to degree m exist:

Definition 1. Let μ be a positive Borel measure on $\mathbb{R}^N$ and $m \geq 1$ such that

$$\int_{\mathbb{R}^N} \|x\|^k \, \mu(dx) < \infty$$

for $0 \leq k \leq m$ holds true. A *cubature formula of degree* m for μ is given by an integer $k \geq 1$, points $x_1, \ldots, x_k \in \operatorname{supp} \mu$, and weights $\lambda_1, \ldots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} P(x)\mu(dx) = \sum_{i=1}^{k} \lambda_i P(x_i)$$

for all polynomials on $\mathbb{R}^N$ with degree less or equal m, where $\operatorname{supp} \mu$ denotes the closed support of the measure μ, i.e. the complement of the biggest open set $O \subset \mathbb{R}^N$ with $\mu(O) = 0$.

Cubature formulas of degree m have been proved to exist for Borel measures μ, where the $(m+1)$st moments exist; see [1] and [6]. The result in the case of compact $\operatorname{supp} \mu$ is classical, and due to Tchakaloff (see [10]), hence we refer to the assertion as Tchakaloff's Theorem.

We collect some basic notions and results from convex analysis (see for instance [9]): fix $N \geq 1$, for some set $S \subset \mathbb{R}^N$ the convex hull of S, i.e. the smallest convex

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set in $\mathbb{R}^N$ containing S, is denoted by $\text{conv}(S)$, and the (topological) closure of $\text{conv}(S)$ is denoted by $\overline{\text{conv}}(S)$. The closure of a convex set is convex. Note that the convex hull of a compact set is always closed, but there are closed sets whose convex hull is no longer closed (see [9]).

Closed convex sets can also be described by their *supporting hyperplanes.* Given a convex set C let $y \in \partial C := \overline{C} \setminus \text{int}(C)$ be a boundary point. There is a linear functional l_y and a real number β_y such that the hyperplane defined by $l_y = \beta_y$ contains y, and C is contained in the closed half-space $l_y \leq \beta_y$. Hyperplanes and half-spaces with this property are called supporting hyperplanes and supporting half-spaces, respectively. Moreover, $\overline{C}$ is the intersection of all its supporting half-spaces. Furthermore, if C is not contained in any hyperplane of $\mathbb{R}^N$ (i.e. it has non-empty interior), then a point $x \in C$ is contained in a supporting hyperplane if and only if $x \in C$ is not an interior point of C (see [9], Th. 11.6). This means that we can characterize the boundary of C as those points, which lie at least in one supporting hyperplane of C.

In the case of a convex cone C the supporting hyperplanes can be chosen to be homogeneous, i.e. to be of the form $l_y = 0$. We denote the convex cone generated by some set $A \subset \mathbb{R}^N$ by $\text{cone}(A)$ and its closure by $\overline{\text{cone}}(A)$.

We also introduce the notion of the *relative interior* of a convex set C: a point $x \in C$ lies in the relative interior $\text{ri}(C)$ if for every $y \in C$ there is $\epsilon > 0$ such that $x - \epsilon(y - x) \in C$. In particular we have that the relative interior of a convex set C coincides with the relative interior of its closure $\overline{C}$. Interior points of C lie in the relative interior; see [9]. This remains true even if C lies in an affine subspace of $\mathbb{R}^N$, and a point of C lies in the interior with respect to the subspace topology.

Given a measure μ on some measurable space $(\Omega, \mathcal{F})$ and a Borel measurable map $\phi : \Omega \to \mathbb{R}^N$, we denote by $\phi_*\mu$ the *push-forward Borel measure* on $\mathbb{R}^N$, which is defined via

$$\phi_*\mu(A) := \mu(\phi^{-1}(A))$$

for all Borel sets $A \subset \mathbb{R}^N$.

Theorem 1. *Let μ be a positive Borel measure on $\mathbb{R}^N$, such that the first moments exist, i.e.*

$$\int_{\mathbb{R}^N} \|x\| \, \mu(dx) < \infty,$$

and let $A \subset \mathbb{R}^N$ be a measurable set with $\mu(\mathbb{R}^N \setminus A) = 0$. Then the first moment $E = \int_{\mathbb{R}^N} x\mu(dx)$, where x denotes the vector $(x_1, \dots, x_N)$, is contained in $\text{cone}(A)$.

Proof. We first assume that there is no $B \subset A$ with $\mu(A \setminus B) = 0$ such that B is contained in a hyperplane, since otherwise we could work in a lower-dimensional space instead (with A replaced by B). Fix some $y \in \overline{K} \setminus \text{int}(K)$ in the boundary of $K = \text{cone}(A)$. Then all linear functionals $l_y : \mathbb{R}^N \to \mathbb{R}$ corresponding to the supporting half-spaces $l_y \leq 0$ at y are certainly integrable and we have

$$l_y(E) = \int_{\mathbb{R}^N} l_y(x)\mu(dx) \leq 0;$$

consequently $E \in \overline{\text{cone}}(A)$.

By existence of the first moments, for each $\delta > 0$ we have $\mu(\mathbb{R}^N \setminus B(0, \delta)) < \infty$, where $B(0, \delta)$ denotes the centered ball with radius δ. Given l_y as above, we may conclude that $\mu(\{x \in A | l_y(x) < 0\}) > 0$, since otherwise the complement in A of the intersection of A with the hyperplane $l_y = 0$ would have measure 0, a contradiction to the assumption above. Then we can find $\epsilon > 0$ such that

$$0 < \mu(\{x \in A | l_y(x) \leq -\epsilon\}) < \infty$$

and get

$$l_y(E) = \int_{\mathbb{R}^N} l_y(x)\mu(dx) \leq -\epsilon\mu(\{x \in A | l_y(x) \leq -\epsilon\}) < 0.$$

Hence, $E \in \overline{\text{cone}}(A)$ is an interior point of $\overline{\text{cone}}(A)$. In particular, $E \in \text{cone}(A)$, since the interior lies in the convex cone hull of A. If the first condition is not satisfied, we obtain that E is an interior point of $\text{cone}(A)$ in an affine subspace of $\mathbb{R}^N$ (where the first condition is satisfied), but then E lies in the relative interior of $\text{cone}(A)$ in $\mathbb{R}^N$, which is the desired result. □

Corollary 1. *Let μ be a positive Borel measure on $\mathbb{R}^N$ concentrated in $A \subset \mathbb{R}^N$, i.e. $\mu(\mathbb{R}^N \setminus A) = 0$, such that the first moments exist, i.e.*

$$\int_{\mathbb{R}^N} \|x\| \, \mu(dx) < \infty.$$

Then there exist an integer $1 \leq k \leq N$, points $x_1, \dots, x_k \in A$ and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} f(x)\mu(dx) = \sum_{i=1}^{k} \lambda_i f(x_i)$$

for any monomial f on $\mathbb{R}^N$ of degree 1.

Proof. The corollary follows immediately from Theorem 1 and Caratheodory's Theorem (see [9], Th. 17.1 and Cor. 17.1.2). □

Corollary 2. *Let μ be a positive measure on the measurable space $(\Omega, \mathcal{F})$ concentrated in $A \in \mathcal{F}$, i.e. $\mu(\Omega \setminus A) = 0$, and $\phi : \Omega \to \mathbb{R}^N$ a Borel measurable map. Assume that the first moments of $\phi_*\mu$ exist, i.e.*

$$\int_{\mathbb{R}^N} \|x\| \, \phi_*\mu(dx) < \infty.$$

Then there exist an integer $1 \leq k \leq N$, points $\omega_1, \dots, \omega_k \in A$ and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\Omega} \phi_j(\omega)\mu(d\omega) = \sum_{i=1}^{k} \lambda_i \phi_j(\omega_i)$$

for $1 \leq j \leq N$, where ϕ_j denotes the j-th component of ϕ.

Remark 1. In other words, $A \in \mathcal{F}$ such that $\mu(\Omega \setminus A) = 0$ correspond to $B \subset \phi(\Omega)$ such that $\phi_*\mu(\mathbb{R}^N \setminus B) = 0$.

Remark 2. Note that $\mu(\Omega) = \infty$ is also possible, since we only speak about integrability of N measurable functions $\phi_1, \dots, \phi_N$. If we have $\mu(\Omega) < \infty$, we could add $\phi_{N+1} = 1$, and we obtain in particular $\sum_{i=1}^{k'} \lambda_i' = \mu(\Omega)$ (with possibly a different number $1 \leq k' \leq N+1$ of points x_i' and weights λ_i').

In the setting of Theorem 1 assume that μ is a probability measure on $\mathbb{R}^N$. Then—by the previous consideration—$E = \int_{\mathbb{R}^N} x\mu(dx)$ lies in the convex hull $\text{conv}(A)$. This fact is well known in financial mathematics, since it means that the price range of forward contracts is given by the relative interior of the convex hull of the no-arbitrage bounds of the (discounted) price process (see for instance [2], Th. 1.40).

The result is also well known in the field of geometry of the moment problem; see for instance [4]. As mentioned therein, the result for compactly supported measures essentially even goes back to F. Riesz; see [8].

Proof. We solve the problem with respect to $\phi_* \mu$ on $\mathbb{R}^N$ and obtain $1 \leq k \leq N$, $y_1, \ldots, y_k \in \phi(A)$ and $\lambda_1, \ldots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} f(y)(\phi_* \mu)(dy) = \sum_{i=1}^{k} \lambda_i f(y_i)$$

for all polynomials f of degree 1. Thus we obtain points $\omega_1, \ldots, \omega_k$ with $\phi(\omega_i) = y_i$ for $1 \leq i \leq k$, and furthermore

$$\int_{\mathbb{R}^N} f(y)(\phi_* \mu)(dy) = \int_{\Omega} (f \circ \phi)(\omega) \mu(d\omega)$$

by definition, hence the result. □

In an adequate algebraic framework the previous Theorem 1 yields all cubature results in full generality, and even generalizes those results (see [1], [6] and [7] for related theory and interesting extensive references).

For this purpose we consider polynomials in N (commuting) variables $e_1, \ldots, e_N$ with degree function $\deg(e_i) := k_i$ for $1 \leq i \leq N$ and integers $k_i \geq 1$. Hence, we can associate a degree to monomials $e_{i_1} \cdots e_{i_l}$ with $(i_1, \ldots, i_l) \in \{1, \ldots, N\}^l$ for $l \geq 0$ (note that the monomial associated to the empty sequence is by convenience 1), namely

$$\deg(e_{i_1} \cdots e_{i_l}) = \sum_{r=1}^{l} k_{i_r}.$$

We denote by $\mathbb{A}^N_{\deg \leq m}$ the vector space of polynomials generated by monomials of degree less or equal to m, for some integer $m \geq 1$. We define a continuous map $\phi : \mathbb{R}^N \to \mathbb{A}^N_{\deg \leq m}$, via

$$\phi(x_1, \ldots, x_N) = \sum_{l \geq 0} \sum_{(i_1, \ldots, i_l) \in \{1, \ldots, N\}^l, \ \sum_{r=1}^{l} k_{i_r} \leq m} x_{i_1} \cdots x_{i_l} e_{i_1} \cdots e_{i_l}.$$

Continuity is obvious, since we are given monomials in each coordinate. ϕ is even an embedding and a closed map.

The following example shows the relevant idea in coordinates, since for $N = 1$ and $\deg(e_1) = 1$ we obtain $\mathbb{A}^1_{\deg \leq m} = \mathbb{R}^{m+1}$.

Example 1. Fix $m \geq 1$. Then $\phi(x) = (1, x, x^2, \ldots, x^m)$ is a continuous map $\phi : \mathbb{R}^1 \to \mathbb{R}^{m+1}$. Given a positive Borel measure μ on $\mathbb{R}^1$ such that moments up to degree m exist, i.e.

$$\int_{\mathbb{R}} |x|^k \mu(dx) < \infty$$

for $0 \leq k \leq m$, then $\phi_* \mu$ admits moments up to degree 1. Hence we conclude that there exist $1 \leq k \leq m+1$, points $x_1, \dots, x_k$ and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} P(x) \mu(dx) = \sum_{i=1}^{k} \lambda_i P(x_i)$$

for all polynomials P of degree less than or equal to m.

Theorem 2. *Given $N \geq 1$ and degree function* deg *and $m \geq 1$, fix a finite, positive Borel measure μ on $\mathbb{R}^N$ concentrated in $A \subset \mathbb{R}^N$, i.e. $\mu(\mathbb{R}^N \setminus A) = 0$, such that*

$$\int_{\mathbb{R}^N} |x_{i_1} \cdots x_{i_l}| \, \mu(dx) < \infty$$

for $(i_1, \dots, i_l) \in \{1, \dots, N\}^l$ with $\sum_{r=1}^{l} k_{i_r} \leq m$. Then there exist an integer $1 \leq k \leq \dim \mathbb{A}^N_{\deg \leq m}$, points $x_1, \dots, x_k \in A$ and weights $\lambda_1, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^N} P(x) \mu(dx) = \sum_{i=1}^{k} \lambda_i P(x_i)$$

for $P \in \mathbb{A}^N_{\deg \leq m}$.

Proof. The measure $\phi_* \mu$ admits first moments by assumption, hence we conclude by Corollary 2. □

Remark 3. Tchakaloff's Theorem is a special case of the above theorem with $A = \operatorname{supp} \mu$.

Remark 4. Fix a non-empty, closed set $K \subset \mathbb{R}^N$. We note that a finite sequence of real numbers $m_{i_1 \cdots i_l}$ for $(i_1, \dots, i_l) \in \{1, \dots, N\}^l$ with $\sum_{r=1}^{l} k_{i_r} \leq m$ represents the sequence of moments of a Borel probability measure μ with support $\operatorname{supp} \mu \subset K$, where moments of degree less than or equal to m exist, if and only if

$$\sum_{l \geq 0} \sum_{(i_1, \dots, i_l) \in \{1, \dots, N\}^l, \ \sum_{r=1}^{l} k_{i_r} \leq m} m_{i_1 \cdots i_l} e_{i_1} \cdots e_{i_l} \in \operatorname{conv} \phi(K).$$

The argument in one direction is that any element of $\operatorname{conv} \phi(K)$ is represented as expectation with respect to some probability measure with support in K, for instance the given convex combination. The other direction is Tchakaloff's Theorem in the general form of Theorem 2. Consequently we have a precise geometric characterization of solvability of the Truncated Moment Problem for measures with support in K. Note that one can often describe $\operatorname{conv} \phi(K)$ by finitely many inequalties.

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