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MAXIMALITY OF SUMS OF TWO MAXIMAL MONOTONE OPERATORS

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Abstract. We use methods from convex analysis, relying on an ingenious function of Simon Fitzpatrick, to prove maximality of the sum of two maximal monotone operators on reflexive Banach space under weak transversality conditions.

1. Introduction and preliminaries

The central result of this paper, Theorem 5, marries recent work by Simons and Zalinescu [16] with additional convex analysis to provide an accessible short proof of the maximality of the sum of two maximal monotone operators.

Recall that the *domain* of an extended valued convex function, denoted dom (f), is the set of points with value less than $+\infty$, and that a point s is in the *core* of a set S (denoted by $s \in \operatorname{core} S$) provided that s lies in S and $X = \bigcup_{\lambda>0} \lambda(S-s)$. For a concave function g, we use $\operatorname{dom} g = \operatorname{dom}(-g)$. Recall that $x^* \in X^*$ is a *subgradient* of $f : X \to (-\infty, +\infty]$ at $x \in \operatorname{dom} f$ provided that $f(y) - f(x) \geq \langle x^*, y-x \rangle$. The set of all subgradients of f at x is called the *subderivative* or *subdifferential* of f at x and is denoted $\partial f(x)$. We use the convention that $\partial f(x) = \emptyset$ for $x \notin \operatorname{dom} f$. We shall need the *indicator* function $\iota_C(x)$ which is zero for x in C and $+\infty$ otherwise, the *Fenchel conjugate* $f^*(x^*) := \sup_x\{\langle x, x^* \rangle - f(x)\}$ and the *infimal convolution* $f^* \odot \frac{1}{2}\|\cdot\|_*^2(x^*) := \inf\{f^*(y^*) + \frac{1}{2}\|z^*\|_*^2 \colon x^* = y^* + z^*\}$. When f is convex and closed and x is in the domain of f, $x^* \in \partial f(x)$ exactly when $f(x) + f^*(x^*) = \langle x, x^* \rangle$.

We say a multifunction $F : X \mapsto 2^{X^*}$ is *monotone* provided that for any $x, y \in X$, $x^* \in F(x)$ and $y^* \in F(y)$,

$$\langle y^* - x^*, y - x \rangle \geq 0,$$

and we say that T is *maximal monotone* if its graph is not properly included in any other monotone graph. The subdifferential of a convex lower semicontinuous (lsc) function on a Banach space is a typical example of a maximal monotone multifunction (see [4, 6, 13] wherein other notation and usage may also be followed upon). Indeed we reserve the notation J for the duality map

$$J(x) := \frac{1}{2}\,\partial\|x\|^2 = \{x^* \in X^* \colon \|x\|^2 = \|x^*\|^2 = \langle x, x^* \rangle\}.$$

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Further applications and more extended discussion of the techniques in this note can be found in [1].

Proposition 1 ([4, 6, 13]). *For a closed convex function F, let $f_J := f + \frac{1}{2}\|\cdot\|^2$. Then $f_J^* = \left(f + \frac{1}{2}\|\cdot\|^2\right)^* = f^* \odot \frac{1}{2}\|\cdot\|_*^2$ is everywhere continuous. Also,*

$$v^* \in \partial f(v) + J(v) \Leftrightarrow f_J^*(v^*) + f_J(v) - \langle v, v^* \rangle \leq 0.$$

For any monotone mapping T, we associate the *Fitzpatrick function* introduced by Simon Fitzpatrick in [8] but then neglected for many years until re-popularized in papers by Penot [10], Buracik-Svaiter [7], and others. Some more history may be found in [1]. Fitzpatrick's function is

$$F_T(x, x^*) := \sup\{\langle x, y^* \rangle + \langle x^*, y \rangle - \langle y, y^* \rangle : y^* \in T(y), y \in \operatorname{dom} T\},$$

which is clearly lower semicontinuous and convex as an affine supremum. Moreover,

Proposition 2 ([8, 6]). *For a maximal monotone operator T*

$$F_T(x, x^*) \geq \langle x, x^* \rangle,$$

with equality if and only if $x^* \in T(x)$.

We recall the version of the Hahn-Banach theorem we need:

Theorem 3 (Hahn-Banach Sandwich, [4, 6, 13]). *Suppose f and $-g$ are proper extended real-valued lsc convex functions on a Banach space X and that $f(x) \geq g(x)$, for all x in X. Assume that*

$$0 \in \operatorname{core}\left(\operatorname{dom}(f) - \operatorname{dom}(g)\right).$$

Then there is a continuous linear function λ such that

$$f(x) - g(y) \geq \langle \lambda, x - y \rangle,$$

for all $x \in \operatorname{dom} f, y \in \operatorname{dom} -g$ in X.

Proof. The value function $h(u) := \inf_X f(x) - g(x - u)$ is convex. It is continuous at 0—indeed the constraint qualification and semi-continuity of the data force h to be bounded above around zero—by a Baire category type argument [17, 14, 6]. Hence there is some $-\lambda \in \partial h(0)$. This provides the linear part of the asserted affine separator. Indeed, we have

$$f(x) - g(u - x) \geq h(u) - h(0) \geq \lambda(u - 0),$$

as required. □

The next result, implicit in the literature [14], avoids needing to renorm a reflexive space to have a single-valued duality map with a single-valued inverse.

Proposition 4 ([14, 11, 12, 6]). *A monotone multifunction T is maximal if and only if the mapping $T(\cdot + w) + J$ is surjective for all w in X.* [*When J and J^{-1} are both single valued, a monotone mapping T is maximal if and only if $T + J$ is surjective.*]

Proof. We prove the "if" part. The "only if" part is completed in Corollary 7. Assume (w, w^*) is monotonically related to the graph of T. By hypothesis, we may solve $w^* \in T(x + w) + J(x)$. Thus $w^* = t^* + j^*$, where $t^* \in T(x + w), j^* \in J(x)$.

$$0 \leq \langle w^* - t^*, w - (w + x) \rangle = -\langle w^* - t^*, x \rangle = -\langle j^*, x \rangle = -\|x\|^2 \leq 0.$$

Hence $x = 0, j^* = 0$, and we are done.

The refined equivalence when J, J^{-1} are single-valued may be found in [14, Thm. 10.6]. $\square$

2. The main result

We now prove our asserted result whose proof—originally very hard and due to Rockafellar [12]—has been revisited over many years culminating in [14, 7, 10, 16, 6], among others. The proof we give is perhaps the first to avoid using either renorming [12] or some preliminary minimax arguments [14]:

Theorem 5. *Let X be any reflexive space with given norm. Let T be maximal monotone and f closed and convex. Suppose that*

$$0 \in \operatorname{core}\{\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(\partial f)\}.$$

Then

(a) $\partial f + T + J$ *is surjective;*
(b) $\partial f + T$ *is maximal monotone;*
(c) ∂f *is maximal monotone.*

Proof. (a) As in [16], we consider the Fitzpatrick function $F_T(x, x^*)$ and further introduce $f_J(x) := f(x) + 1/2\|x\|^2$. Let $G(x, x^*) := -f_J(x) - f_J^*(-x^*)$. Observe that

$$F_T(x, x^*) \geq \langle x, x^* \rangle \geq G(x, x^*)$$

pointwise thanks to Proposition 2, and the *Fenchel-Young inequality*: for any function $f(x) + f^*(x^*) \geq \langle x, x^* \rangle, \forall x, x^*$. Now, the (CQ)

$$0 \in \operatorname{core}\{\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(\partial f)\}$$

assures that the *Sandwich theorem* applies. Indeed, by Proposition 1, f_J^* is everywhere finite and in consequence zero is in the core of $\operatorname{dom} F_T - \operatorname{dom} G$.

Then there are $w \in X$ and $w^* \in X^*$ such that

$$F_T(x, x^*) - G(z, z^*) \geq w(x^* - z^*) + w^*(x - z) \tag{1}$$

so that for all $x^* \in T(x), x \in \operatorname{dom}(T)$ and for all z^*, z we have

$$\langle x^* - w^*, x - w \rangle + [f_J(z) + f_J^*(-z^*) + \langle z, z^* \rangle] \geq \langle w^* - z^*, w - z \rangle.$$

Now use the fact that $-w^* \in \operatorname{dom}(\partial f_J^*)$, by Proposition 1, to deduce that for some v, $-w^* \in \partial f_J(v)$ and so

$$\langle x^* - w^*, x - w \rangle + [f_J(v) + f_J^*(-w^*) + \langle v, w^* \rangle] \geq \langle w^* - w^*, w - v \rangle = 0.$$

The second term on the left is zero, and so $w^* \in T(w)$ by maximality. Substitution of $x = w$ and $x^* = w^*$ in (1), and rearranging yields

$$\langle w^*, w \rangle + \{\langle -z^*, w \rangle - f_J^*(-z^*)\} + \{\langle z, -w^* \rangle - f_J(z)\} \leq 0,$$

for all z, z^*. Taking the supremum over z and z^* produces $\langle w^*, w \rangle + f_J(w) + f_J^*(-w^*) \leq 0$. This shows $-w^* \in \partial f_J(w) = \partial f(w) + J(w)$ on using the sum formula for subgradients, implicit in Proposition 1.

Thus, $0 \in (T + \partial f_J)(w)$ and, since all range translations of $T + \partial f$ may be used, $\partial f + T + J$ is surjective, which completes (a). Additionally, since all domain translations may be used, $\partial f + T$ is maximal by the easy part of Proposition 4, which yields (b).

Finally, setting $T \equiv 0$ we recover the reflexive case of the maximality for a lsc convex function ∂f which is (c). □

Note that we have exploited the beautiful inequality

$$F_T(x, x^*) + f(x) + f^*(-x^*) \geq 0 \qquad \forall x \in X, x^* \in X^*, \tag{2}$$

valid for *any* maximal monotone T and *any* convex function f.

3. Some corollaries

We first recover the so-called *Brezis-Attouche theorem*:

Corollary 6 ([14]). *The sum of two maximal monotone operators T_1 and T_2 is maximal monotone if* $0 \in \operatorname{core}[\operatorname{conv}\operatorname{dom}(T_1) - \operatorname{conv}\operatorname{dom}(T_2)]$.

Proof. Theorem 5 applies to the maximal monotone mapping $T(z) := (T_1(x), T_2(y))$ and the indicator function $f(x, y) = \iota_{\{x=y\}}$. Finally, check that the given transversality condition implies the needed (CQ). We obtain that $T + J_{X\otimes X} + \partial\iota_{\{x=y\}}$ is surjective. Thus, so is $T_1 + T_2 + 2J$, and we are done. □

We next recover the *Rockafellar-Minty surjectivity theorem*:

Corollary 7. *A maximal monotone on a reflexive space has* $\operatorname{range}(T + J) = X^*$.

Proof. Let $f \equiv 0$ in Theorem 5. □

Recall that T is *coercive on C* if $\inf_{y^*\in T(y)}\langle y, y^*\rangle/\|y\| \to \infty$ as $y \in C$ tends in norm to infinity, and using the convention that $\inf \emptyset = +\infty$.

A *variational inequality* requests a solution $y \in C$ and $y^* \in T(y)$ to

$$\langle y^*, x - y\rangle \geq 0 \qquad \forall x \in C.$$

We denote the variational inequality by $V(T; C)$.

Corollary 8. *Suppose T is maximal monotone on a reflexive Banach space and is coercive on the closed convex set C. Suppose also that* $0 \in \operatorname{core}(C - \operatorname{conv}\operatorname{dom}(T))$. *Then $V(T, C)$ has a solution.*

Proof. Without loss, we assume 0 lies in C. Let $f := \iota_C$, the indicator function. For $n = 1, 2, 3\cdots$, let $T_n := T + J/n$. We solve

$$0 \in (T_n + \partial\iota_C)(y_n) = \left((T + \partial\iota_C) + \frac{1}{n}J\right)(y_n)$$

and take limits as n goes to infinity. More precisely, we observe that using Theorem 5 we find y_n in C, and $y_n^* \in T(y_n), j_n^* \in J(y_n)/n$ with

$$\langle y_n^*, x - y_n\rangle \geq -\langle j_n^*, x - y_n\rangle \qquad \forall x \in C.$$

Then coercivity (of $T + \partial\iota_C$) implies that $\|y_n\|$ remains bounded and so $j_n^* \to 0$. On taking a subsequence we may assume $y_n \rightharpoonup y, y_n^* \to y^*$. Since T is demi-closed [6], it follows that $y^* \in T(y)$ and that

$$\langle y^*, x - y\rangle \geq 0 \qquad \forall x \in C.$$

□

Letting $C = X$ we deduce:

Corollary 9. *Every coercive maximal monotone multifunction on a Banach space is surjective if (and only if) the space is reflexive.*

Proof. To complete the proof we recall that, by James' theorem, surjectivity of J is equivalent to reflexivity of the corresponding space. □

Details of these and other corollaries are to be found in [1].

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