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# COMMUTANTS OF CERTAIN ANALYTIC OPERATOR ALGEBRAS

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Abstract. We prove that algebraic commutants of maximal subdiagonal algebras and of analytic operator algebras determined by flows in a $\sigma$-finite von Neumann algebra are self-adjoint.

## 1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For a subset $E$ of $\mathcal{B}(\mathcal{H})$, we denote by $E'$ the algebraic commutant, that is,

$$E' = \{X \in \mathcal{B}(\mathcal{H}) : AX = XA,\ \forall A \in E\}.$$

If $T \in \mathcal{B}(\mathcal{H})$, we call $\{T\}'$ the algebraic commutant of $T$. The well-known theorem of Fuglede states that if $N$ is normal and $X$ commutes with $N$, so does $X^*$. That is, the algebraic commutant $\{N\}'$ of $N$ is self-adjoint. Note that $\{N\}'$ is the same as the commutant of the algebra generated by $N$ and $I$, which is non-self-adjoint in general. Thus it may be asked which subalgebras have a self-adjoint commutant. For example, if all elements in a subalgebra are normal or the algebra itself is self-adjoint, then its algebraic commutant is self-adjoint. In general, this problem is not particularly interesting. However special cases of this problem are interesting. F. Gilfeather and D.R. Larson in [6] showed that the algebraic commutant of a nest subalgebra of a von Neumann algebra is self-adjoint. We note that a nest subalgebra of a von Neumann algebra is a kind of analytic operator algebra. Thus it is interesting to consider this problem for general analytic operator algebras.

In [2], W. Arveson introduced the notion of subdiagonal algebras to give a unified theory of non-self-adjoint operator algebras, including the algebra of bounded analytic matrix-valued (or more generally, operator-valued) functions and nest subalgebras of von Neumann algebras.

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Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra acting on $\mathcal{H}$. We denote by $\mathcal{M}_*$ the space of all $\sigma$-weakly continuous linear functionals of $\mathcal{M}$. For a von Neumann subalgebra $\mathfrak{D}$ of $\mathcal{M}$, let $\Phi$ be a faithful normal conditional expectation from $\mathcal{M}$ onto $\mathfrak{D}$. A subalgebra $\mathfrak{A}$ of $\mathcal{M}$, containing $\mathfrak{D}$, is called a subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ if

(i) $\mathfrak{A} \cap \mathfrak{A}^* = \mathfrak{D}$,

(ii) $\Phi$ is multiplicative on $\mathfrak{A}$, and

(iii) $\mathfrak{A} + \mathfrak{A}^*$ is $\sigma$-weakly dense in $\mathcal{M}$.

The algebra $\mathfrak{D}$ is called the diagonal of $\mathfrak{A}$. Although subdiagonal algebras are not assumed to be $\sigma$-weakly closed in [2], the $\sigma$-weak closure of a subdiagonal algebra is again a subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ (Remark 2.1.2 in [2]). Thus we assume that our subdiagonal algebras are always $\sigma$-weakly closed.

We say that $\mathfrak{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to $\Phi$ in case that $\mathfrak{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to $\Phi$. Put $\mathfrak{A}_0 = \{X \in \mathfrak{A} : \Phi(X) = 0\}$ and $\mathfrak{A}_m = \{X \in \mathcal{M} : \Phi(AXB) = \Phi(BXA) = 0,\ \forall A \in \mathfrak{A},\ B \in \mathfrak{A}_0\}$. By Theorem 2.2.1 in [2], we recall that $\mathfrak{A}_m$ is a maximal subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ containing $\mathfrak{A}$. If there is a faithful normal finite trace $\tau$ on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$, we say that $\mathfrak{A}$ is finite subdiagonal.

On the other hand, let $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ be a flow of $\mathbb{R}$ on $\mathcal{M}$, i.e. $\{\alpha_t\}_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms of $\mathcal{M}$ such that, for each $X \in \mathcal{M}$, $t \to \alpha_t(X)$ is $\sigma$-weakly continuous. Write $H^\infty(\alpha) = \{X \in \mathcal{M} : sp_\alpha(X) \subseteq [0, \infty)\}$, where $sp_\alpha(\cdot)$ is an Arveson spectrum (Section 3). Then $H^\infty(\alpha)$ is a $\sigma$-weakly closed subalgebra of $\mathcal{M}$ satisfying that $H^\infty(\alpha) + (H^\infty(\alpha))^*$ is $\sigma$-weakly dense in $\mathcal{M}$. The structure of $H^\infty(\alpha)$ was studied by several authors (cf. [3, 13, 14, 16]). It is known that if there is a faithful normal conditional expectation from $\mathcal{M}$ onto $H^\infty(\alpha) \cap (H^\infty(\alpha))^*$, then $H^\infty(\alpha)$ is a maximal subdiagonal algebra of $\mathcal{M}$. Moreover, if $\mathcal{A}$ is a nest subalgebra of $\mathcal{M}$, then there is an inner flow $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$, that is, $\alpha$ is implemented by a continuous unitary group $\{U_t : t \in \mathbb{R}\} \subset \mathcal{M}$, such that $\mathcal{A} = H^\infty(\alpha)$ (cf. Theorem 4.2.3 in [14]).

In this note we prove that algebraic commutants of maximal subdiagonal algebras and of analytic operator algebras are self-adjoint.

## 2. The commutant of a maximal subdiagonal algebra

We consider the algebraic commutant of a maximal subdiagonal algebra $\mathfrak{A}$ with respect to $\Phi$. The following result was proved in [9]. For completeness, we give the proof here also.

**Lemma 1.** *Let $\mathfrak{A}$ be a finite subdiagonal algebra with respect to $\Phi$ of $\mathcal{M}$. Then $\mathfrak{A}' = \mathcal{M}'$.*

*Proof.* It is trivial that $\mathfrak{A}' \supseteq \mathcal{M}'$. Now let $X \in \mathfrak{A}'$ and $T \in \mathcal{M}$. Then for any $\epsilon > 0$, we have that $T^*T + \epsilon I$ is a positive invertible operator in $\mathcal{M}$. Note that $\mathfrak{A}$ is maximal subdiagonal (cf. [5]). By Theorem 4.2.1 in [2], there is an invertible operator $A$ in $\mathfrak{A}$ so that $T^*T + \epsilon I = A^*A$. Then

$$X^*(T^*T + \epsilon I)X = X^*A^*AX = A^*X^*XA \leq \|X\|^2 A^*A = \|X\|^2(T^*T + \epsilon I).$$

It follows that $X^*T^*TX \leq \|X\|^2 T^*T$ by letting $\epsilon \to 0$. In particular, $X^*EX \leq \|X\|^2 E$ for every positive projection $E$ in $\mathcal{M}$. It follows that $(I-E)X^*EX(I-E) \leq$

0, which implies that $EX(I-E)=0$. Thus $EX=XE$ for every projection $E\in\mathcal{M}$, which implies that $X\in\mathcal{M}'$. The proof is complete. $\square$

We next recall Haagerup's reduction theory [7]. Since $\mathcal{M}$ is $\sigma$-finite, there exists a faithful normal state $\varphi$ of $\mathcal{M}$ such that $\varphi\circ\Phi=\varphi$. Let $\sigma^\varphi=\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ be the modular automorphism group of $\mathcal{M}$ associated with $\varphi$. We know that $\mathfrak{A}$ is $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ invariant from Theorem 2.4 in [10]. Let $G$ be the discrete subgroup $\bigcup_{n\geq 1}2^{-n}\mathbb{Z}$ of $\mathbb{R}$. We consider the crossed product $\mathcal{M}\rtimes_{\sigma^\varphi}G$ with respect to $\sigma^\varphi$. Then we have that $\mathcal{M}\rtimes_{\sigma^\varphi}G$ is a von Neumann algebra on $\ell^2(G,\mathcal{H})$ generated by the operators $\pi(X)$, $X\in\mathcal{M}$, and $\lambda(s)$, $s\in G$, defined by the equations

$$(\pi(X)\xi)(t)=\sigma_{-t}^\varphi(X)\xi(t),\quad \xi\in\ell^2(G,\mathcal{H}),\ t\in G,$$

and

$$(\lambda(s)\xi)(t)=\xi(t-s),\quad \xi\in\ell^2(G,\mathcal{H}),\ t\in G.$$

Note that $\pi$ is a normal faithful representation of $\mathcal{M}$ on $\ell^2(G,\mathcal{H})$. Let $\hat{\varphi}$ be the dual weight of $\varphi$ on $\mathcal{M}\rtimes_{\sigma^\varphi}G$. Then $\hat{\varphi}$ is again a faithful normal state on $\mathcal{M}\rtimes_{\sigma^\varphi}G$. Haagerup's reduction theorem asserts that there is an increasing sequence $\{\mathcal{R}_n\}_{n\geq 1}$ of von Neumann subalgebras of $\mathcal{M}\rtimes_{\sigma^\varphi}G$ with the following properties:

(i) each $\mathcal{R}_n$ is finite;
(ii) $\bigcup_{n\geq 1}\mathcal{R}_n$ is $\sigma$-weakly dense in $\mathcal{M}\rtimes_{\sigma^\varphi}G$;
(iii) for each $n\geq 1$ there is a faithful normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M}\rtimes_{\sigma^\varphi}G$ onto $\mathcal{R}_n$ such that $\hat{\varphi}\circ\mathcal{E}_n=\hat{\varphi}$, $\quad\mathcal{E}_n\circ\mathcal{E}_{n+1}=\mathcal{E}_n$, $\quad n\geq 1$, and $\lim\limits_{n\to\infty}\|\psi\circ\mathcal{E}_n-\psi\|=0$ for all $\psi\in(\mathcal{M}\rtimes_{\sigma^\varphi}G)_*$.

We refer the readers to [7] and [17] for more details.

We now can extend $\Phi$ to a normal faithful conditional expectation $\hat{\Phi}$ from $\mathcal{M}\rtimes_{\sigma^\varphi}G$ onto $\mathfrak{D}\rtimes_{\sigma^\varphi}G$, which is naturally identified as a von Neumann subalgebra of $\mathcal{M}\rtimes_{\sigma^\varphi}G$.

Let $\hat{\mathfrak{A}}$ be the $\sigma$-weakly closed subalgebra generated by $\{\pi(X):X\in\mathfrak{A}\}$ and $\{\lambda(s):s\in G\}$. Since $\mathfrak{A}$ is $\sigma_t^\varphi|_{t\in\mathbb{R}}$ invariant by Theorem 2.4 in [10], $\hat{\mathfrak{A}}$ is the $\sigma$-weak closure of the set of all linear combinations of $\lambda(s)\pi(X)$, $s\in G$, $X\in\mathfrak{A}$. The following lemma was proved in [17].

**Lemma 2.** *$\hat{\mathfrak{A}}$ is a maximal subdiagonal algebra with respect to $\hat{\Phi}$.*

Let $\mathfrak{A}_n=\mathcal{R}_n\cap\hat{\mathfrak{A}}=\mathcal{E}_n(\hat{\mathfrak{A}})$. We have the following lemma (Lemma 2 in [17]).

**Lemma 3.** *$\mathfrak{A}_n$ is a finite subdiagonal algebra in $\mathcal{R}_n$ with respect to $\hat{\Phi}|_{\mathcal{R}_n}$ and $\bigcup_{n\geq 1}\mathfrak{A}_n$ is $\sigma$-weakly dense in $\hat{\mathfrak{A}}$.*

We now have the main theorem in this section.

**Theorem 1.** *Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra and let $\mathfrak{A}$ be a maximal subdiagonal algebra with respect to $\Phi$ of $\mathcal{M}$. Then the commutant $\mathfrak{A}'$ of $\mathfrak{A}$ is self-adjoint, that is, $\mathfrak{A}'=\mathcal{M}'$.*

*Proof.* We first claim that $\hat{\mathfrak{A}}'=(\mathcal{M}\rtimes_{\sigma^\varphi}G)'$. In fact, let $X\in\hat{\mathfrak{A}}'$. Then $X\in\mathfrak{A}_n'$ for all $n\in\mathbb{N}$. By Lemmas 1 and 3, we have $X^*\in\mathfrak{A}_n'$. Note that for every $Y\in\hat{\mathfrak{A}}$, we have $\mathcal{E}_n(Y)\in\mathfrak{A}_n$ for all $n\in\mathbb{N}$ and $Y=\lim\limits_n\mathcal{E}_n(Y)$ $\sigma$-weakly from Haagerup's theory. Then it follows that $X^*Y=YX^*$, which implies that $X^*\in\hat{\mathfrak{A}}'$.

Now let $X \in \mathfrak{A}'$. We define an operator $\hat{X}$ on $\ell^2(G, \mathcal{H})$ by

$$(\hat{X}\xi)(s) = X\xi(s),\ s \in G, \xi \in \ell^2(G, \mathcal{H}).$$

Then we have that $\hat{X} \in \hat{\mathfrak{A}}'$. In fact, for $\xi \in \ell^2(G, \mathcal{H})$, $t,\ s \in \mathbb{R}$ and $Y \in \mathfrak{A}$,

$$\begin{aligned}(\hat{X}\pi(Y)\xi)(t) &= X(\pi(Y)\xi)(t) = X\sigma^{\varphi}_{-t}(Y)\xi(t)\\ &= \sigma^{\varphi}_{-t}(Y)X\xi(t) = \sigma^{\varphi}_{-t}(Y)((\hat{X}\xi)(t))\\ &= (\pi(Y)\hat{X}\xi)(t)\end{aligned}$$

and

$$\begin{aligned}(\hat{X}\lambda_s\xi)(t) &= X((\lambda_s\xi)(t)) = X\xi(t-s)\\ &= (\hat{X}\xi)(t-s) = (\lambda_s(\hat{X}\xi))(t)\\ &= (\lambda_s\hat{X}\xi)(t).\end{aligned}$$

It follows that $(\hat{X})^* \in \hat{\mathfrak{A}}'$, which implies that $(\hat{X})^*\pi(Y) = \pi(Y)(\hat{X})^*$ for all $Y \in \mathfrak{A}$. Note that $(\hat{X})^* = (X^*)\hat{}$. Then $(X^*)\hat{}\pi(Y) = \pi(Y)(X^*)\hat{}$ for all $Y \in \mathfrak{A}$. In particular, $X^*Y = YX^*$. Thus we have $X^* \in \mathfrak{A}'$. Note that $\mathfrak{A} + \mathfrak{A}^*$ is $\sigma$-weakly dense in $\mathcal{M}$. It then follows that $X \in \mathcal{M}'$. Hence $\mathfrak{A}' = \mathcal{M}'$. The proof is complete. □

## 3. The commutant of an analytic operator algebra

In this section we consider the algebraic commutant of an analytic operator algebra determined by a flow on $\mathcal{M}$. We need Arveson's theory of spectral subspaces and, so we recall the definitions here. Let $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ be a flow on $\mathcal{M}$, i.e. a $\sigma$-weakly continuous one parameter group of $*$-automorphisms of $\mathcal{M}$. For each element $X \in \mathcal{M}$ and a function $f \in L^1(\mathbb{R})$, we define the convolution $f \underset{\alpha}{*} X$ by

$$f \underset{\alpha}{*} X = \int_{-\infty}^{+\infty} f(t)\alpha_t(X)dt.$$

For $f \in L^1(\mathbb{R})$, let $Z(f) = \{t \in \mathbb{R} : \hat{f}(t) = 0\}$, where $\hat{f}(t) = \int_{-\infty}^{+\infty} e^{-ist}f(s)ds$ is the Fourier transform of $f$. For $X \in \mathcal{M}$, we define the Arveson spectrum of $X$ with respect to the flow $\alpha$ to be the set

$$\bigcap\{Z(f) : f \underset{\alpha}{*} X = 0\}$$

and denote it by $sp_\alpha(X)$. For any subset $S$ of $\mathbb{R}$ we define the spectral subspace $M^\alpha(S)$ to be the $\sigma$-weak closure of the set $\{X \in \mathcal{M} : sp_\alpha(X) \subset S\}$. We refer the readers to [3, 4, 13, 14] for the elementary properties of spectra and spectral subspaces. Put $H^\infty(\alpha) = M^\alpha([0, \infty))$ and $H^\infty_0(\alpha) = M^\alpha((0, \infty))$. It is known that $H^\infty_0(\alpha)$ is a two-sided ideal of $H^\infty(\alpha)$. Let $\mathfrak{D} = H^\infty(\alpha) \cap (H^\infty(\alpha))^*$ be the fixed point subalgebra of $\alpha$. We recall that $\mathcal{M}$ is said to be $\mathbb{R}$-finite relative to $\alpha$ if there is a separating family of $\alpha$-invariant normal states on $\mathcal{M}$. At the opposite extreme, we say that $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$ in case there are no invariant normal states.

**Lemma 4.** *If $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$, then $H^\infty(\alpha) = H^\infty_0(\alpha)$.*

*Proof.* Since $\mathcal{M}$ is $\sigma$-finite, without loss of generality by choosing an appropriate representation for $\mathcal{M}$, we may assume that $\mathcal{M}$ has a cyclic and separating vector in $\mathcal{H}$.

If $H^\infty(\alpha) \neq H_0^\infty(\alpha)$, then there is an element $f \in \mathcal{M}_*$ such that $f(A) = 0$ for all $A \in H_0^\infty(\alpha)$ and $f(T) \neq 0$ for some $T \in H^\infty(\alpha)$. Since $\mathcal{M}$ has a separating vector in $\mathcal{H}$, there are vectors $x, y \in \mathcal{H}$ such that $f(A) = (Ax, y)$ for all $A \in \mathcal{M}$ by Proposition 7.4.5 and Corollary 7.3.3 in [12]. Let $\mathfrak{M} = [Ax : A \in H^\infty(\alpha)]$ (resp. $\mathfrak{M}_0 = [Ax : A \in H_0^\infty(\alpha)]$) be the closed subspace generated by $\{Ax : A \in H^\infty(\alpha)\}$ (resp. $\{Ax : A \in H_0^\infty(\alpha)\}$) of $\mathcal{H}$. Then both $\mathfrak{M}$ and $\mathfrak{M}_0$ are invariant subspaces for $H^\infty(\alpha)$. We have $\mathfrak{M}_0 \subsetneqq \mathfrak{M}$ since $(Tx, y) \neq 0$ for some $T \in H^\infty(\alpha)$ and $(Tx, y) = 0$ for all $T \in H_0^\infty(\alpha)$. It is trivial that $H_0^\infty(\alpha)\mathfrak{M} \subseteq \mathfrak{M}_0$ since $H_0^\infty(\alpha)$ is a two-sided ideal of $H^\infty(\alpha)$. On the other hand, since $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$, by Corollary 5.7 in [14], $\mathfrak{M}$ is completely normalized in the sense of Definition 5.1 in [14], that is,

$$\mathfrak{M} = \bigwedge_{s<0} [M^\alpha([s, \infty))\mathfrak{M}] = \bigvee_{s>0} [M^\alpha([s, \infty))\mathfrak{M}].$$

Note that $M^\alpha([s, \infty)) \subseteq H_0^\infty(\alpha)$ for all $s > 0$. We then have that $[M^\alpha([s, \infty))\mathfrak{M}] \subseteq \mathfrak{M}_0$ for all $s > 0$ which implies that $\mathfrak{M} \subseteq \mathfrak{M}_0$. This is a contradiction. Hence $H^\infty(\alpha) = H_0^\infty(\alpha)$. The proof is complete. □

We recall that the crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}$ determined by $\mathcal{M}$ and $\alpha$ is the von Neumann algebra on the Hilbert space $L^2(\mathbb{R}, \mathcal{H})$ generated by the operators $\pi(X)$, $X \in \mathcal{M}$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by the equations

$$(\pi(X)f)(t) = \alpha_{-t}(X)f(t), \quad f \in L^2(\mathbb{R}, \mathcal{H}),\ t \in \mathbb{R},$$

and

$$(\lambda(s)f)(t) = f(t - s), \quad f \in L^2(\mathbb{R}, \mathcal{H}),\ t \in \mathbb{R}.$$

It is clear that $\pi(\alpha_t(X)) = \lambda(t)\pi(X)\lambda(t)^*$ for all $X \in \mathcal{M}$ and $t \in \mathbb{R}$. For any $Y \in \mathcal{M} \rtimes_\alpha \mathbb{R}$, we define $\beta_t(Y) = \lambda(t)Y\lambda(t)^*$, $\forall t \in \mathbb{R}$. Then $\beta = \{\beta_t\}_{t\in\mathbb{R}}$ is an inner flow on $\mathcal{M} \rtimes_\alpha \mathbb{R}$. We know that $H^\infty(\beta)$ is a nest subalgebra in $\mathcal{M} \rtimes_\alpha \mathbb{R}$ by Theorem 4.2.3 in [14].

Let $\mathcal{A}$ be the $\sigma$-weakly closed subalgebra of $\mathcal{M} \rtimes_\alpha \mathbb{R}$ generated by $\{\pi(X) :\ X \in H^\infty(\alpha)\}$ and $\{\lambda(t) : t \in \mathbb{R}\}$. Since $\pi(\alpha_t(X)) = \beta_t(\pi(X))$ for $X \in \mathcal{M}$ and $t \in \mathbb{R}$, $\mathcal{A}$ is a subalgebra of $H^\infty(\beta)$. It is noted that $\mathcal{A} + \mathcal{A}^*$ is $\sigma$-weakly dense in $\mathcal{M} \rtimes_\alpha \mathbb{R}$ since $H^\infty(\alpha) + H^\infty(\alpha)^*$ is $\sigma$-weakly dense in $\mathcal{M}$ by Theorem 3.15 in [14].

**Lemma 5.** *$H_0^\infty(\beta) \subset \mathcal{A}$.*

*Proof.* We know that $H_0^\infty(\beta)$ is the $\sigma$-weak closure of the set $\{X \in \mathcal{M} \rtimes_\alpha \mathbb{R} :$ $sp_\beta(X)$ is compact in $(0, +\infty)\}$ by Lemma 2.8 in [16]. Let $X \in \mathcal{M} \rtimes_\alpha \mathbb{R}$ be such that $sp_\beta(X)$ is compact in $(0, +\infty)$. Choose $f \in L^1(\mathbb{R})$ with compactly supported Fourier transform such that support $supp\hat{f}$ of $\hat{f}$ is in $(0, \infty)$ and such that $f \underset{\beta}{*} X = X$. Note that since $\mathcal{A} + \mathcal{A}^*$ is $\sigma$-weakly dense in $\mathcal{M} \rtimes_\alpha \mathbb{R}$, there are nets $\{A_i\}$, $\{B_i\}$ in $\mathcal{A}$ such that $\lim_i (A_i + B_i^*) = X$ $\sigma$-weakly. It follows that $\lim_i (f \underset{\beta}{*} A_i + f \underset{\beta}{*} B_i^*) = f \underset{\beta}{*} X = X$. However, we have $f \underset{\beta}{*} B_i^* = 0$ since $sp_\beta(f \underset{\beta}{*} B_i^*) \subset supp\hat{f} \cap sp_\beta(B_i^*) = \emptyset$. Note that $\mathcal{A}$ is $\{\beta_t\}_{t\in\mathbb{R}}$ invariant, it follows that $f \underset{\beta}{*} A_i \in \mathcal{A}$ and then $X \in \mathcal{A}$. The proof is complete. □

The next result might be known, but we were unable to find a reference for it.

**Lemma 6.** *If $\mathcal{M}$ is $\mathbb{R}$-finite relative to $\alpha$, then $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is $\mathbb{R}$-finite relative to $\beta$.*

*Proof.* By considering a covariant representation of the pair $(\mathcal{M}, \alpha)$ in the sense of Definition 2.5 and Proposition 2.6 in [14], we may assume that $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ is implemented by a continuous unitary representation $t \to U_t$ of $\mathbb{R}$ on $\mathcal{H}$, that is, $\alpha_t(X) = U_t X U_t^*$ for all $x \in \mathcal{M}$. Then by Definition 13.2.6 in [12], the crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is a von Neumann algebra on $L^2(\mathcal{R}, \mathcal{H})$ $(= \mathcal{H} \otimes L^2(\mathbb{R}))$ generated by $\{A \otimes I : a \in \mathcal{M}\}$ and $\{U_t \otimes l_t : t \in \mathbb{R}\}$, where $l_t$ is the shift operator on $L^2(\mathbb{R})$ defined by $(l_t f)(s) = f(s-t)$, $t, s \in \mathbb{R}$, $f \in L^2(\mathbb{R})$. It is known that $\beta$ is implemented by $\{U_t \otimes l_t\}_{t\in\mathbb{R}}$.

Since $L^2(\mathbb{R})$ is separable, there is an orthonormal basis $\{e_n : n = 1, 2, \cdots\}$ of $L^2(\mathbb{R})$. If we define $\mu(T) = \sum_{n=1}^{\infty} \frac{1}{2^n}(Te_n, e_n)$, $\forall T \in \mathcal{B}(L^2(\mathbb{R}))$, then $\mu$ is a faithful normal state of $\mathcal{B}(L^2(\mathbb{R}))$. Let $\varphi$ be a faithful normal state of $\mathcal{M}$ such that $\varphi \circ \alpha_t = \varphi$ for all $t \in \mathbb{R}$. It follows from Proposition 11.2.7 in [12] that $\varphi \overline{\otimes} \mu$ is a normal state on $\mathcal{M} \overline{\otimes} \mathcal{B}(L^2(\mathbb{R})) \supset \mathcal{M} \rtimes_\alpha \mathbb{R}$. We claim that $\varphi \overline{\otimes} \mu$ is faithful. In fact, we way identify $\mathcal{M} \overline{\otimes} \mathcal{B}(L^2(\mathbb{R}))$ with$\{(A_{ij}) \in \mathcal{B}(\mathcal{H} \otimes L^2(\mathbb{R})) : A_{ij} \in \mathcal{M}\}$ by Remark 11.2.3 in [12]. For any $(A_{ij}) \in \mathcal{M} \overline{\otimes} \mathcal{B}(L^2(\mathbb{R}))$, we define $\psi((A_{ij})) = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi(A_{nn})$. Then $\psi$ is a faithful normal state on $\mathcal{M} \overline{\otimes} \mathcal{B}(L^2(\mathbb{R}))$. Since for any $A \in \mathcal{M}$ and $T \in \mathcal{B}(L^2(\mathbb{R}))$, $A \otimes T$ identifies with $(t_{ij} A)$, where $t_{ij} = (Te_j, e_i)$ for all $i, j = 1, 2, \cdots$, $\varphi \overline{\otimes} \mu(A \otimes T) = \varphi(A)\mu(T) = \psi((t_{ij} A))$. It follows that $\varphi \overline{\otimes} \mu = \psi$ and then $\varphi \overline{\otimes} \mu$ is faithful.

We note that if $A \in \mathcal{M}$, then $\varphi \overline{\otimes} \mu(\beta_s(A \otimes I)) = \varphi \overline{\otimes} \mu(A \otimes I)$ for all $s \in \mathbb{R}$. On the other hand, $\beta_t(U_s \otimes l_s) = U_s \otimes l_s$; then $\varphi \overline{\otimes} \mu(\beta_t(U_s \otimes l_s)) = \varphi \overline{\otimes} \mu(U_s \otimes l_s)$, $s, t \in \mathbb{R}$. We now have that $\varphi \overline{\otimes} \mu$ is a faithful normal state on $\mathcal{M} \rtimes_\alpha \mathbb{R}$ such that $\varphi \overline{\otimes} \mu \circ \beta_s = \varphi \overline{\otimes} \mu$ for all $s \in \mathbb{R}$. It follows that $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is $\mathbb{R}$-finite relative to $\beta$. The proof is complete. □

**Lemma 7.** $H^\infty(\beta) = \mathcal{A}$.

*Proof.* We first assume that $\mathcal{M}$ is $\mathbb{R}$-finite relative to $\{\alpha_t\}_{t\in\mathbb{R}}$. Then there is a faithful normal expectation $\Phi$ from $\mathcal{M}$ onto $\mathcal{D}$. Now there is a faithful normal expectation $\Psi$ from $\mathcal{M} \rtimes_\alpha \mathbb{R}$ onto $\mathcal{N}$ such that $\Psi(\pi(A)) = \pi(\Phi(A))$ for all $A \in \mathcal{M}$ by Lemma 6, where $\mathcal{N}$ is the fixed point algebra of $\beta$. We have that $\mathcal{N}$ is generated by $\{\pi(A) : A \in \mathfrak{D}\}$ and $\{\lambda(t) : t \in \mathbb{R}\}$. In fact, take any $D \in \mathcal{N}$. There is a net $A_i = \sum_{j=1}^{n_i} \pi(X_j^i)\lambda(t_j^i)$ such that $\lim_i A_i = D$ $\sigma$-weakly, where $X_j^i \in \mathcal{M}$ and $t_j^i \in \mathbb{R}$. We then have

$$\begin{aligned} D = \Psi(D) = \lim_i \Psi(A_i) &= \lim_i \sum_{j=1}^{n_i} \Psi(\pi(X_j^i))\lambda(t_j^i) \\ &= \lim_i \sum_{j=1}^{n_i} \pi(\Phi(X_j^i))\lambda(t_j^i). \end{aligned}$$

It now follows that $\mathcal{N} = H^\infty(\beta) \cap H^\infty(\beta)^* = \mathcal{A} \cap \mathcal{A}^*$. We know that $H_0^\infty(\beta) \subset \mathcal{A}$ by Lemma 5. Thus $H^\infty(\beta) = \mathcal{N} + H_0^\infty(\beta) \subset \mathcal{A}$ and therefore $H^\infty(\beta) = \mathcal{A}$.

For the general case, there is a projection $E$ in the center of $\mathfrak{D}$ such that $E\mathcal{M}E$ is $\mathbb{R}$-finite, while $(I-E)\mathcal{M}(I-E)$ is completely non-$\mathbb{R}$-finite relative to $\alpha$ from Remark

3.4 in [14]. We note that in this case $\pi(E)(\mathcal{M}\rtimes_\alpha\mathbb{R})\pi(E)$ is $\mathbb{R}$-finite by Lemma 6, and $(I-\pi(E)(\mathcal{M}\rtimes_\alpha\mathbb{R})(I-\pi(E))$ is completely non-$\mathbb{R}$-finite relative to $\beta$. By considering $\alpha$ restricted on $E\mathcal{M}E$, we have $\pi(E)H^\infty(\beta)\pi(E)\subset\pi(E)\mathcal{A}\pi(E)\subset\mathcal{A}$ by Lemma 6 again. On the other hand, if we consider $\alpha$ on $(I-E)\mathcal{M}(I-E)$, then we have $(I-\pi(E))H^\infty(\beta)(I-\pi(E))=(I-\pi(E))H_0^\infty(\beta)(I-\pi(E))\subset H_0^\infty(\beta)$. In particular, $I-\pi(E)\in H_0^\infty(\beta)$ and then $(I-\pi(E))H^\infty(\beta)+H^\infty(\beta)(I-\pi(E))\subset H_0^\infty(\beta)\subset\mathcal{A}$. Thus we have $H^\infty(\beta)\subset\mathcal{A}$, and the proof is complete. □

**Theorem 2.** *The commutant of $H^\infty(\alpha)$ is self-adjoint, that is, $(H^\infty(\alpha))'=\mathcal{M}'$.*

*Proof.* Let $\beta=\{\beta_t\}_{t\in\mathbb{R}}$ be as above. We recall that $H^\infty(\beta)$ is a nest subalgebra of $\mathcal{M}\rtimes_\alpha\mathbb{R}$. Then the commutant of $H^\infty(\beta)$ is self-adjoint by Theorem 2.5 in [6]. Let $X\in(H^\infty(\alpha))'$. Define an operator $\hat{X}$ on $L^2(\mathbb{R},\mathcal{H})$ by

$$(\hat{X}\xi)(t)=X\xi(t),\quad\forall\xi\in L^2(\mathbb{R},\mathcal{H}).$$

Then it is trivial that $\hat{X}$ is bounded. We claim that $\hat{X}\in(H^\infty(\beta))'$. By Lemma 7, it is sufficient to show that $\hat{X}\in\mathcal{A}'$. For any $Y\in H^\infty(\alpha)$ we have

$$\begin{aligned}(\hat{X}\pi(Y)\xi)(t)&=X\alpha_{-t}(Y)\xi(t)=\alpha_{-t}(Y)X\xi(t)\\&=\alpha_{-t}(Y)(\hat{X}\xi)(t)=(\pi(Y)\hat{X}\xi)(t).\end{aligned}$$

Then $\hat{X}\pi(Y)=\pi(Y)\hat{X}$. On the other hand, for any $s\in\mathbb{R}$,

$$\begin{aligned}(\hat{X}\lambda(s)\xi)(t)&=(\hat{X}\xi)(t-s)\\&=\lambda(s)(\hat{X}\xi)(t)=(\lambda(s)\hat{X})(t).\end{aligned}$$

It follows that $\lambda(s)\hat{X}=\hat{X}\lambda(s)$ for any $s\in\mathbb{R}$. We thus have $\hat{X}\in(H^\infty(\beta))'$. By Theorem 2.5 in [6], $(\hat{X})^*\in(H^\infty(\beta))'$. In particular, $(\hat{X})^*$ commutes with $\pi(Y)$ for any $Y\in H^\infty(\alpha)$. Note that $(\hat{X})^*=(X^*)\hat{}$. Given $u$ in $\mathcal{H}$, let $f$ be a continuous function in $L^2(\mathbb{R})$ such that $f(0)=1$ and let $\xi(t)=f(t)u$. Then $\xi\in L^2(\mathbb{R},\mathcal{H})$ and

$$\begin{aligned}((X^*)\hat{}\pi(Y)\xi)(t)&=X^*\alpha_{-t}(Y)\xi(t)=X^*\alpha_{-t}(Y)f(t)u\\&=(\pi(Y)(X^*)\hat{}\xi)(t)=\alpha_{-t}(Y)X^*f(t)u.\end{aligned}$$

When $t=0$, we have $X^*Yu=YX^*u$ for all $u\in\mathcal{H}$. Hence $X^*\in(H^\infty(\alpha))'$ and $(H^\infty(\alpha))'=\mathcal{M}'$. The proof is complete. □

*Remark* 1. We know that if $\mathcal{A}$ is either a subdiagonal algebra or an analytic operator algebra of $\mathcal{M}$, then we have that $\mathcal{A}+\mathcal{A}^*$ is $\sigma$-weakly dense in $\mathcal{M}$. However, if we assume that a subalgebra only satisfies this condition, it may not follow that the algebraic commutant is self-adjoint. For example, from Corollary 1.4 in [1], we know that there is a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A}+\mathcal{A}^*$ is $\sigma$-weakly dense in $\mathcal{B}(\mathcal{H})$ and such that $\mathcal{A}$ is similar to a proper von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. It easily follows that the algebraic commutant of $\mathcal{A}$ is not self-adjoint.

*Remark* 2. We considered two classes of non-self-adjoint operator algebras, subdiagonal algebras and analytic operator algebras determined by flows in von Neumann algebras. If $\mathcal{H}$ is finite dimensional, we know that these two classes of operator algebras are nest subalgebras of von Neumann algebras (cf. Theorem 2.1 in [11]). However, if $\mathcal{H}$ is infinite dimensional, these two classes are different. The analytic operator algebra $H^\infty(\alpha)$ determined by a flow $\alpha$ is a maximal subdiagonal algebra if and only if the flow $\alpha$ is $\mathbb{R}$-finite. There are subdiagonal algebras which are not

analytic operator algebras determined by any flows. We refer the readers to see some examples given in [8].

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