COMMUTANTS OF CERTAIN ANALYTIC OPERATOR ALGEBRAS

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Abstract. We prove that algebraic commutants of maximal subdiagonal algebras and of analytic operator algebras determined by flows in a \( \sigma \)-finite von Neumann algebra are self-adjoint.

1. Introduction

Let \( \mathcal{H} \) be a complex Hilbert space and let \( \mathcal{B}(\mathcal{H}) \) be the algebra of all bounded linear operators on \( \mathcal{H} \). For a subset \( E \) of \( \mathcal{B}(\mathcal{H}) \), we denote by \( E' \) the algebraic commutant, that is,

\[
E' = \{ X \in \mathcal{B}(\mathcal{H}) : AX =XA, \ \forall A \in E \}.
\]

If \( T \in \mathcal{B}(\mathcal{H}) \), we call \( \{ T \}' \) the algebraic commutant of \( T \). The well-known theorem of Fuglede states that if \( N \) is normal and \( X \) commutes with \( N \), so does \( X^* \). That is, the algebraic commutant \( \{ N \}' \) of \( N \) is self-adjoint. Note that \( \{ N \}' \) is the same as the commutant of the algebra generated by \( N \) and \( I \), which is non-self-adjoint in general. Thus it may be asked which subalgebras have a self-adjoint commutant. For example, if all elements in a subalgebra are normal or the algebra itself is self-adjoint, then its algebraic commutant is self-adjoint. In general, this problem is not particularly interesting. However special cases of this problem are interesting. F. Gilfeather and D.R. Larson in [6] showed that the algebraic commutant of a nest subalgebra of a von Neumann algebra is self-adjoint. We note that a nest subalgebra of a von Neumann algebra is a kind of analytic operator algebra. Thus it is interesting to consider this problem for general analytic operator algebras.

In [2], W. Arveson introduced the notion of subdiagonal algebras to give a unified theory of non-self-adjoint operator algebras, including the algebra of bounded analytic matrix-valued (or more generally, operator-valued) functions and nest subalgebras of von Neumann algebras.
Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra acting on $\mathcal{H}$. We denote by $\mathcal{M}_*$ the space of all $\sigma$-weakly continuous linear functionals of $\mathcal{M}$. For a von Neumann subalgebra $\mathcal{D}$ of $\mathcal{M}$, let $\Phi$ be a faithful normal conditional expectation from $\mathcal{M}$ onto $\mathcal{D}$. A subalgebra $\mathfrak{A}$ of $\mathcal{M}$, containing $\mathcal{D}$, is called a subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ if

(i) $\mathfrak{A} \cap \mathfrak{A}^* = \mathcal{D}$,

(ii) $\Phi$ is multiplicative on $\mathfrak{A}$, and

(iii) $\mathfrak{A} + \mathfrak{A}^*$ is $\sigma$-weakly dense in $\mathcal{M}$.

The algebra $\mathcal{D}$ is called the diagonal of $\mathfrak{A}$. Although subdiagonal algebras are not assumed to be $\sigma$-weakly closed in [2], the $\sigma$-weak closure of a subdiagonal algebra is again a subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ (Remark 2.1.2 in [2]). Thus we assume that our subdiagonal algebras are always $\sigma$-weakly closed.

We say that $\mathfrak{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to $\Phi$ in case that $\mathfrak{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to $\Phi$. Put $\mathfrak{A}_0 = \{X \in \mathfrak{A} : \Phi(X) = 0\}$ and $\mathfrak{A}_m = \{X \in \mathcal{M} : \Phi(AXB) = \Phi(BXA) = 0, \forall A \in \mathfrak{A}, B \in \mathfrak{A}_0\}$. By Theorem 2.1.1 in [2], we recall that $\mathfrak{A}_m$ is a maximal subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ containing $\mathfrak{A}$. If there is a faithful normal finite trace $\tau$ on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$, we say that $\mathfrak{A}$ is finite subdiagonal.

On the other hand, let $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ be a flow of $\mathbb{R}$ on $\mathcal{M}$, i.e. $\{\alpha_t\}_{t \in \mathbb{R}}$ is a one-parameter group of *-automorphisms of $\mathcal{M}$ such that, for each $X \in \mathcal{M}$, $t \to \alpha_t(X)$ is $\sigma$-weakly continuous. Write $H^\infty(\alpha) = \{X \in \mathcal{M} : sp_\alpha(X) \subseteq [0, \infty)\}$, where $sp_\alpha(\cdot)$ is an Arveson spectrum (Section 3). Then $H^\infty(\alpha)$ is a $\sigma$-weakly closed subalgebra of $\mathcal{M}$ satisfying that $H^\infty(\alpha) \cap (H^\infty(\alpha))^*$ is $\sigma$-weakly dense in $\mathcal{M}$. The structure of $H^\infty(\alpha)$ was studied by several authors (cf. [3] [13] [14] [16]). It is known that if there is a faithful normal conditional expectation from $\mathcal{M}$ onto $H^\infty(\alpha) \cap (H^\infty(\alpha))^*$, then $H^\infty(\alpha)$ is a maximal subdiagonal algebra of $\mathcal{M}$. Moreover, if $\mathfrak{A}$ is a nest subalgebra of $\mathcal{M}$, then there is an inner flow $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$, that is, $\alpha$ is implemented by a continuous unitary group $\{U_t : t \in \mathbb{R}\} \subset \mathcal{M}$, such that $\mathfrak{A} = H^\infty(\alpha)$ (cf. Theorem 4.2.3 in [14]).

In this note we prove that algebraic commutants of maximal subdiagonal algebras and of analytic operator algebras are self-adjoint.

2. THE COMMUTANT OF A MAXIMAL SUBDIAGONAL ALGEBRA

We consider the algebraic commutant of a maximal subdiagonal algebra $\mathfrak{A}$ with respect to $\Phi$. The following result was proved in [9]. For completeness, we give the proof here also.

**Lemma 1.** Let $\mathfrak{A}$ be a finite subdiagonal algebra with respect to $\Phi$ of $\mathcal{M}$. Then $\mathfrak{A}' = \mathcal{M}'$.

**Proof.** It is trivial that $\mathfrak{A} \supseteq \mathcal{M}'$. Now let $X \in \mathfrak{A}'$ and $T \in \mathcal{M}$. Then for any $\epsilon > 0$, we have that $T^*T + \epsilon I$ is a positive invertible operator in $\mathcal{M}$. Note that $\mathfrak{A}$ is maximal subdiagonal (cf. [2]). By Theorem 4.2.1 in [2], there is an invertible operator $A$ in $\mathfrak{A}$ so that $T^*T + \epsilon I = A^*A$. Then

$$X^*(T^*T + \epsilon I)X = A^*A^*X^*XA \leq \|X\|^2A^*A = \|X\|^2(T^*T + \epsilon I).$$

It follows that $X^*T^*TX \leq \|X\|^2T^*T$ by letting $\epsilon \to 0$. In particular, $X^*EX \leq \|X\|^2E$ for every positive projection $E$ in $\mathcal{M}$. It follows that $(I - E)X^*EX(I - E) \leq \|X\|^2$. Hence, $X^*EX \leq \frac{1}{2}E$ for every projection $E$ in $\mathcal{M}$. Therefore, $\mathfrak{A}' \supseteq \mathcal{M}'$. This completes the proof.
0, which implies that $EX(I-E) = 0$. Thus $EX =XE$ for every projection $E \in \mathcal{M}$, which implies that $X \in \mathcal{M}'$. The proof is complete. □

We next recall Haagerup’s reduction theory [7]. Since $\mathcal{M}$ is $\sigma$-finite, there exists a faithful normal state $\varphi$ of $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$. Let $\sigma^\varphi = \{\sigma^\varphi_t\}_{t \in \mathbb{R}}$ be the modular automorphism group of $\mathcal{M}$ associated with $\varphi$. We know that $\mathfrak{A}$ is $\{\sigma^\varphi_t\}_{t \in \mathbb{R}}$ invariant from Theorem 2.4 in [10]. Let $G$ be the discrete subgroup $\bigcup_{n \geq 1} 2^{-n}\mathbb{Z}$ of $\mathbb{R}$. We consider the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} G$ with respect to $\sigma^\varphi$. Then we have that $\mathcal{M} \rtimes_{\sigma^\varphi} G$ is a von Neumann algebra on $l^2(G, \mathcal{H})$ generated by the operators $\pi(X), X \in \mathcal{M}$, and $\lambda(s), s \in G$, defined by the equations

$$(\pi(X)\xi)(t) = \sigma^\varphi_t(X)\xi(t), \quad \xi \in \ell^2(G, \mathcal{H}), \ t \in G,$$

and

$$(\lambda(s)\xi)(t) = \xi(t-s), \quad \xi \in \ell^2(G, \mathcal{H}), \ t \in G.$$ 

Note that $\pi$ is a normal faithful representation of $\mathcal{M}$ on $\ell^2(G, \mathcal{H})$. Let $\hat{\varphi}$ be the dual weight of $\varphi$ on $\mathcal{M} \rtimes_{\sigma^\varphi} G$. Then $\hat{\varphi}$ is again a faithful normal state on $\mathcal{M} \rtimes_{\sigma^\varphi} G$. Haagerup’s reduction theorem asserts that there is an increasing sequence $\{\mathcal{R}_n\}_{n \geq 1}$ of von Neumann subalgebras of $\mathcal{M} \rtimes_{\sigma^\varphi} G$ with the following properties:

(i) each $\mathcal{R}_n$ is finite;
(ii) $\bigcup_{n \geq 1} \mathcal{R}_n$ is $\sigma$-weakly dense in $\mathcal{M} \rtimes_{\sigma^\varphi} G$;
(iii) for each $n \geq 1$ there is a faithful normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M} \rtimes_{\sigma^\varphi} G$ onto $\mathcal{R}_n$ such that $\hat{\varphi} \circ \mathcal{E}_n = \hat{\varphi}$, $\mathcal{E}_n \mathcal{E}_{n+1} = \mathcal{E}_n$, $n \geq 1$, and

$$\lim_{n \to \infty} \|\psi \circ \mathcal{E}_n - \psi\| = 0 \text{ for all } \psi \in (\mathcal{M} \rtimes_{\sigma^\varphi} G)_*.$$ 

We refer the readers to [7] and [17] for more details.

We now can extend $\Phi$ to a normal faithful conditional expectation $\hat{\Phi}$ from $\mathcal{M} \rtimes_{\sigma^\varphi} G$ onto $\mathcal{D} \rtimes_{\sigma^\varphi} G$, which is naturally identified as a von Neumann subalgebra of $\mathcal{M} \rtimes_{\sigma^\varphi} G$.

Let $\hat{\mathfrak{A}}$ be the $\sigma$-weakly closed subalgebra generated by $\{\pi(X) : X \in \mathfrak{A}\}$ and $\{\lambda(s) : s \in G\}$. Since $\mathfrak{A}$ is $\sigma^\varphi_t$ invariant by Theorem 2.4 in [10], $\hat{\mathfrak{A}}$ is the $\sigma$-weak closure of the set of all linear combinations of $\lambda(s)\pi(X), s \in G, X \in \mathfrak{A}$. The following lemma was proved in [17].

**Lemma 2.** \(\hat{\mathfrak{A}}\) is a maximal subdiagonal algebra with respect to $\hat{\Phi}$.

Let $\mathfrak{A}_n = \mathcal{R}_n \cap \hat{\mathfrak{A}} = \mathcal{E}_n(\hat{\mathfrak{A}})$. We have the following lemma (Lemma 2 in [17]).

**Lemma 3.** $\mathfrak{A}_n$ is a finite subdiagonal algebra in $\mathcal{R}_n$ with respect to $\hat{\Phi}|_{\mathcal{R}_n}$ and $\bigcup_{n \geq 1} \mathfrak{A}_n$ is $\sigma$-weakly dense in $\hat{\mathfrak{A}}$.

We now have the main theorem in this section.

**Theorem 1.** Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra and let $\mathfrak{A}$ be a maximal subdiagonal algebra with respect to $\Phi$ of $\mathcal{M}$. Then the commutant $\mathfrak{A}'$ of $\mathfrak{A}$ is self-adjoint, that is, $\mathfrak{A}' = \mathcal{M}'$.

**Proof.** We first claim that $\hat{\mathfrak{A}}' = (\mathcal{M} \rtimes_{\sigma^\varphi} G)'$. In fact, let $X \in \hat{\mathfrak{A}}'$. Then $X \in \mathfrak{A}_n'$ for all $n \in \mathbb{N}$. By Lemmas 1 and 3, we have $X^* \in \mathfrak{A}_n'$. Note that for every $Y \in \mathfrak{A}$, we have $\mathcal{E}_n(Y) \in \mathfrak{A}_n$ for all $n \in \mathbb{N}$ and $Y = \lim_n \mathcal{E}_n(Y)$ $\sigma$-weakly from Haagerup’s theory. Then it follows that $X^*Y = YX^*$, which implies that $X^* \in \mathfrak{A}'$. 


Now let $X \in \mathfrak{A}$. We define an operator $\hat{X}$ on $\ell^2(G, \mathcal{H})$ by

$$(\hat{X}\xi)(s) = X\xi(s), \quad s \in G, \xi \in \ell^2(G, \mathcal{H}).$$

Then we have that $\hat{X} \in \mathfrak{A}'$. In fact, for $\xi \in \ell^2(G, \mathcal{H})$, $t, s \in \mathbb{R}$ and $Y \in \mathfrak{A}$,

$$(\hat{X}\pi(Y)\xi)(t) = X(\pi(Y)\xi)(t) = X\sigma^\varphi_t(Y)\xi(t) = \sigma^\varphi_t(Y)X\xi(t) = (\pi(Y)\hat{X}\xi)(t)$$

and

$$(\hat{X}\lambda_s\xi)(t) = X((\lambda_s\xi)(t)) = X\xi(t-s) = (\hat{X}\xi)(t-s) = (\lambda_s(\hat{X}\xi))(t) = (\lambda_s\hat{X}\xi)(t).$$

It follows that $(\hat{X})^* \in \mathfrak{A}'$, which implies that $(\hat{X})^*\pi(Y) = \pi(Y)(\hat{X})^*$ for all $Y \in \mathfrak{A}$. Note that $(\hat{X})^* = (X^*)$. Then $(X^*)\pi(Y) = \pi(Y)(X^*)$ for all $Y \in \mathfrak{A}$. In particular, $X^*Y = YX^*$. Thus we have $X^* \in \mathfrak{A}'$. Note that $\mathfrak{A} + \mathfrak{A}^*$ is $\sigma$-weakly dense in $\mathcal{M}$. It then follows that $X \in \mathcal{M}'$. Hence $\mathfrak{A}' = \mathcal{M}'$. The proof is complete. \hfill $\Box$

3. The commutant of an analytic operator algebra

In this section we consider the algebraic commutant of an analytic operator algebra determined by a flow on $\mathcal{M}$. We need Arveson’s theory of spectral subspaces and, so we recall the definitions here. Let $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ be a flow on $\mathcal{M}$, i.e. a $\sigma$-weakly continuous one parameter group of $*$-automorphisms of $\mathcal{M}$. For each element $X \in \mathcal{M}$ and a function $f \in L^1(\mathbb{R})$, we define the convolution $f \ast_X \alpha$ by

$$f \ast_X \alpha = \int_{-\infty}^{+\infty} f(t)\alpha_t(X)dt.$$ 

For $f \in L^1(\mathbb{R})$, let $Z(f) = \{t \in \mathbb{R} : \hat{f}(t) = 0\}$, where $\hat{f}(t) = \int_{-\infty}^{+\infty} e^{-ist}f(s)ds$ is the Fourier transform of $f$. For $X \in \mathcal{M}$, we define the Arveson spectrum of $X$ with respect to the flow $\alpha$ to be the set

$$\bigcap\{Z(f) : f \ast_X \alpha = 0\}$$

and denote it by $sp_\alpha(X)$. For any subset $S$ of $\mathbb{R}$ we define the spectral subspace $M^\alpha(S)$ to be the $\sigma$-weak closure of the set $\{X \in \mathcal{M} : sp_\alpha(X) \subset S\}$. We refer the readers to [3, 4, 13, 14] for the elementary properties of spectra and spectral subspaces. Put $H^\infty(\alpha) = M^\alpha([0, \infty))$ and $H_0^\infty(\alpha) = M^\alpha((0, \infty))$. It is known that $H^\infty_0(\alpha)$ is a two-sided ideal of $H^\infty(\alpha)$. Let $\mathcal{D} = H^\infty(\alpha) \cap (H^\infty(\alpha))^*$ be the fixed point subalgebra of $\alpha$. We recall that $\mathcal{M}$ is said to be $\mathbb{R}$-finite relative to $\alpha$ if there is a separating family of $\alpha$-invariant normal states on $\mathcal{M}$. At the opposite extreme, we say that $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$ in case there are no invariant normal states.

Lemma 4. If $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$, then $H^\infty(\alpha) = H^\infty_0(\alpha)$. 

Proof. Since $\mathcal{M}$ is $\sigma$-finite, without loss of generality by choosing an appropriate representation for $\mathcal{M}$, we may assume that $\mathcal{M}$ has a cyclic and separating vector in $\mathcal{H}$.

If $H^\infty(\alpha) \neq H^\infty(\alpha)_0$, then there is an element $f \in \mathcal{M}_s$ such that $f(A) = 0$ for all $A \in H^\infty(\alpha)$ and $f(T) \neq 0$ for some $T \in H^\infty(\alpha)$. Since $\mathcal{M}$ has a separating vector in $\mathcal{H}$, there are vectors $x, y \in \mathcal{H}$ such that $f(A) = (Ax, y)$ for all $A \in \mathcal{M}$ by Proposition 7.4.5 and Corollary 7.3.3 in [14]. Let $\mathfrak{M} = \{Ax : A \in H^\infty(\alpha)\}$ (resp. $\mathfrak{M}_0 = \{Ax : A \in H^\infty(\alpha)_0\}$) be the closed subspace generated by $\{Ax : A \in H^\infty(\alpha)\}$ (resp. $\{Ax : A \in H^\infty(\alpha)_0\}$) of $\mathcal{H}$. Then both $\mathfrak{M}$ and $\mathfrak{M}_0$ are invariant subspaces for $H^\infty(\alpha)$. We have $\mathfrak{M}_0 \subset \mathfrak{M}$ since $(Tx, y) \neq 0$ for some $T \in H^\infty(\alpha)$ and $(Tx, y) = 0$ for all $T \in H^\infty(\alpha)_0$. It is trivial that $H^\infty(\alpha)_0 \mathfrak{M} \subset \mathfrak{M}_0$ since $H^\infty(\alpha)_0$ is a two-sided ideal of $H^\infty(\alpha)$. On the other hand, since $\mathcal{M}$ is completely non-$\mathbb{R}$-finite relative to $\alpha$, by Corollary 5.7 in [14], $\mathfrak{M}$ is completely normalized in the sense of Definition 5.1 in [14], that is,

$$\mathfrak{M} = \bigwedge_{s < 0} [M^\alpha([s, \infty)))\mathfrak{M}] = \bigvee_{s > 0} [M^\alpha([s, \infty)))\mathfrak{M}].$$

Note that $M^\alpha([s, \infty)) \subset H^\infty(\alpha)_0$ for all $s > 0$. We then have that $[M^\alpha([s, \infty))\mathfrak{M}] \subset \mathfrak{M}_0$ for all $s > 0$ which implies that $\mathfrak{M} \subset \mathfrak{M}_0$. This is a contradiction. Hence $H^\infty(\alpha) = H^\infty(\alpha)_0$. The proof is complete.

We recall that the crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}$ determined by $\mathcal{M}$ and $\alpha$ is the von Neumann algebra on the Hilbert space $L^2(\mathbb{R}, \mathcal{H})$ generated by the operators $\pi(X)$, $X \in \mathcal{M}$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by the equations

$$(\pi(X)f)(t) = \alpha_{-t}(X)f(t), \quad f \in L^2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R},$$

and

$$(\lambda(s)f)(t) = f(t-s), \quad f \in L^2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R}.$$

It is clear that $\pi(\alpha_t(X)) = \lambda(t)\pi(X)\lambda(t)^*$ for all $X \in \mathcal{M}$ and $t \in \mathbb{R}$. For any $Y \in \mathcal{M} \rtimes_\alpha \mathbb{R}$, we define $\beta_t(Y) = \lambda(t)Y\lambda(t)^*$, $\forall t \in \mathbb{R}$. Then $\beta = \{\beta_t\}_{t \in \mathbb{R}}$ is an inner flow on $\mathcal{M} \rtimes_\alpha \mathbb{R}$. We know that $H^\infty(\beta)$ is a nest subalgebra in $\mathcal{M} \rtimes_\alpha \mathbb{R}$ by Theorem 4.2.3 in [14].

Let $\mathcal{A}$ be the $\sigma$-weakly closed subalgebra of $\mathcal{M} \rtimes_\alpha \mathbb{R}$ generated by $\{\pi(X) : X \in H^\infty(\alpha)\}$ and $\{\lambda(t) : t \in \mathbb{R}\}$. Since $\pi(\alpha_t(X)) = \beta_t(\pi(X))$ for $X \in \mathcal{M}$ and $t \in \mathbb{R}$, $\mathcal{A}$ is a subalgebra of $H^\infty(\beta)$. It is noted that $\mathcal{A} + \mathcal{A}^*$ is $\sigma$-weakly dense in $\mathcal{M} \rtimes_\alpha \mathbb{R}$ since $H^\infty(\alpha) + H^\infty(\alpha)^*$ is $\sigma$-weakly dense by Theorem 3.15 in [14].

**Lemma 5.** $H^\infty_0(\beta) \subset \mathcal{A}$.

Proof. We know that $H^\infty_0(\beta)$ is the $\sigma$-weak closure of the set $\{X \in \mathcal{M} \rtimes_\alpha \mathbb{R} : sp_{\beta}(X) \text{ is compact in } (0, +\infty)\}$ by Lemma 2.8 in [15]. Let $X \in \mathcal{M} \rtimes_\alpha \mathbb{R}$ be such that $sp_{\beta}(X)$ is compact in $(0, +\infty)$. Choose $f \in L^1(\mathbb{R})$ with compactly supported Fourier transform such that support $supp\hat{f}$ of $\hat{f}$ is in $(0, \infty)$ and such that $f_X = X$. Note that since $\mathcal{A} + \mathcal{A}^*$ is $\sigma$-weakly dense in $\mathcal{M} \rtimes_\alpha \mathbb{R}$, there are nets $\{A_i\}$, $\{B_i\}$ in $\mathcal{A}$ such that $lim_{i}(A_i + B_i^*) = X$ $\sigma$-weakly. It follows that $lim_{i}(f * A_i + f * B_i^*) = f_X = X$. However, we have $f * B_i^* = 0$ since $sp_{\beta}(f * B_i^*) \subset supp\hat{f} \cap sp_{\beta}(B_i^*) = \emptyset$. Note that $A$ is $\{\beta_t\}_{t \in \mathbb{R}}$ invariant, it follows that $f * A_i \in \mathcal{A}$ and then $X \in \mathcal{A}$. The proof is complete. □
The next result might be known, but we were unable to find a reference for it.

Lemma 6. If $\mathcal{M}$ is $\mathbb{R}$-finite relative to $\alpha$, then $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is $\mathbb{R}$-finite relative to $\beta$.

Proof. By considering a covariant representation of the pair $(\mathcal{M}, \alpha)$ in the sense of Definition 2.5 and Proposition 2.6 in [14], we may assume that $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ is implemented by a continuous unitary representation $t \mapsto U_t$ of $\mathbb{R}$ on $\mathcal{H}$, that is, $\alpha_t(X) = U_tXU_t^*$ for all $x \in \mathcal{M}$. Then by Definition 13.2.6 in [12], the crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is a von Neumann algebra on $L^2(\mathbb{R}, \mathcal{H})$ ($= \mathcal{H} \otimes L^2(\mathbb{R})$) generated by $\{A \otimes I : a \in \mathcal{M}\}$ and $\{U_t \otimes I : t \in \mathbb{R}\}$, where $U_t$ is the shift operator on $L^2(\mathbb{R})$ defined by $(l_t f)(s) = f(t+s)$, $t, s \in \mathbb{R}$, $f \in L^2(\mathbb{R})$. It is known that $\beta$ is implemented by $\{U_t \otimes I : t \in \mathbb{R}\}$.

Since $L^2(\mathbb{R})$ is separable, there is an orthonormal basis $\{e_n : n = 1, 2, \cdots\}$ of $L^2(\mathbb{R})$. If we define $\mu(T) = \sum_{n=1}^{\infty} \frac{1}{\|e_n\|}(Te_n, e_n)$, $\forall T \in \mathcal{B}(L^2(\mathbb{R}))$, then $\mu$ is a faithful normal state of $\mathcal{B}(L^2(\mathbb{R}))$. Let $\phi$ be a faithful normal state of $\mathcal{M}$ such that $\phi \circ \alpha_t = \phi$ for all $t \in \mathbb{R}$. It follows from Proposition 11.2.7 in [12] that $\phi \otimes \mu$ is a normal state on $\mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R})) \supset \mathcal{M} \rtimes_\alpha \mathbb{R}$. We claim that $\phi \otimes \mu$ is faithful. In fact, we way identify $\mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}))$ with $\{A_{ij} \in \mathcal{B}(\mathcal{H} \otimes L^2(\mathbb{R})) : A_{ij} \in \mathcal{M}\}$ by Remark 11.2.3 in [12]. For any $(A_{ij}) \in \mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}))$, we define $\psi((A_{ij})) = \sum_{n=1}^{\infty} \frac{1}{\|e_n\|}\phi(A_{nn})$. Then $\psi$ is a faithful normal state on $\mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}))$. Since for any $A \in \mathcal{M}$ and $T \in \mathcal{B}(L^2(\mathbb{R}))$, $A \otimes T$ identifies with $(t_{ij} A)$, where $t_{ij} = (Te_{ij}, e_i)$ for all $i, j = 1, 2, \cdots$, $\phi \otimes \mu(A \otimes T) = \phi(A)\mu(T) = \psi((t_{ij} A))$. It follows that $\phi \otimes \mu = \psi$ and then $\phi \otimes \mu$ is faithful.

We note that if $A \in \mathcal{M}$, then $\phi \otimes \mu(\beta_s(A \otimes I)) = \phi \otimes \mu(A \otimes I)$ for all $s \in \mathbb{R}$. On the other hand, $\beta_t(U_s \otimes I_s) = U_s \otimes I_s$; then $\phi \otimes \mu(\beta_t(U_s \otimes I_s)) = \phi \otimes \mu(U_s \otimes I_s)$, $s, t \in \mathbb{R}$. We now have that $\phi \otimes \mu$ is a faithful normal state on $\mathcal{M} \rtimes_\alpha \mathbb{R}$ such that $\phi \otimes \mu \circ \beta_s = \phi \otimes \mu$ for all $s \in \mathbb{R}$. It follows that $\mathcal{M} \rtimes_\alpha \mathbb{R}$ is $\mathbb{R}$-finite relative to $\beta$. The proof is complete. \hfill $\Box$


Proof. We first assume that $\mathcal{M}$ is $\mathbb{R}$-finite relative to $\{\alpha_t\}_{t \in \mathbb{R}}$. Then there is a faithful normal expectation $\Phi$ from $\mathcal{M}$ onto $\mathcal{D}$. Now there is a faithful normal expectation $\Psi$ from $\mathcal{M} \rtimes_\alpha \mathbb{R}$ onto $\mathcal{N}$ such that $\Psi(\pi(A)) = \pi(\Phi(A))$ for all $A \in \mathcal{M}$ by Lemma 6, where $\mathcal{N}$ is the fixed point algebra of $\beta$. We have that $\mathcal{N}$ is generated by $\{\pi(A) : A \in \mathcal{D}\}$ and $\{\lambda(t) : t \in \mathbb{R}\}$. In fact, take any $D \in \mathcal{N}$. There is a net $A_i = \sum_{i=1}^{n_i} \pi(X_{i,j}^i)\lambda(t_{i,j}^i)$ such that $\lim_{i} A_i = D$ $\sigma$-weakly, where $X_{i,j}^i \in \mathcal{M}$ and $t_{i,j}^i \in \mathbb{R}$. We then have

$$D = \Psi(D) = \lim_i \Psi(A_i) = \lim_i \sum_{j=1}^{n_i} \Psi(\pi(X_{i,j}^i))\lambda(t_{i,j}^i).$$

It now follows that $\mathcal{N} = H^\infty(\beta) \cap H^\infty(\beta)^* = A \cap A^*$. We know that $H^\infty_0(\beta) \subset A$ by Lemma 5. Thus $H^\infty(\beta) = \mathcal{N} + H^\infty_0(\beta) \subset A$ and therefore $H^\infty(\beta) = A$.

For the general case, there is a projection $E$ in the center of $\mathcal{D}$ such that $E \mathcal{M} E$ is $\mathbb{R}$-finite, while $(I-E)\mathcal{M}(I-E)$ is completely non-$\mathbb{R}$-finite relative to $\alpha$ from Remark
3.4 in [4]. We note that in this case \( \pi(E)({\mathcal M} \times_{\alpha} \mathbb{R})\pi(E) \) is \( \mathbb{R} \)-finite by Lemma 6, and \((I - \pi(E))({\mathcal M} \times_{\alpha} \mathbb{R})(I - \pi(E))\) is completely non-\( \mathbb{R} \)-finite relative to \( \beta \). By considering \( \alpha \) restricted on \( EME \), we have \( \pi(E)H^\infty(\beta)\pi(E) \subset \pi(E)\mathcal{A}\pi(E) \subset \mathcal{A} \) by Lemma 6 again. On the other hand, if we consider \( \alpha \) on \((I - E)\mathcal{M}(I - E)\), then we have \((I - \pi(E))H^\infty(\beta)(I - \pi(E)) = (I - \pi(E))H^\infty_0(\beta)(I - \pi(E)) \subset H^\infty_0(\beta)\). In particular, \( I - \pi(E) \in H^\infty_0(\beta) \) and then \((I - \pi(E))H^\infty(\beta) + H^\infty(\beta)(I - \pi(E)) \subset H^\infty_0(\beta) \subset \mathcal{A} \). Thus we have \( H^\infty(\beta) \subset \mathcal{A} \), and the proof is complete.

**Theorem 2.** The commutant of \( H^\infty(\alpha) \) is self-adjoint, that is, \((H^\infty(\alpha))^* = \mathcal{M}'\).

**Proof.** Let \( \beta = \{\beta_t\}_{t \in \mathbb{R}} \) be as above. We recall that \( H^\infty(\beta) \) is a nest subalgebra of \( \mathcal{M} \times_{\alpha} \mathbb{R} \). Then the commutant of \( H^\infty(\beta) \) is self-adjoint by Theorem 2.5 in [6]. Let \( X \in (H^\infty(\alpha))^* \). Define an operator \( \hat{X} \) on \( L^2(\mathbb{R}, \mathcal{H}) \) by

\[
(\hat{X}\xi)(t) = X\xi(t), \quad \forall \xi \in L^2(\mathbb{R}, \mathcal{H}).
\]

Then it is trivial that \( \hat{X} \) is bounded. We claim that \( \hat{X} \in (H^\infty(\beta))^* \). By Lemma 7, it is sufficient to show that \( X \in \mathcal{A}' \). For any \( Y \in H^\infty(\alpha) \) we have

\[
(\hat{X}\pi(Y)\xi)(t) = X\alpha_{-t}(Y)\xi(t) = \alpha_{-t}(Y)X\xi(t) = \alpha_{-t}(Y)(\hat{X}\xi)(t) = (\pi(Y)\hat{X}\xi)(t).
\]

Then \( \hat{X}\pi(Y) = \pi(Y)\hat{X} \). On the other hand, for any \( s \in \mathbb{R} \),

\[
(\hat{X}\lambda(s)\xi)(t) = (\hat{X}\xi)(t - s) = \lambda(s)(\hat{X}\xi)(t) = (\lambda(s)\hat{X})(t).
\]

It follows that \( \lambda(s)\hat{X} = \hat{X}\lambda(s) \) for any \( s \in \mathbb{R} \). We thus have \( \hat{X} \in (H^\infty(\beta))^* \). By Theorem 2.5 in [6], \( (\hat{X})^* \in (H^\infty(\beta))^* \). In particular, \( (\hat{X})^* \) commutes with \( \pi(Y) \) for any \( Y \in H^\infty(\alpha) \). Note that \( (\hat{X})^* = (X^*) \). Given \( u \in \mathcal{H} \), let \( f \) be a continuous function in \( L^2(\mathbb{R}) \) such that \( f(0) = 1 \) and let \( \xi(t) = f(t)u \). Then \( \xi \in L^2(\mathbb{R}, \mathcal{H}) \) and

\[
((X^*)\pi(Y)\xi)(t) = X^*\alpha_{-t}(Y)\xi(t) = X^*\alpha_{-t}(Y)f(t)u = (\pi(Y)X^*)\xi(t) = \alpha_{-t}(Y)X^*f(t)u.
\]

When \( t = 0 \), we have \( X^*Yu = YX^*u \) for all \( u \in \mathcal{H} \). Hence \( X^* \in (H^\infty(\alpha))^* \) and \( (H^\infty(\alpha))^* = \mathcal{M}' \). The proof is complete.

**Remark 1.** We know that if \( \mathcal{A} \) is either a subdiagonal algebra or an analytic operator algebra of \( \mathcal{M} \), then we have that \( \mathcal{A} + \mathcal{A}^* \) is \( \sigma \)-weakly dense in \( \mathcal{M} \). However, if we assume that a subalgebra only satisfies this condition, it may not follow that the algebraic commutant is self-adjoint. For example, from Corollary 1.4 in [1], we know that there is a subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{H}) \) such that \( \mathcal{A} + \mathcal{A}^* \) is \( \sigma \)-weakly dense in \( \mathcal{B}(\mathcal{H}) \) and such that \( \mathcal{A} \) is similar to a proper von Neumann subalgebra of \( \mathcal{B}(\mathcal{H}) \). It easily follows that the algebraic commutant of \( \mathcal{A} \) is not self-adjoint.

**Remark 2.** We considered two classes of non-self-adjoint operator algebras, subdiagonal algebras and analytic operator algebras determined by flows in von Neumann algebras. If \( \mathcal{H} \) is finite dimensional, we know that these two classes of operator algebras are nest subalgebras of von Neumann algebras (cf. Theorem 2.1 in [11]). However, if \( \mathcal{H} \) is infinite dimensional, these two classes are different. The analytic operator algebra \( H^\infty(\alpha) \) determined by a flow \( \alpha \) is a maximal subdiagonal algebra if and only if the flow \( \alpha \) is \( \mathbb{R} \)-finite. There are subdiagonal algebras which are not
analytic operator algebras determined by any flows. We refer the readers to see some examples given in [8].

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References


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