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MINIMIZING EULER CHARACTERISTICS OF SYMPLECTIC FOUR-MANIFOLDS

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Abstract. We prove that the minimal Euler characteristic of a closed symplectic four-manifold with given fundamental group is often much larger than the minimal Euler characteristic of almost complex closed four-manifolds with the same fundamental group. In fact, the difference between the two is arbitrarily large for certain groups.

It was first proved by Dehn [2] that every finitely presentable group Γ can be realized as the fundamental group of a closed oriented smooth four-manifold. Taking the minimum over the Euler characteristics of all such manifolds, one obtains an interesting numerical invariant $q^{DIFF}(\Gamma)$ of finitely presentable groups; see for example [4, 5, 7]. As mentioned in [7], there are geometric variants $q^{GEO}(\Gamma)$ of this definition, obtained by minimizing the Euler characteristic only over those four-manifolds with fundamental group Γ which carry a specified geometric structure. One trivially has

$$q^{DIFF}(\Gamma) \leq q^{GEO}(\Gamma)$$

for all geometric structures. Moreover, the inequality is often strict.

For a simple example of a geometric invariant, consider almost complex four-manifolds. Every finitely presentable group is the fundamental group of an almost complex four-manifold [6], but the minimal Euler characteristic over almost complex four-manifolds is strictly larger than $q^{DIFF}(\Gamma)$ for many Γ. Nevertheless, in this case it is easy to see that the difference between the smooth and geometric invariants is universally bounded independently of Γ; compare [6].

The purpose of this paper is to show that in the symplectic category this boundedness fails. Recall that Gompf [3] proved that every finitely presentable Γ can be realised as the fundamental group of a closed symplectic four-manifold. Thus we can define $q^{SYMP}(\Gamma)$ to be the minimal Euler characteristic of a closed symplectic four-manifold with fundamental group Γ. Then we have

Theorem 1. *For every $c > 0$ there exists a finitely presentable group Γ satisfying*

$$q^{SYMP}(\Gamma) \geq q^{DIFF}(\Gamma) + c\ .$$

Proof. We shall use the sequence F_r of free groups of rank r. It suffices to show that the difference

$$q^{SYMP}(F_r) - q^{DIFF}(F_r)$$

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grows linearly with the rank r. We know from [7] that $q^{DIFF}(F_r) = -2(r-1)$, because, on the one hand, this value is the obvious lower bound $2-2b_1$ for the Euler characteristic of any closed four-manifold with fundamental group F_r, and, on the other hand, this value is realized by the connected sum of r copies of $S^1 \times S^3$.

To estimate $q^{SYMP}(F_r)$ let X be a closed symplectic four-manifold with fundamental group F_r and with minimal Euler characteristic. The minimality of the Euler characteristic implies that X is symplectically minimal in the sense that it contains no symplectically embedded (-1)-spheres. Let us assume for the moment that the positive part $b_2^+(X)$ of the intersection form of X is strictly larger than 1. Then a result of Taubes [10] implies $c_1^2(X) \geq 0$; see also [8]. We expand this inequality as follows:

$$0 \leq c_1^2 = 2\chi + 3\sigma = 4 - 4b_1 + 5b_2^+ - b_2^- \leq 4 - 4b_1 + 5b_2^+ \ .$$

This yields $b_2^+ \geq \frac{4}{5}(b_1 - 1)$, and thus

$$\chi = 2 - 2b_1 + b_2 \geq 2 - 2b_1 + b_2^+ \geq -\frac{6}{5}(b_1 - 1) \ .$$

Therefore we have

$$q^{SYMP}(F_r) \geq -\frac{6}{5}(r-1) \ , \tag{1}$$

showing that the difference $q^{SYMP}(F_r) - q^{DIFF}(F_r)$ grows linearly with r.

It remains to remove the assumption $b_2^+(X) > 1$. As X is symplectic, the only other possibility is $b_2^+(X) = 1$. If this happens, consider a d-fold covering X_d of X, with $d > 1$. This is symplectic with free fundamental group of rank $1+d(r-1)$. The multiplicativity of the signature and of the Euler characteristic in finite coverings then imply $b_2^+(X_d) = db_2^+(X) = d > 1$. We can not apply Taubes's inequality to X_d because a priori we do not know that X_d is symplectically minimal. Instead of proving this, we argue as follows. The minimal model Y_d of X_d has the same b_1 and b_2^+ as X_d. Taubes's inequality $c_1^2 \geq 0$ applied to Y_d gives

$$0 \leq c_1^2(Y_d) \leq 4 - 4b_1(Y_d) + 5b_2^+(Y_d) = d(9 - 4r) \ .$$

It follows that $r \leq 2$. In the cases $r \leq 1$, inequality (1) is trivial. In the case $r = 2$ it reduces to $q^{SYMP}(F_2) \geq -1$, which is true because in this case $\chi(X) = 2 - 2b_1(X) + b_2(X) = -2 + b_2(X) \geq -1$. $\square$

This result was motivated by the recent paper of Baldridge and Kirk [1], concerned with a systematic study of $q^{SYMP}(\Gamma)$. The lower bounds for $q^{SYMP}(\Gamma)$ given in [1] are never better than $q^{DIFF}(\Gamma) + 2$, because only the condition $b_2^+ \geq 1$ and the existence of almost complex structures on symplectic manifolds are used.

It turns out that the bound (1) holds in almost complete generality.

Theorem 2. *Let Γ be a finitely presentable group. The inequality*

$$q^{SYMP}(\Gamma) \geq -\frac{6}{5}(b_1(\Gamma) - 1) \tag{2}$$

holds for Γ if and only if Γ is not the fundamental group of a closed oriented surface of genus ≥ 2.

Proof. First of all, if Γ is the fundamental group of a closed oriented surface of genus $g \geq 2$, then it was proved in [7] that $q^{DIFF}(\Gamma) = 4(1-g) = 2(2 - b_1(\Gamma))$. The manifold $S^2 \times \Sigma_g$ realizes the minimum and is symplectic, so that $q^{SYMP}(\Gamma) = 2(2 - b_1(\Gamma)) < \frac{6}{5}(1 - b_1(\Gamma))$.

Suppose now that Γ is not a surface group. If a symplectic manifold with fundamental group Γ realizing the smallest possible Euler characteristic has $b_2^+ > 1$, then Taubes's inequality $c_1^2 \geq 0$ for minimal symplectic manifolds with $b_2^+ > 1$ implies (2), as in the proof of Theorem 1. If the symplectic minimizer for Γ has $b_2^+ = 1$, then for arbitrary Γ we may not be able to use covering tricks as in the proof of Theorem 1. However, because Γ is not a surface group, our manifold cannot be ruled. Therefore we can use Liu's extension [9] of Taubes's inequality to minimal non-ruled symplectic manifolds with $b_2^+ = 1$ to reach the same conclusion as before. $\square$

Gompf [3] asked whether a non-ruled symplectic four-manifold necessarily has non-negative Euler characteristic. This question is still open. A positive answer would of course provide a vast generalization of the results proved here. If a finitely presentable group Γ satisfies $q^{DIFF}(\Gamma) < 0$, then one knows a lot of its properties. For example, Γ cannot embed non-trivially in itself with finite index, it is non-amenable, and has a subgroup of finite index surjecting onto F_2; see [4, 7]. Thus there are many group-theoretic constraints for a negative answer to Gompf's question.

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