

## ON THE IRREDUCIBILITY OF THE HILBERT SCHEME OF SPACE CURVES

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ABSTRACT. Denote by  $H_{d,g,r}$  the Hilbert scheme parametrizing smooth irreducible complex curves of degree  $d$  and genus  $g$  embedded in  $\mathbb{P}^r$ . In 1921 Severi claimed that  $H_{d,g,r}$  is irreducible if  $d \geq g + r$ . As it has turned out in recent years, the conjecture is true for  $r = 3$  and 4, while for  $r \geq 6$  it is incorrect. We prove that  $H_{g,g,3}$ ,  $H_{g+3,g,4}$  and  $H_{g+2,g,4}$  are irreducible, provided that  $g \geq 13$ ,  $g \geq 5$  and  $g \geq 11$ , correspondingly. This augments the results obtained previously by Ein (1986), (1987) and by Keem and Kim (1992).

### 1. INTRODUCTION

In 1921 Severi asserted that the Hilbert scheme  $H_{d,g,r}$  parametrizing smooth integral complex curves of degree  $d$  and genus  $g$  embedded in  $\mathbb{P}^r$  is irreducible for  $d \geq g + r$  [Sev21]. Harris [Ein87, Prop. 9] and Keem [Kee94] show by examples that the claim is incorrect for  $r \geq 6$ . Meanwhile Ein proved Severi's claim for  $r = 3$  and 4. Shortly after, Keem and Kim gave a different proof for the case of  $r = 3$  in [KK92], where they also proved the irreducibility of  $H_{g+2,g,3}$  if  $g \geq 5$  and of  $H_{g+1,g,3}$  if  $g \geq 11$ . The goal of this paper is to extend the results in the cases of  $r = 3$  and  $r = 4$ . We focus on the case  $\rho(d, g, r) > 0$ , where  $\rho(d, g, r) = g - (r + 1)(g - d + r)$  is the Brill-Noether number. Our first result extends the irreducibility range of  $H_{d,g,3}$ .

**Theorem 3.1.**  *$H_{g,g,3}$  is irreducible provided that  $g \geq 13$ .*

The second result extends the irreducibility range about curves in  $\mathbb{P}^4$ .

**Theorem 3.2.** (a)  *$H_{g+3,g,4}$  is irreducible if  $g \geq 5$ .*

(b)  *$H_{g+2,g,4}$  is irreducible if  $g \geq 11$ .*

Our approach follows the one established through the works of Arbarello-Cornalba, Ein and Keem-Kim. Let  $g$ ,  $r$  and  $d$  be non-negative integers. Consider the moduli space  $\mathcal{M}_g$  of smooth curves of genus  $g$ . For any given  $p \in \mathcal{M}_g$  there exist a neighborhood  $U \subset \mathcal{M}_g$  of  $p$  and a smooth connected variety  $\mathcal{M}$  which is a finite ramified covering  $h : \mathcal{M} \rightarrow U$ , and also varieties  $\mathcal{C}$ ,  $\mathcal{W}_d^r$  and  $\mathcal{G}_d^r$  proper over  $\mathcal{M}$  with the following properties.

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- (1)  $\xi : \mathcal{C} \rightarrow \mathcal{M}$  is a universal curve, i.e. for every  $p \in \mathcal{M}$ ,  $\xi^{-1}(p)$  is a smooth curve of genus  $g$  isomorphic to  $h(p)$ ,
- (2)  $\mathcal{W}_d^r$  parametrizes the pairs  $(p, L)$ , where  $L$  is a line bundle of degree  $d$  and  $h^0(L) \geq r + 1$ ,
- (3)  $\mathcal{G}_d^r$  parametrizes the couples  $(p, \mathcal{D})$ , where  $\mathcal{D}$  is a linear series of dimension  $r$  and degree  $d$  on  $h(p)$ .

The main point in the above construction is the existence of a universal family over  $\mathcal{M}$ . Intuitively, the varieties  $\mathcal{W}_d^r$  and  $\mathcal{G}_d^r$  can be viewed as patching of the varieties  $W_d^r(C)$  and  $G_d^r(C)$  when the curve  $C$  moves in a subset of  $\mathcal{M}_g$ . Also, there exists the so-called *relative Picard scheme*  $\text{Pic}\mathcal{C}$ , which is just the relative analogue of  $\text{Pic}(C)$  defined for a fixed curve  $C$ . For further details see [AC81], [AC83] and [ACGH].

Let  $\mathcal{G}$  be the union of components of  $\mathcal{G}_d^r$  whose general element  $(p, \mathcal{D})$  represents a very ample linear series  $\mathcal{D}$  on the curve  $C = \xi^{-1}(p)$ . Since an irreducible component of  $H_{d,g,r}$  is a  $\text{PGL}(r+1)$ -fiber bundle over a component of  $\mathcal{G}$ , to establish irreducibility of  $H_{d,g,r}$  it is sufficient to prove that  $\mathcal{G}$  is irreducible. Regarding the existence of  $H_{d,g,r}$ , or equivalently of  $\mathcal{G}$ , we remark that for  $d \geq g+r$  it follows by the Halphen's theorem [Har77, IV., Proposition 6.1], while for  $d < g+r$  an answer, sufficient for our purposes, is given by Serres in [Ser84], namely:

**Proposition 1.1.** *For all non-negative integers  $g, r, d$  such that  $r \geq 3$ ,  $d \geq r+1$  and*

$$g - d + r \geq \max\{0, 1 - \rho(d, g, r)\},$$

*there exists a regular component  $V$  of  $H_{d,g,r}$  which has the expected number of moduli. A general point of  $V$  corresponds to an embedding  $C \hookrightarrow \mathbb{P}^r$  by a complete linear system (i.e.  $h^0(C, \mathcal{O}_C(1)) = r+1$ ), the normal bundle  $N_C$  satisfies  $H^1(C, N_C) = 0$  and the cup-product map*

$$\mu_0(C) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) \rightarrow H^0(C, K_C)$$

*is of maximal rank.*

The following facts will be used in what follows. For proofs see [AC81] and [AC83].

**Proposition 1.2.** (1) *The dimension of any component of  $\mathcal{G}_d^r$  is at least  $3g - 3 + \rho(d, g, r)$ .*

- (2) *Assuming that  $\rho(d, g, r) \geq 1$  and  $g \geq 2$ , there exists a unique component  $\mathcal{G}_0 \subset \mathcal{G}$  such that  $\mathcal{G}_0$  has general moduli and further the linear series of its general element is very ample (this component is called *principal component*).*
- (3)  *$\mathcal{G}_d^1$  is a smooth and irreducible variety of dimension  $3g - 3 + \rho(d, g, 1)$ .*
- (4) *The dimension of any component of  $\mathcal{G}_d^2$  whose general point contains a birationally very ample linear series is  $3g - 3 + \rho(d, g, 2)$ .*
- (5) *If  $\rho(d, g, 2) \geq 1$  then  $\mathcal{G}_d^2$  has a unique component such that its general element carries a birationally very ample linear series.*
- (6) *If  $\mathcal{W} \subset \mathcal{W}_d^r$  is an irreducible component whose general element is a pair of a curve  $C$  and a very ample line bundle  $L$  such that  $h^0(C, L) = r+1$ , then for the dimension of the Zariski tangent space to  $\mathcal{W}$  at  $(C, L)$  we have*

$$\dim T_{(C,L)}\mathcal{W} = 3g - 3 + \rho(d, g, r) + h^1(C, N_{C, \mathbb{P}^r}),$$

*where  $N_{C, \mathbb{P}^r}$  is the normal sheaf of  $C \hookrightarrow \mathbb{P}^r$  of the embedding induced by  $L$ .*

Next we recall a result of Ein, which is in fact the technical crux of his proof of Severi’s claim for  $r = 3$  and 4.

**Proposition 1.3.** *Let  $C$  be a smooth curve and  $L$  a very ample line bundle on  $C$  of degree  $d$  and dimension  $r \geq 3$ . Consider the embedding  $C \hookrightarrow \mathbb{P}^r$  induced by  $L$ . Denote  $\delta := h^1(C, L)$ . Then*

- (1) *if  $\delta \leq 2$ , then  $h^1(C, N_{C, \mathbb{P}^r}) = 0$ ;*
- (2) *if  $\delta \geq 2$ , then  $h^1(C, N_{C, \mathbb{P}^r}) \leq (r - 2)(\delta - 2)$ .*

We will also need an estimate of the dimension of a component  $\mathcal{W}$  of  $\mathcal{W}_d^r$  used in Keem-Kim’s proof of the irreducibility of  $H_{d,g,3}$ .

**Proposition 1.4.** *If  $\mathcal{G}' \subset \mathcal{G}_d^r$ , with  $r \geq 2$ , is a closed subvariety whose general element carries a special birationally very ample linear series, then*

$$\dim \mathcal{G}' \leq 3d + g - 4r - 1.$$

## 2. PRELIMINARY RESULTS

First we give an upper bound for the dimension of an irreducible component  $\mathcal{W} \subset \mathcal{W}_d^r$ .

**Proposition 2.1.** *Let  $d, g$  and  $r$  be positive integers such that  $2 \leq r$ ,  $0 < d \leq g + r - 2$  and let  $\mathcal{W}$  be an irreducible component of  $\mathcal{W}_d^r$ . Let  $b$  be the degree of the base locus of the linear series  $|D|$  on  $C$ , for a general element  $(C, |D|) \in \mathcal{W}$ . Assume also that for general  $(C, |D|) \in \mathcal{W}$  the curve  $C$  is not hyperelliptic. If the moving part of  $|D|$  is*

- (a.1) *very ample and  $r \geq 3$ , then  $\dim \mathcal{W} \leq 3d + g + 1 - 5r - 2b$ ;*
- (a.2) *birationally very ample, then  $\dim \mathcal{W} \leq 3d + g - 1 - 4r - 2b$ ;*
- (b) *compounded, then  $\dim \mathcal{W} \leq 2g - 1 + d - 2r$ .*

*Proof.* (a.1) Consider first the projection  $\mathcal{W} \times_{\mathcal{M}} \mathcal{C}_b \rightarrow \mathcal{W}_{d-b}^{r-b}$  given by

$$((C, |D|), (P_1 + P_2 + \dots + P_b)) \mapsto ((C, |D - (P_1 + P_2 + \dots + P_b)|).$$

There exists a component  $\mathcal{W}_b \subset \mathcal{W}_{d-b}^{r-b}$  whose general element carries a basepoint free linear series. For its dimension we have  $\dim \mathcal{W}_b = \dim \mathcal{W} - b$ , where  $b$  is the dimension of the fiber. Now the dimension of  $\dim \mathcal{W}_b$  is estimated using Proposition 1.2 and Proposition 1.3. Consider a general element  $l_b := (C, |D|) \in \mathcal{W}_b$ , so the linear series  $|D|$  of degree  $d - b$  is very ample on  $C$ . Then

$$\dim \mathcal{W}_b \leq \dim T_l \mathcal{W}_b = 3g - 3 + \rho(d - b, g, r) + h^1(C, N_{C, \mathbb{P}^r}).$$

By Proposition 1.3,  $h^1(C, N_{C, \mathbb{P}^r}) \leq (r - 2)(g - d + b + r - 2)$ , which yields the desired inequality.

(a.2) Similar to (a.1), but we apply Proposition 1.4 instead.

(b) Consider a general  $l := (C, |D|) \in \mathcal{W}$ , and assume that  $|D|$  is a compounded linear series on  $C$ . This means that the morphism  $\Phi_{|D|} : C \rightarrow \Gamma = \Phi_{|D|}(C)$  is of degree  $n \geq 2$ . Let  $\gamma$  be the geometric genus of  $\Gamma$  and consider first the case  $\gamma \geq 1$  (in fact, we can assume that  $\Gamma$  is smooth). Denote by  $\mathcal{X}_{n,\gamma}$  the set of points in moduli space  $\mathcal{M}_g$  representing smooth curves which are  $n : 1$  covers of smooth curves of genus  $\gamma$ . By the well-known Riemann’s moduli count for  $\gamma \geq 1$ ,

$$\dim \mathcal{X}_{n,\gamma} \leq 2g - 2 + (2n - 3)(1 - \gamma).$$

According to H.Martens’s theorem [ACGH, IV.5], the dimension of the fiber  $W_d^r(C)$  of  $\mathcal{W}$  over a point  $C \in \mathcal{X}_{n,\gamma}$  does not exceed  $d - 2r$ ; therefore for  $\gamma \geq 1$  we find

$$\begin{aligned} \dim \mathcal{W} &\leq \mathcal{X}_{n,\gamma} + \dim W_b^r(C) \\ &\leq 2g - 2 + (2n - 3)(1 - \gamma) + d - 2r \\ &< 2g - 1 + d - 2r. \end{aligned}$$

If  $\gamma = 0$ ,  $C$  is an  $n$ -sheeted covering of  $\mathbb{P}^1$ , and the complete linear series  $g_{d-b}^r$  is transformed into a complete linear series  $g_{\frac{d-b}{n}}^r = rg_1^1$  on  $\mathbb{P}^1$ . Pulling it back, we get an injective map  $W_{d-b}^r \rightarrow W_n^1$ . Since  $\dim W_n^1 \leq \dim \mathcal{G}_n^1 = 2g + 2n - 5$  and  $n = \frac{d-b}{r}$ , we find that

$$\dim \mathcal{W} \leq \dim \mathcal{G}_n^1 + b = 2g - 5 + 2\frac{d}{r} - 2\frac{b}{r} + b.$$

By Clifford’s inequality  $0 \leq b \leq d - 2r - 1$ , and since  $-2\frac{b}{r} + b$  is a non-decreasing function in  $b$  for  $r \geq 2$ , we get

$$\begin{aligned} \dim \mathcal{W} &\leq 2g - 5 + 2\frac{d}{r} - 2\frac{d - 2r - 1}{r} + d - 2r - 1 \\ &= 2g - 2 + (d - 2r) + \frac{2}{r} \\ &\leq 2g - 1 + (d - 2r). \end{aligned}$$

This completes the proof of the proposition. □

The next lemma deals with a specific situation which occurs in the course of the proof.

**Lemma 2.2.** *Let  $\mathcal{W} \subset \mathcal{W}_e^s$ ,  $0 < e < 2g - 2$ , be an irreducible component whose general element  $l \in \mathcal{W}$  represents a birationally very ample but not very ample line bundle  $L$  on a curve  $C$ , and assume further that the moving part of its dual, i.e.  $K_C \otimes L^{-1}$ , is birationally very ample. Consider the open subset  $\mathcal{V} \subset \mathcal{W}$  consisting of  $(C, L) \in \mathcal{W}$  for which  $L$  is basepoint free, birationally very ample on  $C$  and  $h^0(C, L) = s + 1$ . Since for  $(C, L) \in \mathcal{V}$  the line bundle  $L$  cannot be very ample, there exist points  $p, p' \in C$  such that  $(C, L(-p - p')) \in \mathcal{W}_{e-2}^{s-1} = \mathcal{W}_1 \cup \dots \cup \mathcal{W}_m$ , where  $\mathcal{W}_1, \dots, \mathcal{W}_m$  are the irreducible components of  $\mathcal{W}_{e-2}^{s-1}$ . Let  $X \subset \mathcal{V} \times_{\mathcal{M}} \mathcal{C}_2$  be the subset defined as  $X := \{(L, p + q) \in \mathcal{V} \times_{\mathcal{M}} \mathcal{C}_2 \mid h^0(C, L(-p - q)) \geq s\}$  and consider the projection*

$$\begin{aligned} X &\rightarrow \mathcal{W}_{e-2}^{s-1}, \\ (C, L; p + p') &\mapsto (C, L(-p - p')). \end{aligned}$$

Let  $\mathcal{W}_1$  be the component of  $\mathcal{W}_{e-2}^{s-1}$ , where the elements of  $X$  are generically projected. Then

$$\dim \mathcal{W} = \dim \mathcal{V} \leq \dim \mathcal{W}_1.$$

*Proof.* For the proof it is enough to show that for any  $(C, L) \in \mathcal{V}$  and  $p, p' \in C$  such that  $(C, L(-p - p')) \in \mathcal{W}_1$ , there are at most finitely many  $(C, M) \in \mathcal{V}$  and  $q, q' \in C$  such that

$$L(-p - p') \cong M(-q - q').$$

Assume that  $(C, L; p + p'), (C, M; q + q') \in X$  have the above property. A simple calculation gives

$$\begin{aligned} h^0(C, K_C \otimes L^{-1}(p + p' - q - q')) &= h^0(C, K_C \otimes M^{-1}) \\ &= h^0(C, K_C \otimes L^{-1}) \\ &= h^0(C, K_C \otimes L^{-1}(p + p')) - 1. \end{aligned}$$

By assumption, the moving part of  $K_C \otimes L^{-1}$  is birationally very ample, hence the moving part of  $K_C \otimes L^{-1}(p + p')$  will also be birationally very ample. Therefore, for the given  $C$  there exist at most finitely many pairs  $(q, q')$  such that the above equality holds. But giving such a pair  $(q, q')$  determines the line bundle  $M$  up to isomorphism by

$$L(-p - p' + q + q') \cong M,$$

and it follows that there are at most finitely many line bundles  $M$  with this property. This implies  $\dim \mathcal{W} = \dim \mathcal{V} = \dim X \leq \dim \mathcal{W}_1$ .  $\square$

We will also use the next lemma.

**Lemma 2.3.** *Let  $g_d^r$  be a birationally very ample linear series of degree  $d \geq g$  on a smooth curve of genus  $g$ . Then*

$$r \leq \frac{1}{3}(2d - g + 1).$$

*Proof.* The lemma is obtained using Castelnuovo's inequality

$$g \leq \binom{m}{2}(r - 1) + m\varepsilon,$$

where  $m := \lfloor \frac{d-1}{r-1} \rfloor$  and  $\varepsilon = d - 1 - m(r - 1)$ .  $\square$

### 3. IRREDUCIBILITY OF $H_{g,g,3}$ , $H_{g+3,g,4}$ AND $H_{g+2,g,4}$

**Theorem 3.1.**  *$H_{g,g,3}$  is irreducible if  $g \geq 13$ .*

*Proof.* As was pointed out in the beginning, it is sufficient to prove that there is a unique irreducible component  $\mathcal{G}' \subset \mathcal{G}_g^3$  whose general element carries a very ample linear series. The existence of such a component follows by Proposition 1.1. Further, by Proposition 1.2 we have  $\dim \mathcal{G}' \geq 3g - 3 + \rho(g, g, 3)$ . Let  $r$  be the dimension  $r = h^0(C, |\mathcal{D}|) - 1$ , where  $(C, \mathcal{D}) \in \mathcal{G}'$  is a general element, and let  $\mathcal{W}' \subset \mathcal{W}_g^r$  be the component whose general elements contain the completions  $|\mathcal{D}|$ . Thus,  $\dim \mathcal{G}' = \dim \mathcal{W}' + \dim \text{Gr}(4, r + 1)$ , and by Proposition 2.1 we find

$$4g - 15 = 3g - 3 + \rho(g, g, 3) \leq \dim \mathcal{G}' \leq 4g - r - 11,$$

which implies

$$(1) \quad r \leq 4.$$

Let  $\mathcal{W}_0 \subset \mathcal{W}_{g-2}^{r-1}$  be the dual of  $\mathcal{W}'$  in the Picard scheme  $\text{Pic}C$ .

The curve  $C$  in the general element  $(C, K_C(-\mathcal{D}))$  of  $\mathcal{G}'$  cannot be hyperelliptic since a hyperelliptic curve cannot have a very ample special linear series. Then it is easy to see that the linear series of the general element of  $\mathcal{W}_0$  is birationally very

ample, after the clearing of a possible base part. If we assume the opposite we find by Proposition 2.1 that

$$\begin{aligned} 4g - 15 &\leq \dim \mathcal{W}' + 4(r - 3) \\ &= \dim \mathcal{W}_0 + 4(r - 3) \\ &\leq 2g - 1 + ((g - 2) - 2(r - 1)) + 4(r - 3) \\ &= 3g + 2r - 13, \end{aligned}$$

and this implies  $g \leq 2r + 2$ . The last is impossible due to (1) and the assumption  $g \geq 13$ . This means that the moving part of the linear series in a general element  $(C, |D|) \in \mathcal{W}_0$  is birationally very ample. Let  $b$  be the degree of its base locus. Again applying Proposition 2.1, we obtain

$$4g - 15 \leq \dim \mathcal{W}_0 + 4(r - 3) \leq 4g - 15 - 2b,$$

which implies  $b = 0$ , i.e.  $|D|$  is basepoint free and also

$$\dim \mathcal{W}' = \dim \mathcal{W}_0 = 4g - 4r - 3.$$

Now I claim that we can only have  $r = 3$ . Indeed, by inequality (1) it is enough to check that  $r \neq 4$  since  $r \geq 3$  by default. So, assume that  $r = 4$ . Then

$$\dim \mathcal{W}_0 = 4g - 4r - 3 = 4g - 19.$$

For a general  $(C, L) \in \mathcal{W}_0$  the line bundle  $L$  cannot be very ample, because if it were very ample, then applying Proposition 2.1 we would get

$$\dim \mathcal{W}_0 \leq 3(g - 2) + g + 1 - 5(4 - 1) = 4g - 20,$$

which is impossible. Thus, for a general  $(C, L) \in \mathcal{W}_0$  the line bundle  $L$  is basepoint free and birationally very ample but not very ample on  $C$ . Let  $\mathcal{W}_1 \subset \mathcal{W}_{g-4}^2$  be a component chosen like the component  $\mathcal{W}_1$  in Lemma 2.2. Therefore,

$$4g - 19 = \dim \mathcal{W}_0 \leq \dim \mathcal{W}_1.$$

Now we estimate the dimension of  $\mathcal{W}_1$ . If its general element has a compounded linear series, applying Proposition 2.1 we get

$$4g - 19 \leq \dim \mathcal{W}_1 \leq 2g - 1 + g - 4 - 4 = 3g - 9,$$

which is impossible due to the assumption  $g \geq 13$ . So, it remains for its general element to carry a birationally very ample linear series, but then by the same proposition we must have

$$4g - 19 \leq \dim \mathcal{W}_1 \leq 3(g - 4) + g - 8 - 1 = 4g - 21,$$

which is again a contradiction. This proves that  $r = 3$  is the only possibility.

Now we can complete the proof of the theorem. For any component  $\mathcal{G}' \subset \mathcal{G}_g^3$  whose general element carries a very ample linear series, we have established that:

- its general element has linear series that is necessarily complete;
- the general element of its dual  $\mathcal{W}_0 \subset \mathcal{W}_{g-2}^2$  carries a basepoint free and birationally very ample linear series.

Let  $\mathcal{G}$  be the union of components of  $\mathcal{G}_{g-2}^2$  whose general element has a birationally very ample linear series. By Proposition 1.2,  $\dim \mathcal{G} = 4g - 15$ , and since  $\dim \mathcal{W}_0 = 4g - 15$ , it turns out that  $\mathcal{W}_0$  is an irreducible component of  $\mathcal{G}$ . But since  $\rho(g - 2, g, 2) = \rho(g, g, 3) \geq 1$ , as we assume  $g \geq 13$ , it follows again by Proposition 1.2 that  $\mathcal{G}$  is irreducible. On its turn this means that the union of irreducible

components of  $\mathcal{G}_g^3$  whose general element has a very ample linear series must be irreducible. Therefore  $H_{g,g,3}$  is irreducible.  $\square$

**Theorem 3.2.** (a)  $H_{g+3,g,4}$  is irreducible if  $g \geq 5$ .  
 (b)  $H_{g+2,g,4}$  is irreducible if  $g \geq 11$ .

*Proof.* (a) For the first part it is sufficient to check that  $\mathcal{G}_{g+3}^4$  has a unique component whose general element represents a curve and a very ample linear series on it. Let  $\mathcal{G}' \subset \mathcal{G}_{g+3}^4$  be a component with this property; the existence of such components follows from Proposition 1.1. For its dimension we have

$$\dim \mathcal{G}' \geq 3g - 3 + \rho(g + 3, g, 4) = 4g - 8.$$

I claim now that the linear series in a general element of  $\mathcal{G}'$  is complete. To see this denote by  $\mathcal{W}' \subset \mathcal{W}_{g+3}^r$  the component containing the completed linear series  $|\mathcal{D}|$  for  $(C, \mathcal{D}) \in \mathcal{G}'$ , i.e.  $\mathcal{G}'$  is generically a Grassmannian fiber bundle over  $\mathcal{W}'$  with fiber  $\text{Gr}(5, r + 1)$ , where  $r = h^0(C, |\mathcal{D}|) - 1$ . We need to check that  $r = 4$ . Assume that  $r \geq 5$ . Then for a general  $l := (C, |\mathcal{D}|) \in \mathcal{W}'$  the linear series  $|\mathcal{D}|$  is very ample on  $C$ , and using Proposition 2.1 we obtain

$$4g - 8 \leq \dim \mathcal{G}' = \dim \mathcal{W}' + 5(r - 4) \leq 4g - 10,$$

which is impossible.

Assume that  $\mathcal{G}' \subset \mathcal{G}_{g+3}^4$  is a component different from the principal component  $\mathcal{G}_0$  dominating the moduli space. This implies for the image  $\pi(\mathcal{G}') \subset \mathcal{M}_g$  of the forgetful map  $\pi : (C, \mathcal{D}) \mapsto C$  that  $\dim \pi(\mathcal{G}') \leq 3g - 4$ . From

$$3g - 3 + \rho(g + 3, g, 4) \leq \dim \mathcal{G}'$$

we obtain for the dimension of the fiber  $\pi^{-1}(C)$  over general  $C \in \pi(\mathcal{G}')$  that  $\dim \pi^{-1}(C) = \dim G_{g+3}^4(C) \geq \rho(g + 3, g, 4) + 1$ . By [ACGH, Ch. IV., Theorem 4.1]

$$\dim G_{g+3}^4(C) \leq \dim T_{\mathcal{D}}G_{g+3}^4(C) = \rho(g + 3, g, 4) + \dim \ker \mu_0(C),$$

where  $\mu_0(C)$  is the cup-product map

$$\mu_0(C) : H^0(C, \mathcal{D}) \otimes H^0(C, K_C(-\mathcal{D})) \rightarrow H^0(C, K_C).$$

Since the general  $\mathcal{D} \in G_{g+3}^4(C)$  is complete, we find  $h^0(C, K(-\mathcal{D})) = 1$ , and therefore the map  $\mu_0(C)$  is injective, i.e.  $\dim \ker \mu_0(C) = 0$ . This implies

$$\rho(g + 3, g, 4) + 1 \leq \dim G_{g+3}^4(C) \leq \rho(g + 3, g, 4) + \dim \ker \mu_0(C) = \rho(g + 3, g, 4),$$

which is a contradiction. This means that  $\mathcal{G}'$  must coincide with  $\mathcal{G}_0$ . This completes the proof of (a).

(b) Just like in the proofs of the preceding irreducibility statements,  $\mathcal{G}_{g+2}^4$  contains a principal component  $\mathcal{G}_0$  whose general element represents a curve and a complete very ample linear series on it. Assume that  $\mathcal{G}'$  is another component of  $\mathcal{G}_{g+2}^4$  for whose general element  $(C, \mathcal{D}) \in \mathcal{G}'$  the linear series  $\mathcal{D}$  is very ample on  $C$ . Let  $r$  be the dimension of the completion  $\mathcal{D}$ , and let  $\mathcal{W}' \subset \mathcal{W}_{g+2}^r$  be the component whose general elements are the completions  $(C, |\mathcal{D}|)$ . Thus,  $\dim \mathcal{G}' = \dim \mathcal{W}' + \dim \text{Gr}(5, r + 1)$ . Using Proposition 2.1 we find

$$4g - 13 \leq \dim \mathcal{G}' = \dim \mathcal{W}' + \dim \text{Gr}(5, r + 1) \leq 4g - 13,$$

which implies

$$(2) \quad \dim \mathcal{W}' = 4g - 5r + 7.$$

As we want to dismiss the existence of  $\mathcal{G}'$ , we will show first that  $r \geq 5$ , and then we will see that the latter also leads to a contradiction, proving in this way that  $\mathcal{G}_0 \subset \mathcal{G}_{g+2}^4$  is the only component whose general element carries a very ample series. Denote by  $\mathcal{W}_0 \subset \mathcal{W}_{g-4}^{r-3}$  the dual of the  $\mathcal{W}'$  variety in the Picard scheme.

I claim that  $r \geq 5$ . To see this assume the opposite, i.e.  $r = 4$ . Then  $\mathcal{G}' \equiv \mathcal{W}' \subset \mathcal{W}_{g+2}^4$ , and the dimension of the dual  $\mathcal{W}_0 \subset \mathcal{W}_{g-4}^1 \subset \mathcal{G}_{g-4}^1$  is  $\dim \mathcal{W}_0 = \dim \mathcal{W}' = 4g - 13 = \dim \mathcal{G}_{g-4}^1$ . As has been pointed out, the linear series in a general element of the principal component  $\mathcal{G}_0$  is complete, and therefore for the dual  $\mathcal{X}_0$  of  $\mathcal{G}_0$  we have  $\mathcal{X}_0 \subset \mathcal{G}_{g-4}^1$  and  $\dim \mathcal{X}_0 = \dim \mathcal{G}_0 = 4g - 13 = \dim \mathcal{G}_{g-4}^1$ . But according to Theorem 1.2, the variety  $\mathcal{G}_{g-4}^1$  is smooth and irreducible of dimension  $4g - 13$ , which means that  $\mathcal{W}_0$  and  $\mathcal{X}_0$  must coincide. Hence, their dual components  $\mathcal{G}'$  and  $\mathcal{G}_0$  also coincide. Therefore  $r \geq 5$ .

I claim now that the general element of  $\mathcal{W}_0 \subset \mathcal{W}_{g-4}^{r-3}$  cannot be a compounded linear series. To see this, assume the opposite. By (2) and Proposition 2.1, it follows that

$$4g - 5r + 7 = \dim \mathcal{W}' = \dim \mathcal{W}_0 \leq 3g - 2r + 1,$$

which gives  $\frac{g+6}{3} \leq r$ . Then applying Lemma 2.3 for a general element  $(C, g_{g+2}^r) \in \mathcal{W}' \subset \mathcal{W}_{g+2}^r$ , we get

$$r \leq \frac{g+3}{3},$$

which is a contradiction.

Thus, the general element of  $\mathcal{W}_0$  carries a birationally very ample linear series, after clearing a possible base part, and we can apply Proposition 2.1 to obtain

$$4g - 5r + 7 = \dim \mathcal{W}_0 \leq 3(g - 4) + g - 4(r - 3) - 1 = 4g - 4r - 1.$$

This gives  $r \geq 8$ .

It is easy to see that the moving part of the linear series in a general element of  $\mathcal{W}_0$  cannot be very ample, since if it were very ample, applying Proposition 2.1 for  $\mathcal{W}_0 \subset \mathcal{W}_{g-4}^{r-3}$  we would get

$$4g - 5r + 7 = \dim \mathcal{W}_0 \leq 3(g - 4) + g + 1 - 5(r - 3) = 4g - 5r + 4,$$

which is absurd.

Further, set  $r_0 = r - 3$  and  $d_0 = g - 4$  and assume that the degree of the base locus of the general element of  $\mathcal{W}_0$  is  $b_0$ . Consider the map

$$\begin{aligned} \mathcal{W}_0 \times_{\mathcal{M}} \mathcal{C}_{b_0} &\rightarrow \mathcal{W}_{d_0-b_0}^{r_0-b_0}, \\ ((C, L), (P_1 + \dots + P_{b_0})) &\mapsto (C, L(-P_1 - \dots - P_{b_0})). \end{aligned}$$

Let  $\mathcal{V}'_0 \subset \mathcal{W}_{d_0-b_0}^{r_0-b_0}$  be the closed subset defined as

$$\mathcal{V}'_0 := \{(C, L) \in \mathcal{W}_{d_0-b_0}^{r_0-b_0} \mid h^0(C, L) \geq r_0 + 1\},$$

and let  $\mathcal{V}_0 \subset \mathcal{V}'_0$  be the component whose general elements are the general elements of  $\mathcal{W}_0$  but without base loci. For the dimensions of  $\mathcal{V}_0$  and  $\mathcal{W}_0$  we have

$$\dim \mathcal{W}_0 \leq \dim \mathcal{V}_0 + b_0.$$

Also, the general element of  $\mathcal{V}_0$  is necessarily base-point free and birationally very ample but not very ample. Consider the mapping

$$\begin{aligned} \mathcal{V}_0 \times_{\mathcal{M}} \mathcal{C}_2 &\rightarrow \mathcal{W}_{d_0-b_0-2}^{r_0-2}, \\ ((C, L), (P + Q)) &\mapsto (C, L(-P - Q)). \end{aligned}$$



Let  $\mathcal{W}_1 \subset \mathcal{W}_{d_0-2-b_0}^{r_0-1}$  be a component of maximal dimension (if there are more than one) whose elements contain the projections of the general elements of  $\mathcal{V}_0$ . Set  $r_1 = r_0 - 1$  and  $d_1 = d_0 - 2 - b_0$ , i.e.

$$\mathcal{W}_1 \subset \mathcal{W}_{d_1}^{r_1}.$$

Remark that its dual  $\mathcal{W}'_1 \subset \mathcal{W}_{g+2+b_0+2}^{r+b_0+1}$  arises by “adding” some effective divisors of degree  $b_0 + 2$  to the linear series of  $\mathcal{W}' \subset \mathcal{W}_{g+2}^r$ , and  $\dim \mathcal{W}'_1 = \dim \mathcal{W}_1$ . Also, by Lemma 2.2 we have

$$\dim \mathcal{W}_0 \leq \dim \mathcal{W}_1.$$

Now we proceed with  $\mathcal{W}_1$  just as we did with  $\mathcal{W}_0$ . If its general element has a very ample or compounded linear series (after clearing base locus), we reach a numerical contradiction using Proposition 2.1. If the moving part is birationally very ample but not very ample, we apply the same procedure to get

$$\mathcal{W}_2 \subset \mathcal{W}_{d_2}^{r_2},$$

where  $r_2 = r_1 - 1$ ,  $d_2 = d_1 - 2 - b_1$ , with  $b_1$  the degree of the base locus. We would like to have just as before

$$\dim \mathcal{W}_1 \leq \dim \mathcal{W}_2.$$

For this we must guarantee that the general element of  $\mathcal{W}'_1$  has a birationally very ample moving part, so we could apply Lemma 2.2. But this is almost immediate: if we suppose the opposite, it follows by Proposition 2.1 that

$$\begin{aligned} \dim \mathcal{W}_1 &= \dim \mathcal{W}'_1 \\ &\leq 2g - 1 + (d_1 - 2r_1), \end{aligned}$$

and this leads to a numerical contradiction due to  $\dim \mathcal{W}_1 \geq \dim \mathcal{W}_0 = 4g - 5r + 7$ .

In this way we can construct  $\mathcal{W}_2, \mathcal{W}_3, \dots$ , unless the process terminates due to entering a case where the general element of  $\mathcal{W}_i$  is very ample or compounded. Precisely, if the general element of  $\mathcal{W}_i$  is compounded, we obtain

$$\begin{aligned} 4g - 5r + 7 &= \dim \mathcal{W}_0 \leq \dim \mathcal{W}_i + B_i \\ &\leq 2g - 1 + (d_i - 2r_i) + B_i \\ &= 2g - 1 + (d_0 - 2i - B_i - 2(r_0 - i)) + B_i \\ &= 2g - 1 + (d_0 - 2r_0), \end{aligned}$$

where  $B_i := b_0 + \dots + b_{i-1}$ . This yields a numerical contradiction just as before. The case in which the general element has a linear series with a very ample moving part is dealt with in a similar way. Remark that, as soon as we construct  $\mathcal{W}_i$ , we get that the moving part of the linear series in a general element of  $\mathcal{W}'_i$  is birationally very ample, which allows us to apply Lemma 2.2 and construct  $\mathcal{W}_{i+1}$ . We continue the inductive process until  $i$  becomes larger than  $\frac{r}{2} - 4$ . Then for  $i = [\frac{r}{2} - 3]$  we can apply Proposition 2.1 for birationally very ample linear series to get

$$\begin{aligned} 4g - 5r + 7 &\leq \dim \mathcal{W}_i + B_i \\ &\leq 3(d_i - b_i) + g - 4r_i - 1 + B_i \\ &= 3(d_0 - 2i - B_{i+1}) + g - 4(r_0 - i) - 1 + B_i \\ &\leq 3d_0 + g - 4r_0 - 2i - 1 \\ &\leq 3(g - 4) + g - 4(r - 3) - 2\left(\frac{r-1}{2} - 3\right) - 1 \\ &= 4g - 5r + 6, \end{aligned}$$

which is a contradiction. This precludes the existence of a component  $\mathcal{G}' \subset \mathcal{G}_{g+2}^4$  different from the principal component  $\mathcal{G}_0$ , and completes the proof.  $\square$

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