

ON THE MANIN-MUMFORD CONJECTURE  
FOR ABELIAN VARIETIES WITH A PRIME  
OF SUPERSINGULAR REDUCTION

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ABSTRACT. We give a short proof of the “prime-to- $p$  version” of the Manin-Mumford conjecture for an abelian variety over a number field, when it has supersingular reduction at a prime dividing  $p$ , by combining the methods of Bogomolov, Hrushovski, and Pink-Roessler. Our proof here is quite simple and short, and neither  $p$ -adic Hodge theory nor model theory is used. The observation is that a power of a lift of the Frobenius element at a supersingular prime acts on the prime-to- $p$  torsion points via nontrivial homothety.

Let  $A$  be an abelian variety over a number field  $K$ . For an integer  $n \geq 1$ , let  $[n]: A \rightarrow A$  be the multiplication-by- $n$  isogeny on  $A$ , and  $A[n]$  the kernel of  $[n]$  acting on the set of  $\overline{K}$ -valued points  $A(\overline{K})$  of  $A$ , where  $\overline{K}$  denotes an algebraic closure of  $K$ . Let  $\text{Tor}(A) := \bigcup_{n \geq 1} A[n]$  be the group of torsion points on  $A$ . For a prime number  $p$ , let  $\text{Tor}_p(A) := \bigcup_{n \geq 1} A[p^n]$  (resp.  $\text{Tor}^p(A) := \bigcup_{(n,p)=1} A[n]$ ) denote the subgroup of  $\text{Tor}(A)$  consisting of elements which have  $p$ -power order (resp. order prime to  $p$ ).

The Manin-Mumford conjecture states that, for an irreducible closed subvariety  $X$  of  $A$ , if  $\text{Tor}(A) \cap X$  is Zariski dense in  $X$ , then  $X$  is a translate of an abelian subvariety of  $A$ . For a prime number  $p$ , we can also consider the “ $p$ -primary version” (resp. “prime-to- $p$  version”) by considering  $\text{Tor}_p(A)$  (resp.  $\text{Tor}^p(A)$ ) instead of  $\text{Tor}(A)$ . This conjecture was originally proposed by Lang for curves embedded in their Jacobian varieties ([L1]). A first partial result was obtained by Bogomolov, who proved the “ $p$ -primary version” for each  $p$  using  $p$ -adic Hodge theory ([Bog]). The first full proof was obtained by Raynaud by rigid analytic methods ([Ra1], [Ra2]). Up to now, several different proofs and generalizations are known by many people such as Hindry, Hrushovski, Ullmo, Szpiro, Zhang, Pink, Roessler ([Hi], [Hr], [U], [Z], [PR1], [PR2], [Ro], see also [Tz]).

The aim of this paper is to give a short proof of the “prime-to- $p$  version” of the Manin-Mumford conjecture for an abelian variety over a number field, when it has supersingular reduction at a prime dividing  $p$ , by combining the methods of Bogomolov, Hrushovski, and Pink-Roessler (see Remark 3 for a difference between the proof here and previously known ones).

Recall that an abelian variety  $A$  over a field  $k$  of characteristic  $p > 0$  is *supersingular* if and only if  $A \otimes_k \overline{k}$  is isogenous to a power of an elliptic curve  $E$  over  $\overline{k}$  such

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that  $E(\bar{k})$  has no nontrivial  $p$ -torsion point. There are many other characterizations (e.g. [LO]).

**Theorem 1.** *Let  $A$  be an abelian variety of dimension  $g$  over a number field  $K$ . Assume that  $A$  has supersingular reduction at a prime  $v$  dividing  $p$ . Let  $F$  be the residue field at  $v$ . Let  $X$  be an irreducible closed subvariety of  $A$  defined over  $K$ . If  $\text{Tor}^p(A) \cap X$  is Zariski dense in  $X$ , then  $X$  is a translate of an abelian subvariety of  $A$ , that is,  $X = A' + a$ , where  $A'$  is an abelian subvariety of  $A$  and  $a \in A$ .*

*Proof.* Let  $A_F$  be the reduction of  $A$  at  $v$ , which is a supersingular abelian variety over  $F$ . Let  $q$  be the cardinality of  $F$ , which is a power of  $p$ . Let  $\sigma \in \text{Gal}(\bar{F}/F)$  be the  $q$ -th power Frobenius automorphism of  $\bar{F}$ . For a prime number  $\ell \neq p$ , the characteristic polynomial of  $\sigma$  acting on the  $\ell$ -adic Tate module  $V_\ell A_F := (\varprojlim_n A_F[\ell^n]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$  has coefficients in  $\mathbb{Z}$  and is independent of  $\ell \neq p$ . Let  $\alpha_1, \dots, \alpha_{2g}$  be the eigenvalues  $\sigma$  acting on  $V_\ell A_F$ . It is known that, for each  $i$  and each embedding  $\iota: \mathbb{Q}(\alpha_i) \hookrightarrow \mathbb{C}$ , the complex absolute value of  $\iota(\alpha_i)$  is  $q^{1/2}$  (e.g. [Mi], [Mu]).

Since  $A_F$  is supersingular, we know that  $\alpha_i^2/q$  are algebraic numbers whose absolute values are 1 at all finite and infinite primes. Hence  $\alpha_i^2/q$  are roots of unity (e.g. [L2, Chapter V, §1]). Therefore, there is an integer  $m \geq 1$  such that  $\alpha_1^{2m} = \dots = \alpha_{2g}^{2m} = q^m$ . Since  $\sigma$  acts semisimply on  $V_\ell A_F$  ([Ta]), this means that  $\sigma^{2m} \cdot x = [q^m](x)$  for all  $x \in \text{Tor}^p(A_F)$ . Take a lift  $\tilde{\sigma} \in \text{Gal}(\bar{K}/K)$  of  $\sigma$ . Since  $\text{Tor}^p(A)$  is naturally isomorphic to  $\text{Tor}^p(A_F)$ , we have  $\tilde{\sigma}^{2m} \cdot x = [q^m](x)$  for all  $x \in \text{Tor}^p(A)$ . On the other hand, since  $X$  is defined over  $K$ , the set  $X(\bar{K})$  is stable by the action of  $\text{Gal}(\bar{K}/K)$ . Hence we have  $[q^m](x) \in X$  for all  $x \in \text{Tor}^p(A) \cap X$ . Since  $\text{Tor}^p(A) \cap X$  is Zariski dense in  $X$ , we conclude that  $[q^m](X) = X$ .

Therefore,  $X$  is a translate of an abelian subvariety of  $A$  by the same argument as in [Bog, Théorème 3]. We reproduce the proof here for the completeness of the paper. After replacing  $A$  by its quotient by  $\text{Stab}_A(X) := \{a \in A \mid X + a = X\}$ , we may assume  $\text{Stab}_A(X)$  is trivial. We shall prove  $X$  is a point. Let  $d$  be the dimension of  $X$ , and put  $n := q^m$ . Consider the action of  $[n]$  on the  $\ell$ -adic cycle class  $[X] \in H^{2(g-d)}(A \otimes_K \bar{K}, \mathbb{Q}_\ell(g-d))$  for some  $\ell$ . Since  $[n]: A \rightarrow A$  is an étale Galois covering of  $A$  and the Galois group is  $A[n]$  acting on  $A$  by translation, we have  $[n]^{-1}(X) = \bigcup_{a \in A[n]} (X + a)$ . Since  $\text{Stab}_A(X)$  is trivial, we know that  $X + a$  are different for all  $a \in A[n]$ . Hence we have  $[n]^*[X] = \sum_{a \in A[n]} [X + a] = n^{2g}[X]$ . Since  $[n]^*$  acts on  $H^{2(g-d)}(A \otimes_K \bar{K}, \mathbb{Q}_\ell(g-d))$  via multiplication by  $n^{2(g-d)}$ , we have  $n^{2(g-d)} = n^{2g}$ . Therefore, we have  $d = 0$  and  $X$  is a point.  $\square$

*Remark 2.* By a theorem of Elkies, if  $A$  is isogenous to a power of an elliptic curve and  $K$  admits a real embedding, there are infinitely many primes of supersingular reduction ([E1], [E2]). However, no general result seems known up to now for a general abelian variety over a number field.

*Remark 3.* Here we compare the proof above and the proofs previously obtained by Bogomolov, Hrushovski, and Pink-Roessler. Bogomolov proved, for each  $p$ , that there exists an element  $\sigma \in \text{Gal}(\bar{K}/K)$  which acts on the  $p$ -adic Tate module via multiplication by an integer  $n > 1$  ([Bog, Corollaire 2]). The proof of this fact relies on a study of  $p$ -adic Galois representations which have Hodge-Tate decomposition at a prime dividing  $p$ . It seems very difficult to determine  $\sigma$  and  $n$  explicitly. On the other hand, Hrushovski considered a lift of the Frobenius element at a

prime of good reduction. However, the Frobenius element does not act on the Tate module via homothety in general. Hrushovski overcame this difficulty by using the model theory of difference fields ([Hr]). Pink-Roessler, inspired by Hrushovski's work, obtained a new proof using only classical algebraic geometry ([PR1], see also [PR2]). Their proof is very short and illuminating, but we need a trick for replacing  $A$  by a power of it (see the proof of [PR1, Theorem 1.2]). The observation of this paper is that a power of a lift of the Frobenius element at a supersingular prime acts on the prime-to- $p$  torsion points via nontrivial homothety.

*Remark 4.* It is true but not automatic that a combination of the “ $p$ -primary version” and the “prime-to- $p$  version” implies the full Manin-Mumford conjecture. We need some Galois theoretic results for proving such an implication (see [Ra2, §8.1], see also [Ro]). It seems difficult to obtain the full Manin-Mumford conjecture by simply generalizing the method of this paper because it seems difficult to control the action of the Galois group on  $\mathrm{Tor}_p(A)$ . Note that, in [PR1, §3], to cover all torsion points, Pink-Roessler took lifts of the Frobenius elements at *two* primes of good reduction of different characteristics, and used a deep ramification-theoretic result of Serre. In [Ro], Roessler used a result of Boxall instead of Serre's result. Finally, we note that the idea of finding an element in the Galois group acting on the Tate module via nontrivial homothety to prove the Manin-Mumford conjecture is classical and due to Lang ([L1, §3], see also [Tz, Conjecture 5.1]).

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