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THE SUFFICIENCY OF ARITHMETIC PROGRESSIONS FOR THE $3x+1$ CONJECTURE

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ABSTRACT. Define $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $T(x) = (3x+1)/2$ if x is odd and $T(x) = x/2$ if x is even. The $3x+1$ Conjecture states that the T-orbit of every positive integer contains 1. A set of positive integers is said to be *sufficient* if the T-orbit of every positive integer intersects the T-orbit of an element of that set. Thus to prove the $3x+1$ Conjecture it suffices to prove it on some sufficient set. Andaloro proved that the sets $1+2^n\mathbb{N}$ are sufficient for $n \leq 4$ and asked if $1+2^n\mathbb{N}$ is also sufficient for larger values of n. We answer this question in the affirmative by proving the stronger result that $A+B\mathbb{N}$ is sufficient for any nonnegative integers A and B with $B \neq 0$, i.e. every nonconstant arithmetic sequence forms a sufficient set. We then prove analagous results for the Divergent Orbits Conjecture and Nontrivial Cycles Conjecture.

1. Introduction and statement of main results

A dynamical system on a set X is obtained by iterating a map $f : X \to X$. For any function $f : X \to X$ and any $x \in X$, the *f-orbit* of x is the set $\{f^k(x) : k \in \mathbb{N}\}$, where f^0 is the identity map and $f^k = f \circ f^{k-1}$ for all positive integers k. If $x, y \in X$ whose f-orbits are not disjoint, we will say that x and y *merge* or that x *merges with* y. Thus, x and y merge if and only if there exist some nonnegative integers k and j such that $f^k(x) = f^j(y)$. The relation "merges with" is an equivalence relation on X. A subset $S \subseteq X$ is called *sufficient* for f on X if S meets every equivalence class, i.e. if every element of X merges with some element of S. Since all of the elements in an equivalence class merge, the long-term behavior of their orbits must be identical. Thus to determine the long-term behavior of the orbits of the elements of X, it suffices to determine the long-term behavior of the orbits of the elements of some sufficient set S.

The famous $3x+1$ Conjecture has been an open problem in dynamical systems theory for more than 65 years (see [Lag], [Wir]). The conjecture states that the T-orbit of every positive integer contains 1, where T is given by $T(x) = x/2$ if x is even and $T(x) = (3x+1)/2$ if x is odd. For any positive integer x, if x merges with 1, then $T^k(x) = T^j(1)$ for some nonnegative integers k and j. Since $T^j(1) = 1$ if j is even and $T^j(1) = 2$ if j is odd, either $T^k(x) = 1$ or $T^{k+1}(x) = 1$. In either case the T-orbit of x contains 1. Thus, the $3x+1$ Conjecture is true if and only if every positive integer merges with 1. Therefore, to prove the conjecture it suffices to

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prove it holds on some set S that is sufficient for T on $\mathbb{Z}^+$, since if every element of S merges with 1, then every positive integer also merges with 1. For the remainder of the paper, the term "sufficient" with no qualification will mean "sufficient for T on $\mathbb{Z}^+$".

Andaloro [And] proved that the set $1 + 16\mathbb{N}$ is sufficient. As any superset of a sufficient set is also sufficient, it follows that $1 + 2^n\mathbb{N}$ is sufficient for $n \leq 4$. In the same paper, Andaloro asks if $1 + 2^n\mathbb{N}$ is sufficient for any or all $n > 4$.

All of the sets $1 + 2^n\mathbb{N}$ are members of the more general family of sets of the form $A + B\mathbb{N}$, where A and B are natural numbers and $B \neq 0$. We will call any such set an *arithmetic progression* since it is the set of terms in the nonconstant arithmetic sequence $A, A + B, A + 2B, \ldots$. It is natural to ask which of these sets are sufficient. We answer this question with our main result.

Theorem 1.1. *Every arithmetic progression is sufficient.*

We postpone the proof until Section 3.

There are only two ways that the T-orbit of a positive integer x can fail to contain 1. If x has an unbounded T-orbit, we say the T-orbit of x is divergent. The well-known Divergent Orbits Conjecture states that no positive integer has an unbounded T-orbit. Let S be a set of positive integers with the following property: if no element of S has an unbounded orbit, then the Divergent Orbits Conjecture is true. We say such a set S is *sufficient for the Divergent Orbits Conjecture.*

If $T^k(x) = x$ for some $k > 1$, then we say the T-orbit of x is a *cycle*. The second way the $3x + 1$ Conjecture could be false is if the T-orbit of some positive integer x is a cycle other than the trivial cycle, $\{1, 2\}$. The well-known Nontrivial Cycles Conjecture states that the only cycle of positive integers is the trivial one. As with the previous conjecture, we say S is *sufficient for the Nontrivial Cycles Conjecture* when S has the following property: if no element of S has an orbit that contains a nontrivial cycle, then the Nontrivial Cycles Conjecture is true.

Clearly the truth of these two conjectures would imply the $3x + 1$ Conjecture. Our result carries over to these conjectures.

Corollary 1.2. *Every arithmetic progression is sufficient for the Divergent Orbits Conjecture.*

Corollary 1.3. *Every arithmetic progression is sufficient for the Nontrivial Cycles Conjecture.*

Thus to prove any of the three conjectures, the $3x+1$ Conjecture, the Divergent Orbits Conjecture, or the Nontrivial Cycles Conjecture, it suffices to show that it holds for the elements of some arithmetic progression. Hence, studying the behavior of T on particular arithmetic sequences may lead to a proof of the conjecture itself.

Though not directly applicable to the $3x + 1$ Conjecture itself, the arguments in this paper can be used to show that every negative arithmetic progression (the negation of all terms in an arithmetic progression) is sufficient for T on $\mathbb{Z}^-$. Additionally, the union of an arithmetic progression, a negative arithmetic progression, and $\{0\}$ forms a sufficient set for T on $\mathbb{Z}$. The proofs of these cases are omitted, as they are identical to the proof of Theorem 1.1.

In the next section we provide previously known facts and notation used in our proofs.

2. Background information and notation

For convenience, we include here notation and results from Wirsching's book [Wir] and other sources that will be used in the proofs of our results.

For any $a, b, c \in \mathbb{Z},\ c > 0$, we will write $a \underset{c}{\equiv} b$ as an abbreviation for $a = b \pmod{c}$.

Define the set of *feasible vectors* to be $\mathcal{F} = \bigcup_{k=0}^{\infty} \mathbb{N}^{k+1}$. Let $s \in \mathcal{F}$. Then $s = (s_0, s_1, ..., s_k)$ for some nonnegative integers k and $s_0, s_1, ..., s_k$. The *length* of s, written $l(s)$, is k. The *norm* of s, written $||s||$, is $l(s) + \sum_{i=0}^{l(s)} s_i$.

In our proof, it will be essential to consider the inverses of the piecewise components of T. The even and odd components are denoted $T_0(x) = x/2$ and $T_1(x) = (3x+1)/2$ respectively, giving $T_0^{-1}(x) = 2x$ and $T_1^{-1}(x) = 2 \cdot 3^{-1}x - 3^{-1}$ as functions on the rational numbers. For $s \in \mathcal{F}$ with $s = (s_0, s_1, ..., s_k)$, Wirsching calls the function $v_s : \mathbb{Z}^+ \to \mathbb{Q}$ given by

$$v_s = T_0^{-s_0} \circ T_1^{-1} \circ T_0^{-s_1} \circ T_1^{-1} \circ \ldots \circ T_1^{-1} \circ T_0^{-s_k}$$

a *back-tracing function*. If $v_s(x) \in \mathbb{Z}^+$, then we say s is an *admissible vector* for x. Define $\mathcal{E}(x) = \{s \in \mathcal{F} : s \text{ is admissible for } x\}$. Wirsching proves a lemma that justifies calling v_s the back-tracing function.

Lemma 2.1 ([Wir, Lemma 2.17])**.** *If $s \in \mathcal{E}(x)$, then $T^{||s||}(v_s(x)) = x$.*

Wirsching also works out a convenient explicit formula for v_s in terms of the entries of s.

Lemma 2.2 ([Wir, Lemma 2.13])**.** *Let $s \in \mathcal{F}$. Define*

$$c(s) = \frac{2^{||s||}}{3^{l(s)}}$$

and

$$r(s) = \sum_{j=0}^{l(s)-1} 2^{j+s_0+s_1+\ldots+s_j} 3^{-(j+1)}.$$

Then $v_s(x) = c(s)x - r(s)$.

It is also useful to concatenate smaller vectors into larger vectors. Let $s = (s_0, s_1, s_2, ..., s_k)$ and $t = (t_0, t_1, t_2, ..., t_m)$. Define $s \cdot t = (s_0, s_1, s_2, ..., s_{k-1}, s_k + t_0, t_1, t_2, ..., t_m)$. Then $v_{s \cdot t}$ represents the function that back-traces first along t and then along s, as indicated by the following lemma.

Lemma 2.3 ([Wir, Corollary 2.10])**.** *For any $s, t \in \mathcal{F}$, $v_{s \cdot t} = v_s \circ v_t$.*

In contrast to concatenation, it is also useful to back-trace through part of a longer back-tracing vector. Define a *terminal part* of a vector $s = (s_0, s_1, s_2, ..., s_k)$ to be any vector of the form $(s_j, s_{j+1}, ..., s_k)$ for $j \in \{0, 1, ..., k\}$. Wirsching proves a lemma regarding these.

Lemma 2.4 ([Wir, Lemma 2.16])**.** *Let $s \in \mathcal{F}$ and let $x \in \mathbb{Z}^+$. If $s \in \mathcal{E}(x)$, then for any terminal part t of s, $t \in \mathcal{E}(x)$.*

Additionally, a condition relating divisibility by 3 and back-tracing will be needed.

Lemma 2.5 ([Wir, Lemma 3.4]). *Let $x \in \mathbb{Z}^+$. Then $x \underset{3}{\not\equiv} 0$ if and only if there exists some $s \in \mathcal{E}(x)$ with $l(s) \geq 1$.*

Note that T_0 and T_1 are both nonconstant linear functions and therefore are bijections on $\mathbb{R}$. Thus, T_0, T_1, and their inverses generate a group of real-valued functions. This group is described by Misiurewicz and Rodrigues [MR].

Lemma 2.6 ([MR, Theorem 5.1]). *Let G be the group of real-valued functions generated by T_0, T_1, T_0^{-1}, and T_1^{-1}. Then*

$$G = \left\{ h : h(x) = 2^n 3^m x + \frac{k}{2^i 3^j} \text{ for some } n, m, k \in \mathbb{Z} \text{ and } i, j \in \mathbb{N} \right\}.$$

Finally, the following number theoretic result follows from arguments given in [Ber].

Lemma 2.7. *For every $n \in \mathbb{N}$, for every odd $\tau \in \mathbb{N}$, and for every $a \in \mathbb{Z}$, there exists an increasing (possibly empty) sequence of natural numbers $t_1, t_2, ..., t_h$ with $t_h < n$ such that $\sum_{i=1}^{h} 2^{t_i} \tau^i \underset{2^n}{\equiv} a$.*

3. Proof

Much of the proof will consist of studying the actions of functions T_0 and T_1 on $\mathbb{Z}/b\mathbb{Z}$, where b is a positive integer relatively prime to 2 and 3. Let b be such an integer and let $\mathbb{Z}_b = \mathbb{Z}/b\mathbb{Z}$. Note that a function of the form

$$h(x) = 2^n 3^m x + k \tag{3.1}$$

induces a well-defined permutation of $\mathbb{Z}_b$ for any $n, m, k \in \mathbb{Z}$. For any such function h, define $\overline{h}$ to be the corresponding permutation of $\mathbb{Z}_b$. We adopt the usual convention of using x to stand for the congruence class of the integer x in expressions such as $\overline{h}(x)$. For example, $\overline{T_0} : \mathbb{Z}_b \to \mathbb{Z}_b$ and $\overline{T_1} : \mathbb{Z}_b \to \mathbb{Z}_b$ are given by $\overline{T_0}(x) = 2^{-1}x$ and $\overline{T_1}(x) = 2^{-1}3x + 2^{-1}$, and both are in the form of (3.1). Note also that if h and g are two functions in the form of (3.1), then $h \circ g$ is also of the same form, and $\overline{h \circ g} = \overline{h} \circ \overline{g}$. We classify the group generated by these two permutations for a given b.

Lemma 3.1. *Let b be a positive integer relatively prime to 2 and 3. Let G_b be the group of permutations on $\mathbb{Z}_b$ generated by $\overline{T_0}$ and $\overline{T_1}$. Then*

$$G_b = \left\{ \overline{h} : \ h(x) = 2^n 3^m x + k \ \text{ for some } n, m, k \in \mathbb{Z} \right\}.$$

Proof. Define

$$M = \left\{ \overline{h} : \ h(x) = 2^n 3^m x + k \ \text{ for some } n, m, k \in \mathbb{Z} \right\}$$

and let $H \in M$. Then $H = \overline{h}$ for some h such that $h(x) = 2^n 3^m x + k$ for some $n, m, k \in \mathbb{Z}$. By Lemma 2.6 h is some composition of T_0, T_1, T_0^{-1}, and T_1^{-1}. Thus there exist $e_1, e_2, \ldots, e_g \in \mathbb{Z}$ such that $h = T_1^{e_g} \circ T_0^{e_{g-1}} \circ \ldots \circ T_1^{e_4} \circ T_0^{e_3} \circ T_1^{e_2} \circ T_0^{e_1}$.

Since the set of permutations on $\mathbb{Z}_b$ is a finite group, there exist $\sigma_0, \sigma_1 \in \mathbb{Z}^+$ such that $\overline{T_0}^{\sigma_0} = \overline{T_0}$ and $\overline{T_1}^{\sigma_1} = \overline{T_1}$. Thus by increasing e_i by a multiple of σ_0 for odd i and increasing e_i by a multiple of σ_1 for even i as necessary, we can create a sequence of positive integers $e'_1, e'_2, \ldots, e'_g$ such that $\overline{h} = \overline{T_1}^{e'_g} \circ \overline{T_0}^{e'_{g-1}} \circ \ldots \circ \overline{T_1}^{e'_4} \circ \overline{T_0}^{e'_3} \circ \overline{T_1}^{e'_2} \circ \overline{T_0}^{e'_1}$. Therefore $\overline{h} \in G_b$, so $H \in G_b$. Thus $M \subseteq G_b$.

Conversely, since b is relatively prime to 2 and 3, both 2^{-1} and 3^{-1} are elements of $\mathbb{Z}_b$. Thus, the generators $\overline{T_0}(x) = 2^{-1}x = 2^{-1}3^0x + 0$ and $\overline{T_1}(x) = 2^{-1}3^1x + 2^{-1}$ are both elements of M. Therefore $\left(\overline{T_0} \circ H\right)(x) = 2^{n-1}3^mx + 2^{-1}k$ and $\left(\overline{T_1} \circ H\right)(x) = 2^{n-1}3^{m+1}x + \left(2^{-1}3k + 2^{-1}\right)$, which are also both elements of M. Thus M contains the generators of G_b and is closed under composition with respect to these generators. Hence $G_b \subseteq M$, and we are done. □

We now present an outline of the proof. In order to prove that any arithmetic progression is sufficient, we choose an arbitrary arithmetic progression $A + B\mathbb{N}$ and an arbitrary positive integer x and show that x merges with an element $a \in A + B\mathbb{N}$. This is done in three steps. In the first step (Lemma 3.2), we find a particular k such that we can back-trace from $T^k(x)$ to a. In the second step (Lemma 3.6), we use $T^k(x)$ as a stepping-stone to show that x merges with a number z that is congruent to 0 modulo 2^nb, where $B = 2^n3^mb$ and b is relatively prime to 2 and 3. For the third and final step (Lemma 3.8), we use z as the next stepping-stone to show that x merges with a.

To begin the first step, we observe that in order to back-trace from a number with admissible vectors of positive length, it must not be divisible by 3 by Lemma 2.5.

Lemma 3.2. *For every $x \in \mathbb{Z}^+$, there exists $k \in \mathbb{N}$ such that $T^k(x) \underset{3}{\not\equiv} 0$.*

Proof. Let $x \in \mathbb{Z}^+$. If $x \underset{3}{\not\equiv} 0$, then we are done, since $T^0(x) = x$.

Assume $x \underset{3}{\equiv} 0$. Let k be one more than the exponent of the largest 2-power that divides x. Clearly $T^{k-1}(x) = 2i + 1$ for some $i \in \mathbb{N}$. Thus

$$\begin{aligned}
T^k(x) &= T\left(T^{k-1}(x)\right) \\
&= T(2i+1) \\
&= \frac{3(2i+1)+1}{2} \\
&= 3i + 2 \\
&\underset{3}{\not\equiv} 0.
\end{aligned}$$

□

Thus any positive integer has an iterate that is not divisible by 3. We now present three technical lemmas that will be used in step two. We start by reproving a stronger version of [Wir, Lemma 3.1].

Lemma 3.3. *For any $s \in \mathcal{F}$, there exists $q_0 \in \mathbb{N} - 3\mathbb{N}$ such that for any $q \in \mathbb{Z}^+$,*

$$s \in \mathcal{E}(q) \Leftrightarrow q \underset{3^{l(s)}}{\equiv} q_0.$$

Proof. Let $s \in \mathcal{F}$. We have that $s = \left(s_0, s_1, \ldots, s_{l(s)}\right)$ for some natural numbers $s_0, s_1, \ldots, s_{l(s)}$. Let $q \in \mathbb{Z}^+$. If $l(s) = 0$, then $v_s(q) = 2^{s_0}q \in \mathbb{Z}^+$ so that $s \in \mathcal{E}(q)$. Thus the theorem is satisfied trivially in this case.

Assume $l(s) \geq 1$. There exists $t \in \mathbb{N}$ such that $2t \underset{3^{l(s)}}{\equiv} 1$ since 2 is relatively prime to $3^{l(s)}$. Let $w = 3^{l(s)} r(s)$. By the definition of $r(s)$, we have

$$
\begin{aligned}
w &= 3^{l(s)} \sum_{j=0}^{l(s)-1} 2^{j+s_0+s_1+\ldots+s_j} 3^{-(j+1)} \\
&= 3^{l(s)} \left(2^{l(s)-1+s_0+s_1+\ldots+s_{l(s)-1}} 3^{-l(s)} + \sum_{j=0}^{l(s)-2} 2^{j+s_0+s_1+\ldots+s_j} 3^{-(j+1)} \right) \\
&= 2^{l(s)-1+s_0+s_1+\ldots+s_{l(s)-1}} + 3 \sum_{j=0}^{l(s)-2} 2^{j+s_0+s_1+\ldots+s_j} 3^{l(s)-j-2},
\end{aligned}
\tag{3.2}
$$

which shows that $w \in \mathbb{Z}^+$ and $3 \nmid w$. Define $q_0 = t^{||s||} w$. Since $3 \nmid t$ as well, $q_0 \in \mathbb{N} - 3\mathbb{N}$. By Lemma 2.2,

$$v_s(q) = c(s)\, q - r(s) = 3^{-l(s)} \left(2^{\|s\|} q - 3^{l(s)} r(s) \right) = 3^{-l(s)} \left(2^{\|s\|} q - w \right).$$

We have that

$$
\begin{aligned}
s \in \mathcal{E}(q) &\Leftrightarrow v_s(q) \in \mathbb{Z}^+ \\
&\Leftrightarrow 3^{-l(s)} \left(2^{\|s\|} q - w \right) \in \mathbb{Z}^+ \\
&\Leftrightarrow 3^{l(s)} \mid \left(2^{\|s\|} q - w \right) \\
&\Leftrightarrow 2^{\|s\|} q - w \underset{3^{l(s)}}{\equiv} 0 \\
&\Leftrightarrow q \underset{3^{l(s)}}{\equiv} t^{\|s\|} w \\
&\Leftrightarrow q \underset{3^{l(s)}}{\equiv} q_0,
\end{aligned}
$$

which completes the proof. ☐

The second technical lemma demonstrates that any feasible vector can be concatenated with a vector of length zero to make it admissible for any given natural number relatively prime to 3.

Lemma 3.4. *For any $s \in \mathcal{F}$ and any $y \in \mathbb{N} - 3\mathbb{N}$, there exists a natural number k such that $s \cdot (k) \in \mathcal{E}(y)$.*

Proof. Let $s \in \mathcal{F}$. Let $y \in \mathbb{N} - 3\mathbb{N}$. By Lemma 3.3, there exists $q_0 \in \mathbb{N} - 3\mathbb{N}$ such that for any $q \in \mathbb{Z}^+$, $s \in \mathcal{E}(q) \Leftrightarrow q \underset{3^{l(s)}}{\equiv} q_0$.

Since 2 is a primitive root for $3^{l(s)}$ (cf. [Hua]), we have that $2^k y \underset{3^{l(s)}}{\equiv} q_0$ for some $k \in \mathbb{Z}^+$. Thus $s \in \mathcal{E}\left(2^k y\right)$, so by Lemma 2.3 we have

$$v_{s\cdot(k)}(y) = \left(v_s \circ v_{(k)}\right)(y) = v_s\left(T_0^{-k}(y)\right) = v_s\left(2^k y\right) \in \mathbb{Z}^+.$$

Therefore $s \cdot (k) \in \mathcal{E}(y)$. ☐

For the third technical lemma, again using $v_s(x) = c(s)\, x - r(s)$ as in Lemma 2.2, we prove two facts about $r(s)$. Namely, we show that any finite increasing sequence can appear in the exponents of the 2-powers in the formula for $r(s)$, and for a given positive integer b relatively prime to 2 and 3, all elements of $\mathbb{Z}_b$ can be obtained as $r(s)$ for some s.

Lemma 3.5. *Let b be a positive integer relatively prime to 2 and 3.*

(i) *Let $t_1, t_2, ..., t_d$ be a strictly increasing sequence of natural numbers. Then there exists some $s \in \mathcal{F}$ such that $v_s(x) = 2^{||||} 3^{-d} x - \sum_{i=1}^{d} 2^{t_i} 3^{-i}$.*

(ii) *Let $n, m, k \in \mathbb{Z}$. There exists an $s \in \mathcal{F}$ such that $\overline{v_s} \in G_b$ is given by $\overline{v_s}(x) = 2^n 3^m x + k$. Furthermore, there exist increasing natural numbers $u_1, u_2, \ldots, u_d$ such that $\sum_{i=1}^{d} 2^{u_i} 3^{-i} \underset{b}{\equiv} k$.*

Proof. (i) Define $s_0 = t_1$ and $s_i = t_{i+1} - t_i - 1$ for $i \in \{1, 2, ..., d-1\}$. The sum $\sum_{i=0}^{j} s_i$ is a telescoping sum, so by induction on j, $\sum_{i=0}^{j} s_i = t_{j+1} - j$ for any $j \in \{0, 1, ..., d-1\}$.

Now take $s = (s_0, s_1, ..., s_{d-1}, 0)$. Then $l(s) = d$. By Lemma 2.2, we see that

$$\begin{aligned} v_s(x) &\underset{b}{\equiv} 2^{||s||} 3^{-d} x - \sum_{j=0}^{d-1} 2^{\left(j + \sum_{i=0}^{j} s_i\right)} 3^{-(j+1)} \\ &= 2^{||s||} 3^{-d} x - \sum_{j=0}^{d-1} 2^{t_{j+1}} 3^{-(j+1)}, \end{aligned}$$

and we are done.

(ii) The function $\overline{h}(x) = 2^n 3^m x - k$ is an element of G_b by Lemma 3.1. Since G_b is generated by $\overline{T_0}$ and $\overline{T_1}$, $\overline{h} = \overline{T_1}^{e_g} \circ \overline{T_0}^{e_{g-1}} \circ \ldots \circ \overline{T_1}^{e_4} \circ \overline{T_0}^{e_3} \circ \overline{T_1}^{e_2} \circ \overline{T_0}^{e_1}$ for some $g, e_1, e_2, \ldots, e_g \in \mathbb{N}$. Let σ_0 and σ_1 be the orders of T_0 and T_1 respectively, as in the proof of Lemma 3.1. By decreasing e_i by a multiple of σ_0 for odd i and decreasing e_i by a multiple of σ_1 for even i as necessary, we can create a sequence of nonpositive integers $e'_1, e'_2, \ldots, e'_g$ such that $\overline{h} = \overline{T_1}^{e'_g} \circ \overline{T_0}^{e'_{g-1}} \circ \ldots \circ \overline{T_1}^{e'_4} \circ \overline{T_0}^{e'_3} \circ \overline{T_1}^{e'_2} \circ \overline{T_0}^{e'_1}$.

Thus there exists an $s \in \mathcal{F}$ with $\overline{v_s} = \overline{h}$, so $\overline{v_s}(x) = 2^n 3^m x - k$. Since $v_s(x) = c(s)x - r(s)$, we have

$$r(s) = -v_s(0) \underset{b}{\equiv} -h(0) = k. \tag{3.3}$$

Define

$$u_j = (j-1) + s_0 + s_1 + ... + s_{j-1}$$

for $j \in \{1, \ldots, l(s)\}$. Clearly $u_1, u_2, ..., u_{l(s)-1}$ is strictly increasing, since Lemma 2.2 implies that

$$\begin{aligned} \sum_{j=1}^{l(s)} 2^{u_j} 3^{-j} &= \sum_{j=0}^{l(s)-1} 2^{j+s_0+s_1+...+s_j} 3^{-(j+1)} \\ &= r(s) \\ &\underset{b}{\equiv} k \end{aligned}$$

by (3.3). □

We now complete the second step in the proof of the main result.

Lemma 3.6. *Let b be a positive integer relatively prime to 2 and 3 and let $n \in \mathbb{N}$. For every $x \in \mathbb{Z}^+$, there exists $z \in \mathbb{Z}^+$ such that $z \underset{2^n b}{\equiv} 0$ and $z \underset{3}{\not\equiv} 0$ and x merges with z.*

Proof. Let $x \in \mathbb{Z}^+$. By Lemma 3.2, there exists y such that $y \underset{3}{\not\equiv} 0$ and x merges with y. We will produce an $s \in \mathcal{E}(y)$ such that $v_s(y) \underset{2^n b}{\equiv} 0$ and $v_s(y) \underset{3}{\not\equiv} 0$.

We know that there exists an $s_1 \in \mathcal{F}$ such that $v_{s_1}(x) \underset{b}{\equiv} x+1$ by part (ii) of Lemma 3.5. Thus for any $m \in \mathbb{N}$

$$v_{s_1}^m(x) \underset{b}{\equiv} x+m. \tag{3.4}$$

Lemma 3.4 allows us to choose $n_1 \in \mathbb{N}$ such that $(0,0) \cdot \underbrace{s_1 \cdot s_1 \cdot \ldots \cdot s_1}_{b} \cdot (n_1) \in \mathcal{E}(y)$. Let $s_2 = (0,0) \cdot \underbrace{s_1 \cdot s_1 \cdot \ldots \cdot s_1}_{b} \cdot (n_1)$. Since s_2 is admissible for y, $v_t(y) \in \mathbb{Z}^+$ for any terminal part t of s_2 by Lemma 2.4. So $v_{(n_1)}(y) \underset{b}{\equiv} a$ for some $a \in \{0,1,2,\ldots,b-1\}$. Define $s_3 = \underbrace{s_1 \cdot s_1 \cdot \ldots \cdot s_1}_{b-a} \cdot (n_1)$. This gives us

$$\begin{aligned} v_{s_3}(y) &= v_{\underbrace{s_1 \cdot s_1 \cdot \ldots \cdot s_1}_{b-a} \cdot (n_1)}(y) \\ &= v_{s_1}^{b-a}\left(v_{(n_1)}(y)\right) \text{ by Lemma 2.3} \\ &\underset{b}{\equiv} v_{s_1}^{b-a}(a) \\ &\underset{b}{\equiv} a + (b-a) \text{ by (3.4)} \\ &\underset{b}{\equiv} 0. \end{aligned}$$

Define $s_4 = (0,0) \cdot \underbrace{s_1 \cdot s_1 \cdot \ldots \cdot s_1}_{a}$. By Lemma 2.3,

$$v_{s_4}(v_{s_3}(y)) = v_{s_4 \cdot s_3}(y) = v_{s_2}(y) \in \mathbb{N}.$$

Thus $s_4 \in \mathcal{E}(v_{s_3}(y))$. Since $l(s_4)$ is positive, $v_{s_3}(y) \underset{3}{\not\equiv} 0$ by Lemma 2.5.

Define $s = (n) \cdot s_3$. We have that

$$v_s(y) = v_{(n)\cdot s_3}(y) = v_{(n)}(v_{s_3}(y)) = 2^n v_{s_3}(y) \in \mathbb{N}.$$

Thus $s \in \mathcal{E}(y)$. Define $z = v_s(y)$. Since $v_{s_3}(y) \underset{3}{\not\equiv} 0$, we have $2^n v_{s_3}(y) \underset{3}{\not\equiv} 0$. Thus $z \underset{3}{\not\equiv} 0$.

Since $v_{s_3}(y) \underset{b}{\equiv} 0$, we have $2^n v_{s_3}(y) \underset{2^n b}{\equiv} 0$. Thus $z \underset{2^n b}{\equiv} 0$.

Lemma 2.1 implies that $y = T^{||s||}(v_s(y))$, so $y = T^{||s||}(z)$. Thus y merges with z, and x merges with y, so x merges with z. $\square$

We now provide a technical lemma in preparation for the final step.

Lemma 3.7. *Let $s \in \mathcal{F}$ and let $m, q_0 \in \mathbb{N}$. Then there exists $p_0 \in \mathbb{N} - 3\mathbb{N}$ such that for any $p \in \mathbb{Z}^+$, if $p \underset{3^{l(s)+m}}{\equiv} p_0$, then $s \in \mathcal{E}(p)$ and $v_s(p) \underset{3^m}{\equiv} q_0$.*

Proof. Define $w = 3^{l(s)} r(s)$, as in Lemma 3.3. Again by (3.2), $w \in \mathbb{Z}^+$ and $3 \nmid w$.

Since 2 is relatively prime to $3^{l(s)+m}$, there exists some $t \in \mathbb{Z}^+$ such that $2t \underset{3^{l(s)+m}}{\equiv} 1$. Define $p_0 = t^{||s||}\left(3^{l(s)} q_0 + w\right)$. Note that $3 \nmid p_0$ since $3 \nmid t$ and $3 \nmid w$. Let $p \in \mathbb{Z}^+$ and assume $p \underset{3^{l(s)+m}}{\equiv} p_0$.

$$2^{||s||} p - w \underset{3^{l(s)+m}}{\equiv} 2^{||s||}\left(t^{||s||}\left(3^{l(s)} q_0 + w\right)\right) - w \tag{3.5}$$
$$\underset{3^{l(s)+m}}{\equiv} 2^{||s||} t^{||s||} 3^{l(s)} q_0 + 2^{||s||} t^{||s||} w - w$$
$$\underset{3^{l(s)+m}}{\equiv} 1^{||s||} 3^{l(s)} q_0 + 1^{||s||} w - w$$
$$\underset{3^{l(s)+m}}{\equiv} 3^{l(s)} q_0.$$

Thus $3^{l(s)} \mid 2^{\|s\|} p - w$, so $\frac{2^{||s||} p - w}{3^{l(s)}} \in \mathbb{Z}^+$. From Lemma 2.2 we have that $v_s(p) = \frac{2^{\|s\|}}{3^{l(s)}} p - r(s) = \frac{2^{\|s\|} p - w}{3^{l(s)}}$, so $v_s(p) \in \mathbb{Z}^+$ and $s \in \mathcal{E}(p)$. Division of both sides and the modulus of (3.5) by $3^{l(s)}$ yields that $v_s(p) \underset{3^m}{\equiv} q_0$. □

In the third and final step we use z as our stepping-stone to show that any positive integer x can merge with a, a number that is in whatever congruence class we desire.

Lemma 3.8. *Let b be a positive integer relatively prime to 2 and 3 and let $n, m \in \mathbb{N}$. For every $x \in \mathbb{Z}^+$ and for any $a_0, q_0 \in \mathbb{N}$, there exists an $a \in \mathbb{Z}^+$ such that $a \underset{2^n b}{\equiv} a_0$ and $a \underset{3^m}{\equiv} q_0$ and x merges with a.*

Proof. Let $x \in \mathbb{Z}^+$. By Lemma 3.6, there exists an $z \in \mathbb{Z}^+$ such that $z \underset{2^n b}{\equiv} 0$ and $z \underset{3}{\not\equiv} 0$, and x merges with z. Let $a_0, q_0 \in \mathbb{N}$. We have $a_0 \underset{2^n b}{\equiv} a_1 + 2^n a_2$ for some $a_1 \in \{0, 1, 2, ..., 2^n - 1\}$ and $a_2 \in \{0, 1, 2, ..., b - 1\}$.

We now show that there exists an odd natural number τ such that $3\tau \underset{2^n b}{\equiv} 1$. If $n = 0$, we can take any natural number t with $3t \underset{b}{\equiv} 1$ and either set $\tau = t$ (in the case that t is odd) or $\tau = t + b$ (in the case that t is even). Otherwise if $n > 0$, then any $\tau \in \mathbb{N}$ such that $3\tau \underset{2^n b}{\equiv} 1$ satisfies $2^n b \mid (3\tau - 1)$ which implies that τ is odd.

There exists an increasing sequence of natural numbers $t_1, t_2, ..., t_{h_1}$ with $t_{h_1} < n$ such that $\sum_{i=1}^{h_1} 2^{t_i} \tau^i \underset{2^n}{\equiv} -a_1$ by Lemma 2.7. By the definition of congruence, $\sum_{i=1}^{h_1} 2^{t_i} \tau^i = -a_1 + 2^n i_1$ for some $i_1 \in \mathbb{Z}$. There exist increasing natural numbers $u_1, u_2, ..., u_{h_2}$ such that $\sum_{i=1}^{h_2} 2^{u_i} \tau^i \underset{b}{\equiv} 3^{h_1}(-a_2 - i_1)$ by part (ii) of Lemma 3.5. Therefore $\tau^{h_1} \sum_{i=1}^{h_2} 2^{u_i} \tau^i \underset{b}{\equiv} (-a_2 - i_1)$, so by the definition of congruence, $\tau^{h_1} \sum_{i=1}^{h_2} 2^{u_i} \tau^i = -a_2 - i_1 + b i_2$ for some $i_2 \in \mathbb{Z}$. Define

$$w_i = \begin{cases} t_i \text{ if } 1 \le i \le h_1, \\ n + u_{i-h_1} \text{ if } \ h_1 + 1 \le i \le h_1 + h_2. \end{cases}$$

That is, w_i is the i^{th} term of the sequence $t_1, t_2, \ldots, t_{h_1}, n+u_1, n+u_2, ..., n+u_{h_2}$. Since t_i and u_i are increasing and $t_{h_1} < n$, the sequence $w_1, w_2, \ldots, w_{h_1+h_2}$ is also increasing. So

$$\begin{aligned}
\sum_{i=1}^{h_1+h_2} 2^{w_i}\tau^i &= \sum_{i=1}^{h_1} 2^{w_i}\tau^i + \sum_{i=h_1+1}^{h_1+h_2} 2^{w_i}\tau^i \\
&= \sum_{i=1}^{h_1} 2^{t_i}\tau^i + \sum_{i=1}^{h_2} 2^{w_{i+h_1}}\tau^{i+h_1} \\
&= (-a_1 + 2^n i_1) + \tau^{h_1} \sum_{i=1}^{h_2} 2^{n+u_i}\tau^i \\
&= -a_1 + 2^n i_1 + 2^n \left(\tau^{h_1} \sum_{i=1}^{h_2} 2^{u_i}\tau^i \right) \\
&= -a_1 + 2^n i_1 + 2^n \left(-a_2 - i_1 + b i_2\right) \\
&= -a_1 - 2^n a_2 + 2^n b i_2 \\
&\underset{2^n b}{\equiv} -a_1 - 2^n a_2 \\
&\underset{2^n b}{\equiv} -a_0.
\end{aligned}$$

Thus there exists an increasing sequence $w_1, w_2, \ldots, w_{h_1+h_2}$ such that $\sum_{i=1}^{h_1+h_2} 2^{w_i}\tau^i \underset{2^n b}{\equiv} \sum_{i=1}^{h_1+h_2} 2^{w_i} 3^{-i} \underset{2^n b}{\equiv} -a_0$, so by part (i) of Lemma 3.5 there exists an $s_1 \in \mathcal{F}$ such that

$$v_{s_1}(x) \underset{2^n b}{\equiv} 2^{||s_1||} 3^{-l(s_1)} x + a_0. \tag{3.6}$$

By Lemma 3.7 there exists $p_0 \in \mathbb{N} - 3\mathbb{N}$ such that for all $p \in \mathbb{Z}^+$, if $p \underset{3^{l(s_1)+m}}{\equiv} p_0$, then $s_1 \in \mathcal{E}(p)$ and $v_{s_1}(p) \underset{3^m}{\equiv} q_0$. Since 2 is a primitive root for $3^{l(s_1)+m}$ (cf. [Hua]) and z is a unit mod $3^{l(s_1)+m}$, there exists $n_2 \in \mathbb{Z}^+$ such that $2^{n_2} z \underset{3^{l(s_1)+m}}{\equiv} p_0$. Thus $s_1 \in \mathcal{E}(2^{n_2} z)$ and $v_{s_1}(2^{n_2} z) \underset{3^m}{\equiv} q_0$. Define $s = s_1 \cdot (n_2)$. By Lemma 2.3,

$$v_s(z) = v_{s_1 \cdot (n_2)}(z) = v_{s_1}\left(v_{(n_2)}(z)\right) = v_{s_1}(2^{n_2} z).$$

Therefore $s \in \mathcal{E}(z)$, so $v_s(z) \in \mathbb{Z}^+$ and $v_s(z) \underset{3^m}{\equiv} q_0$. Defining $a = v_s(z)$, we have $a \underset{3^m}{\equiv} q_0$.

Note that $2^{n_2} z$ is divisible by $2^n b$, since z is also divisible. Thus $2^{n_2} z \underset{2^n b}{\equiv} 0$, so that

$$\begin{aligned}
a &= v_s(z) \\
&= v_{s_1}(2^{n_2} z) \\
&\underset{2^n b}{\equiv} v_{s_1}(0) \\
&\underset{2^n b}{\equiv} a_0 \text{ by (3.6).}
\end{aligned}$$

By Lemma 2.1, $T^{\|s\|}(a) = z$. Thus z merges with a, and x merges with z, so x merges with a. $\square$

Since the previous lemma shows that any positive integer x merges with an integer a that can be in any desired congruence classes modulo $2^n b$ and modulo 3^m, we are prepared to prove the main result.

Proof of Theorem 1.1. Let S be an arithmetic progression. Let $x \in \mathbb{Z}^+$. We have that $S = A + B\mathbb{N}$ for some nonnegative integers A and B with $B \neq 0$. There exists some $k \in \mathbb{Z}^+$ such that $kB > A$. We can write $kB = 2^n 3^m b$ for some $n,m \in \mathbb{Z}^+$ and $b \in \mathbb{Z}^+$ relatively prime to 2 and 3. By Lemma 3.8, there exists an $a \in \mathbb{Z}^+$ such that $a \underset{2^n b}{\equiv} A$ and $a \underset{3^m}{\equiv} A$, and x merges with a. Thus $a \underset{2^n b 3^m}{\equiv} A$ since $2^n b$ and 3^m are relatively prime, and both divide $a - A$. Hence $a = A + kBk_1$ for some $k_1 \in \mathbb{Z}$. However, $k_1 > 0$ since a is positive and $kB > A$. Thus $k_1 \in \mathbb{N}$ and $a \in S$. Therefore every positive integer merges with an element of S. So, S is a sufficient set. □

The proofs of the corollaries then easily follow from the main result.

Proof of Corollary 1.2. Let S be an arithmetic progression and assume that for every $a \in S$, $\lim_{j\to\infty} T^j(a) \neq \infty$. Let $x \in \mathbb{Z}^+$. Then x merges with a for some $a \in S$ by Theorem 1.1. Since x and a merge, their orbits are not disjoint, so $T^{j_1}(x) = T^{j_2}(a)$ for some $j_1,j_2 \in \mathbb{Z}^+$. Therefore for every $j \in \mathbb{Z}^+$, $T^{j_1+j}(x) = T^{j_2+j}(a)$. Thus $\lim_{j\to\infty} T^j(x) \neq \infty$ as well. So no positive integer has a divergent orbit, and the Divergent Orbits Conjecture holds. Thus S is sufficient for the Divergent Orbits Conjecture. □

Proof of Corollary 1.3. Let S be an arithmetic progression and assume that no element of S has an orbit that contains a nontrivial cycle. Let $x \in \mathbb{Z}^+$. There exists some $a \in S$ such that x merges with a by Theorem 1.1. Assume the orbit of x contains a nontrivial cycle. Since x and a merge, their orbits are not disjoint, so $T^{j_1}(x) = T^{j_2}(a)$ for some $j_1,j_2 \in \mathbb{Z}^+$. Therefore for every $j \in \mathbb{Z}^+$, $T^{j_1+j}(x) = T^{j_2+j}(a)$, so the orbit of a contains a nontrivial cycle as well, contradicting the assumption. Thus no positive integer has an orbit that contains a nontrivial cycle, and the Nontrivial Cycles Conjecture holds. Thus S is sufficient for the Nontrivial Cycles Conjecture. □

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