ATOMIZATION PROCESS FOR CONVOLUTION OPERATORS ON LOCALLY COMPACT GROUPS

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Abstract. We develop for a large class of locally compact groups a method of approximation of convolution operators on $G$ by finitely supported measures with control of the support and of the operator norm of the approximating measures.

1. Introduction

Let $G$ be a locally compact group, let $1 < p < \infty$ and let $CV_p(G)$ be the Banach space of all $p$-convolution operators on $L^p(G)$. If $\mu$ is a bounded measure on $G$, the corresponding $p$-convolution operator is denoted $\lambda^p_G(f)(\varphi) = \varphi \ast (\Delta^{-1/p} \check{f})$.

If $G$ is amenable, given $T \in CV_p(G)$ and $\Omega$ any open subset containing supp $T$, then the operator $T$ is in the weak closure of the set of all operators $\lambda^p_G(f)$, where $f$ is a continuous function with compact support ($f \in C^{\infty}(G)$) with supp $f \subset \Omega$ and $|||\lambda^p_G(f)|||_p \leq |||T|||_p$ (see [6, Thm. 5]). The following problem is discussed in the present paper: is it true that the operator $T$ is in the weak closure of the set of all operators $\lambda^p_G(f)$, where $\mu$ is a finitely supported measure with supp $\mu \subset \Omega$ and $|||\lambda^p_G(\mu)|||_p \leq |||T|||_p$?

In [7, 8, 9], N. Lohoué investigated a similar question for $G$ abelian. He used in a strong way the dual of $G$, and his approach only applies to commutative groups. Our techniques are different and apply to a large class of non-commutative groups, such as the Heisenberg group. For a given $T \in CV_p(G)$, we construct in an explicit way an approximating net of finitely supported measures converging to $T$. In a preprint of [3] C. Herz announced a similar result for locally compact amenable groups. In the final version of the paper this was suppressed.

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2. Preliminaries

Let $G$ be a locally compact group and $\mathcal{H}$ the set of the closed subgroups of $G$. For each $K$ compact subset of $G$ and $F$ finite family of open subsets of $G$, we set $\mathcal{U}(K,F) = \{ H \in \mathcal{H} : H \cap K = \emptyset, H \cap A \neq \emptyset \ \forall A \in F \}$. We provide $\mathcal{H}$ with the topology generated by $\mathcal{U}(K,F)$. A net of closed subgroups which is converging to $G$ in this topology is called a dense net. We use the terminology of [12]. For example, an increasing sequence $(H_n)_{n \in \mathbb{N}}$ of closed subgroups with $\bigcup_{n=1}^{\infty} H_n$ dense in $G$ is a dense net. Let $m_G$ denote a left Haar measure of $G$.

Theorem 2.1. Let $(H_\alpha)_{\alpha \in I}$ be a dense net of closed subgroups of $G$. For each $\alpha \in I$, there exists a left Haar measure $m_\alpha$ of $H_\alpha$ such that

$$\lim_{\alpha \in I} \int_{H_\alpha} f(h)dm_\alpha(h) = \int_{G} f(x)dx,$$

for all $f \in C_{c0}(G)$.

Moreover, if $f \in C_{c0}(G)$, then

$$\int_{H_\alpha} f(xh)dm_\alpha(h) \longrightarrow \int_{G} f(u)du$$

uniformly on the compact subsets of $G$.

This result is due to K. A. Ross & K. Stromberg [11, Thm. 2] for sequences and to J.-L. Rubio [12, Cor. 3] for nets.

The following lemma will be useful.

Lemma 2.2. Let $\Phi$ be a continuous map of $G$ into $(C_{c0}(G), \| \cdot \|_{\infty})$ and $K$ a compact subset of $G$ such that $\text{supp}(\Phi(y)) \subset K$, for all $y \in G$. Then, for each $\varepsilon > 0$ and $L,V$ compact subsets of $G$, there is $\alpha_0 \in I$ such that, for all $\alpha \geq \alpha_0$,

$$\left| \int_{H_\alpha} \Phi(y)(xh)dm_\alpha(h) - \int_{G} \Phi(y)(u)du \right| < \varepsilon,$$

for all $x \in L$ and $y \in V$.

We also need to transfer convolution operators from groups to subgroups and from subgroups to groups. If $H$ is a closed subgroup of $G$, there is a isometry $i$ of $CV_p(H)$ into $CV_p(G)$ (cf. [5, p. 76]) which satisfies, for all $T \in CV_p(H)$ and $r,s \in C_{c0}(G)$,

$$\langle i(T)r,s \rangle_{L^p(G),L^p'(G)} = \langle T\left(\frac{r}{q^{1/p}}\right)_{x,H},\left(\frac{s}{q^{1/p}}\right)_{x,H} \rangle_{L^p(H),L^p'(H)} \ dx,$$

where $r_{x,H}(h) = r(xh)$. The following properties are satisfied (cf. [1, 5]):

(2.1) $\|i(T)\|_p = \|T\|_p$

and

(2.2) $\text{supp}(i(T)) = \text{supp}(T)$.

Let $H$ be a closed subgroup of $G$, $f \in C_{c0}(G)$ and $\dot{x} \in G/H$. We define

$$T_{H,q}(f)(\dot{x}) = \int_{H} \frac{f(xh)}{q(xh)}dh,$$
where $q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1})$ (cf. [10]).

For each $T \in CV_p(G)$ and $k, l \in C_{oo}(G)$, J. Delaporte and A. Derighetti defined in [3]

$$\langle \Lambda_{k,l}^{(q)}(T) \varphi, \psi \rangle_{L^p(H), L^{p'}(H)} = \langle T\left(\tau_p\left(q^{1/p'}(k \star_H \tau_p\varphi)\right), \tau_{p'}\left(q^{1/p'}(l \star_H \tau_p\psi)\right)\right)_{L^p(G), L^{p'}(G)},$$

for all $\varphi, \psi \in C_{oo}(G)$. Then $\Lambda_{k,l}^{(q)}$ is a linear map of $CV_p(G)$ into $CV_p(H)$ with

$$||\Lambda_{k,l}^{(q)}(T)|| \leq ||T_H(\langle k \rangle)|| ||T_H(\langle l \rangle)||_{p'},$$

$$\Lambda_{k,l}^{(q)}(PM_p(G)) \subset PM_p(H),$$

$$\text{supp}(\Lambda_{k,l}^{(q)}(T)) \subset H \cap \left(\text{supp}(k^{-1} \text{supp}(T) \text{supp}(l))\right).$$

In the present paper, we always suppose $q = 1$ (i.e. $\text{Res}_H(\Delta_G) = \Delta_H$). This property is verified for normal closed subgroups or more generally for neutral closed subgroups of $G$. In this case, we set $\Lambda_{k,l}^{(q)} = \Lambda_{k,l}$ and $T_{H,q} = T_H$.

**Lemma 2.3.** Let $1 < p < \infty$, $G$ a locally compact group, $H$ a closed subgroup of $G$, $f \in C_{oo}(G)$, $k, l \in C_{oo}(G)$, $\varphi \in L^p(G)$ and $\psi \in L^{p'}(G)$. Then, we have

$$\langle \iota(\Lambda_{k,l}(\varphi)), \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \Lambda_{k,l}(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)}$$

$$= \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) \right) \left( T_H(\Delta_G^{1/p'}(y)\psi)\pi(u) - \int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) du \right) dv \right) dy \right) dz$$

$$+ \langle f \ast k \ast \tau_p\varphi, l \ast \tau_{p'}\psi - \tau_p\psi \rangle_{L^p(G), L^{p'}(G)}$$

$$+ \langle f \ast (k \ast \tau_p\varphi - \tau_p\varphi), \tau_p\psi \rangle_{L^p(G), L^{p'}(G)},$$

where $\pi$ is the canonical map of $G$ onto $G/H$.

**Proof.** Let $\beta$ be a Bruhat function of $(G, H)$. On the one hand, we have

$$\langle \iota(\Lambda_{k,l}(\varphi)), \psi \rangle_{L^p(G), L^{p'}(G)}$$

$$= \int_G \langle \Lambda_{k,l}(\varphi), \psi \rangle_{L^p(H), L^{p'}(H)} \pi(x) dx$$

$$= \int_G f(z) \left( \int_G \left( \int_G \beta(x) T_H\left(\Delta_G^{1/p}(z^{-1}y) \varphi\left(T_H\left(\Delta_G^{1/p'}(y)\psi\right)\pi(x)\right)\right) \left(\pi(x)\right) du \right) dv \right) dy \right) dz$$

$$= \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(z^{-1}y) \varphi\left(T_H\left(\Delta_G^{1/p'}(y)\psi\right)\pi(x)\right)\right) (u) du \right) dy \right) dz$$

$$= \int_G f(z) \left( \int_G \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) T_H\left(\Delta_G^{1/p'}(y)\psi\right)(\pi(u)) du \right) dy \right) dz.$$
on the other hand,

\[
\langle \lambda^p_G(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} = \langle f \ast \tau_p \varphi, \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}
\]

and

\[
\int_G f(z) \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u) \left( \int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) \, dv \right) du \right) dy \, dz
= \langle f \ast k \ast \tau_p \varphi, l \ast \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}.
\]

Finally,

\[
\langle i(A_{k,l}(\lambda^p_G(f))) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda^p_G(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)}
= \langle i(A_{k,l}(\lambda^p_G(f))) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle f \ast k \ast \tau_p \varphi, l \ast \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}
+ \langle f \ast k \ast \tau_p \varphi, l \ast \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} - \langle f \ast k \ast \tau_p \varphi, \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}
+ \langle f \ast k \ast \tau_p \varphi, \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda^p_G(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)}
\]

\[
= \int_G f(z) \left( \int_G \Delta_G^{1/p}(u) k(z^{-1}yu) \varphi(u)
\right.
\]
\[
\left. \left( T_H(\Delta_G^{1/p'}(y) \psi)(\pi(u)) - \int_G \Delta_G^{1/p'}(v) l(yv) \psi(v) \, dv \right) du \right) dy \, dz
+ \langle f \ast k \ast \tau_p \varphi, l \ast \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}
+ \langle f \ast (k \ast \tau_p \varphi - \tau_{p'} \psi), \tau_{p'} \psi \rangle_{L^p(G), L^{p'}(G)}.
\]

\[\square\]

3. Atomization process for an inductive limit of discrete groups

Let \(1 < p < \infty\), \(G\) a locally compact group and \((H_\alpha)_{\alpha \in I}\) a dense net of closed subgroups\(^1\) of \(G\).

**Theorem 3.1.** Let \((H_\alpha)_{\alpha \in I}\) be a dense net of closed subgroups of \(G\). Then, there is a net \((\mathcal{L}_\beta)_{\beta \in \mathcal{I}}\) of uniformly bounded endomorphisms of \(CV_p(G)\) such that:

(i) for every \(f \in C_0(G)\), \(\lambda^p_G(f)\) is a cluster point of the net \((\mathcal{L}_\beta(\lambda^p_G(f)))_{\beta \in \mathcal{I}}\) in the weak operator topology;

(ii) for each \(\beta = (\varepsilon, k, l, \alpha) \in \mathcal{I}\), \(\|\mathcal{L}_\beta\| \leq 1 + \varepsilon\);

(iii) for every neighborhood \(U\) of \(\text{supp} f\), there is \(\beta_0 \in \mathcal{I}\) with \(\mathcal{L}_\beta(\lambda^p_G(f)) \subset U \cap H_\alpha\), for \(\beta = (\varepsilon, k, l, \alpha) \geq \beta_0\).

**Proof.** Let

\[
A = \left\{ k \in C_0^+(G) : \int_G k(u) du = 1 \right\}
\]

\(^1\)We suppose \(\text{Res}_{H_\alpha}(\Delta_G) = \Delta_{H_\alpha}\).
and
\[ \mathcal{I} = \left\{ (\varepsilon, k, l, \alpha) : 0 < \varepsilon < 1, k, l \in A, \supp k, \supp l \text{ neigh. of } e \in G, \right. \]
\[ \left. \left| \int_{H_\alpha} k(xh)dm_\alpha(h) - 1 \right| < \varepsilon \text{ for all } x \in \supp k, \right. \]
\[ \left. \left| \int_{H_\alpha} l(xh)dm_\alpha(h) - 1 \right| < \varepsilon \text{ for all } x \in \supp l \right\}. \]

For \( \beta = (\varepsilon, k, l, \alpha), \beta' = (\varepsilon', k', l', \alpha') \in \mathcal{I} \), we define the relation \( \beta \leq \beta' \) if and only if \( \varepsilon' \leq \varepsilon, \alpha \leq \alpha' \), \( \supp k' \subset \supp k \) and \( \supp l' \subset \supp l \). Then, \( (\mathcal{I}, \leq) \) is a directed set.

For each \( \beta = (\varepsilon, k, l, \alpha) \in \mathcal{I} \), we define the endomorphism of \( CV_p(G) \),
\[ L_\beta = i_\alpha \circ \Lambda_{\alpha, k, l}, \]
where \( i_\alpha \) is the canonical isometry of \( CV_p(H_\alpha) \) into \( CV_p(G) \) and \( \Lambda_{\alpha, k, l} \) the map of \( CV_p(G) \) into \( CV_p(H_\alpha) \) defined by \( \text{(2.3)} \).

Let \( \beta = (\varepsilon, k, l, \alpha), \varphi, \psi \in L^p(G) \) and \( \psi \in L^p(G) \). From \( \text{(2.1)} \),
\[ \left| \langle i_\alpha(\Lambda_{\alpha, k, l}(T))\varphi, \psi \rangle_{L^p(G), L^p(G)} \right| \leq |||T|||_p |||T_{H_\alpha}(k)||_p |||T_{H_\alpha}(l)||_p ||\varphi||_p ||\psi||_{p'}. \]

But, the net \( \{T_{H_\alpha}(k)\}_{\alpha \in I} \) is converging to \( \int_{\mathcal{I}} k(u)du = 1 \) uniformly on \( \pi_\alpha(\supp k) \), and \( |||T_{H_\alpha}(k)||_p^p = \int_{G/H_\alpha} |||T_{H_\alpha}(k)(\dot{x})||_p^p d\dot{x} \leq |||T_{H_\alpha}(k)||_p^{p-1} \).

Moreover, for each \( \dot{x} \in \pi_\alpha(\supp k) \), \( |||T_{H_\alpha}(k)(\dot{x})||_p \leq \varepsilon + 1 \). Then, \( |||T_{H_\alpha}(k)||_p \leq (\varepsilon + 1)^{1/p'} \) and \( |||T_{H_\alpha}(l)||_p \leq (\varepsilon + 1)^{1/p} \). Therefore
\[ \left| \langle i_\alpha(\Lambda_{\alpha, k, l}(T))\varphi, \psi \rangle_{L^p(G), L^p(G)} \right| \leq |||T|||_p (\varepsilon + 1) ||\varphi||_p ||\psi||_{p'}, \]
and (ii) is verified.

From \( \text{(2.2)} \) and \( \text{(2.6)} \), for each \( \beta \in \mathcal{I} \),
\[ \supp(L_\beta(T)) = \supp \left( i_\alpha(\Lambda_{\alpha, k, l}(T)) \right) \subset i_\alpha \left( \supp(\Lambda_{\alpha, k, l}(T)) \right) \]
\[ = \supp(\Lambda_{\alpha, k, l}(T)) \subset H_\alpha \cap (\supp(k)^{-1} \supp(T) \supp(l)). \]

Then, (iii) is verified.

Let \( f \in C_{\text{coo}}(G), W \subset G \) a neighborhood of \( \supp(f)^{-1}, \varepsilon_1 > 0, \varphi, \psi \in C_{\text{coo}}(G) \) and \( \beta = (\varepsilon, k, l, \alpha_0) \in \mathcal{I} \).

Let \( 0 < \varepsilon \leq \min(\varepsilon_1, \varepsilon_2) \).

There is a compact neighborhood \( V \) of \( e \) with \( V = V^{-1} \) and \( V \supp(f)V \subset W \). There are \( k', l' \in C_{\text{coo}}(G) \) with \( \|k'\|_1 = \|l'\|_1 = 1 \) such that \( \supp(k') \) and \( \supp(l') \) are neighborhoods of \( e \), \( \supp(k') \subset \supp(k) \cap V \), \( \supp(l') \subset \supp(l) \cap V \),
\[ \|k' \ast \tau_{p'} \varphi - \tau_{p'} \varphi\|_p < \frac{\varepsilon}{1 + 3\|f\|_1 \|\psi\|_{p'}} \]
and
\[ \|l' \ast \tau_{p'} \psi - \tau_{p'} \psi\|_{p'} < \frac{\varepsilon}{1 + 3\|f\|_1 \|\varphi\|_p}. \]

There is \( \alpha_1 \in I \) such that, for all \( \alpha \geq \alpha_1 \), \( |T_{H_\alpha}(k')(\dot{x}) - 1| < \varepsilon \), for each \( \dot{x} \in \pi_\alpha(\supp(k')) \) and \( |T_{H_\alpha}(l')(\dot{x}) - 1| < \varepsilon \), for each \( \dot{x} \in \pi_\alpha(\supp(l')). \)
Moreover, by Lemma 2.2 there is $\alpha_2 \in I$ such that, for all $\alpha \geq \alpha_2$,
\[
\left| \int \left( \Delta_h^{1/p'} (y) \right) (u) \, dm_\alpha (h) - \int \left( \Delta_h^{1/p'} (y) \right) (v) \, dv \right| < \frac{\varepsilon}{1 + 3 \| f \|_1 \| \tau_p \varphi \|_1},
\]
for each $y \in \text{supp}(f) \bigcap \text{supp}(\varphi)^{-1}$ and for each $u \in \text{supp}(\varphi)$.

Let $\alpha' \geq \alpha_0, \alpha_1, \alpha_2$. We define $\beta' = (\varepsilon, k', l', \alpha')$. We have $\beta' \in \mathcal{I}$ and $\beta' \geq \beta$.

We have $\{ y \in G : \exists u \in \text{supp}(\varphi), \exists z \in \text{supp}(f) \text{ with } k(z^{-1}yu) \neq 0 \} \subset \text{supp}(f) \bigcap \text{supp}(\varphi)^{-1}$. Let us make some estimations:
\[
\int_G \left| \int (\int_G \Delta_h^{1/p'} (u) k(z^{-1}yu) \varphi (u)) \right| \, dy \, dz \\
= \int_G \left| \int (\int_G \Delta_h^{1/p'} (u) l(yu) \psi (v)) \, du \right| \, dy \, dz \\
= \int_G \left| \int (|k| \ast |\tau_p \varphi| (y)) \, dy \right| = \| f \|_1 \| \varphi \|_1.
\]

Then,
\[
\int \left| \int (\int_G \Delta_h^{1/p'} (u) k(z^{-1}yu) \varphi (u)) \right| \, dy \, dz \\
\leq \int \left| \int (\int_G \Delta_h^{1/p'} (y) \varphi (u)) \right| \, dy \, dz \\
\leq \int \left| \int (\int_G \Delta_h^{1/p'} (y) \varphi (u)) \right| \, dy \, dz \\
< \frac{\varepsilon}{3}.
\]

Moreover,
\[
\left| \langle f \ast k \ast \tau_p \varphi, l \ast \tau_p \psi - \tau_p \psi \rangle_{L^p (G), L^p (G)} \right| \\
< \| f \ast k \ast \tau_p \varphi \|_{p} \| l \ast \tau_p \psi - \tau_p \psi \|_{p'} < \| f \|_1 \| k \ast \tau_p \varphi \|_{p} \| l \ast \tau_p \psi - \tau_p \psi \|_{p'} \\
< \| f \|_1 \| k \|_1 \| \tau_p \varphi \|_{p} \frac{\varepsilon}{1 + 3 \| f \|_1 \| \varphi \|_p} < \frac{\varepsilon}{3}
\]
and
\[
\left| \langle f \ast (k \ast \tau_p \varphi - \tau_p \varphi), \tau_p \psi \rangle_{L^p (G), L^p (G)} \right| \\
< \| f \ast (k \ast \tau_p \varphi - \tau_p \varphi) \|_{p} \| \tau_p \psi \|_{p'} < \| f \|_1 \| k \ast \tau_p \varphi - \tau_p \varphi \| \| \psi \|_{p'} \\
< \| f \|_1 \| \psi \|_{p'} \frac{\varepsilon}{1 + 3 \| f \|_1 \| \psi \|_{p'}} < \frac{\varepsilon}{3}.
\]
Finally, from Lemma 2.3 we have
\[ \left| \langle \mathcal{L}_\beta (\lambda_G^p(f))\varphi, \psi \rangle_{L^p(G), L^p(G)} - \langle \lambda_G^p(f)\varphi, \psi \rangle_{L^p(G), L^p(G)} \right| < \varepsilon. \]

\[ \square \]

**Corollary 3.2.** Let \( 1 < p < \infty, \) \( G \) a locally compact group, \( \{ H_\alpha \}_{\alpha \in I} \) a dense net of discrete subgroups of \( G. \) Then, for each \( f \in C_{\text{c}}(G), \) there is a net \( (\mu_\beta) \) of finitely supported measures of \( G \) such that:

(i) for every \( f \in C_{\text{c}}(G), \) \( \lambda_G^p(f) \) is a cluster point of the net \( (\lambda_G^p(\mu_\beta))_{\beta \in I} \) in the weak operator topology;

(ii) for each \( \beta = (\varepsilon, k, l, \alpha) \in I, \) \( |||\lambda_G^p(\mu_\beta)||| \leq |||\lambda_G^p(f)|||_p + \varepsilon; \)

(iii) for every neighborhood \( U \) of \( \text{supp} \, f, \) there is \( \beta_0 \in I \) with \( \text{supp} \, \mathcal{L}_\beta (\lambda_G^p(f)) \subset U, \) for \( \beta \geq \beta_0. \)

**Proof.** By Theorem 3.1 we have \( \text{supp} \, \mathcal{L}_\beta (\lambda_G^p(f)) \subset \text{supp} \, \Omega_{k,l}(\lambda_G^p(f)), \) where \( \beta = (\varepsilon, k, l, \alpha). \) But \( \text{supp} \, \Omega_{k,l}(\lambda_G^p(f)) \) is a compact subset of the discrete group \( H_\alpha \) and therefore is a finite set. Finally, there is a finitely supported measure \( \mu_\beta \) with \( \mathcal{L}_\beta (\lambda_G^p(f)) = \lambda_E^p(\mu_\beta). \)

\[ \square \]

**Corollary 3.3.** Let \( 1 < p < \infty, \) \( G \) a locally compact group with \( \{ H_\alpha \}_{\alpha \in I} \) a dense net of discrete subgroups of \( G \) and \( T \in CV_p(G) \) with compact support. Then, there exists a net of finitely supported measures which converges to \( T \) with control of the support and the operator norms of the approximating measures.

**Proof.** There exists a net \( \{ f_\alpha \}_{\alpha \in I} \) of positive compactly supported functions of \( G \) with \( ||f_\alpha||_1 = 1 \) such that \( \lambda_G^p \left( \tau_p (T (\Delta_G^{1/p} f_\alpha)) \right) \) converges weakly to \( T \) with control of the support and of the operator norm (see [6] Prop. 9).

\[ \square \]

**Remark 3.4.**

(1) If \( G \) is amenable and has a dense net of discrete subgroups, every \( T \in CV_p(G) \) can be approximated similarly by finitely supported measures with control of the support and of the operator norm. A similar result was formulated in a preprint of [2] but was suppressed in the published version.

(2) The amenability is not needed for the Corollaries 3.2 and 3.3.

**Example 3.5.** The atomization process is available for the following groups:

(1) \( \mathbb{R} \) with the subgroups \( H_n = 2^{-n} \mathbb{Z}. \)

(2) \( T \) with the subgroups \( H_n = \mathbb{Z}/2^{-n} \mathbb{Z}. \)

(3) The group of the symmetries of the plane \( 0(2) \) with the subgroups \( H_n \) generated by the symmetries of the regular polygons with \( 2^n \) sides.

(4) The Heisenberg groups with the subgroups \( H_n \) of the matrices

\[
\begin{pmatrix}
1 & a 2^{-n} & b 2^{-2n} \\
0 & 1 & c 2^{-n} \\
0 & 0 & 1
\end{pmatrix}, \quad \text{where } a, b, c \in \mathbb{Z}.
\]

This also applies to finite products of groups admitting a dense net of discrete subgroups:

(5) Elementary groups, \( \mathbb{R}^l \times \mathbb{T}^m \times \mathbb{Z}^n \times F \) with \( l, m, n \in \mathbb{N}, \) where \( F \) is finite.

(6) Locally compact abelian groups with no small subgroups [2 Prop. 7.9].
4. Inductive limits

Let $G$ be a locally compact group and $(G^{(m)})_{m=1}^{\infty}$ a dense sequence of closed subgroups of $G$. We suppose that for each $m \in \mathbb{N}$, there is a dense sequence $(G_{m,n}^{(m)})_{n=1}^{\infty}$ of discrete subgroups of $G^{(m)}$.

With this hypothesis, we have:

**Theorem 4.1.** There is a net $(\mathcal{L}_\gamma)_{\gamma \in \mathcal{K}}$ of uniformly bounded endomorphisms of $CV_p(G)$ such that:

1. for every $f \in C_{oo}(G)$, $\lambda^p_G(f)$ is a cluster point of the net $\left(\mathcal{L}_\gamma(\lambda^p_G(f))\right)_{\gamma \in \mathcal{K}}$ in the weak operator topology;
2. for each $\gamma = (\varepsilon, k, l, m, n) \in \mathcal{K}$, $\|\mathcal{L}_\gamma\| \leq 1 + \varepsilon$;
3. for every neighborhood $U$ of supp $f$, there is $\gamma_0 \in \mathcal{K}$ with supp $\mathcal{L}_\gamma(\lambda^p_G(f)) \subset U \cap G_{n}^{(m)}$ for every $\gamma = (\varepsilon, k, l, m, n) \geq \gamma_0$.

**Proof.** For $m, n \in \mathbb{N}$, the maps $i_{m,n} : G_n^{m} \to G^{(m)}$, $i_m : G^{(m)} \to G$ and $\bar{i}_{m,n} = i_n \circ i_{m,n} : G_n^{m} \to G$ denote the canonical inclusions; $\pi_m : G \to G/G^{(m)}$ and $\pi_{m,n} : G^{(m)} \to (G^{(m)}/G_n^{(m)})$ denote the projections.

For $k, l \in C_{oo}(G)$, $\Lambda_{m,n,k,l}$ and $\bar{\Lambda}_{m,n,k,l}$ denote the associate maps of these inclusions.

Let $f \in C_{oo}(G)$, $\varphi \in L^p(G)$ and $\psi \in L^{p'}(G)$.

We consider the set $\mathcal{K}$ of all $\gamma = (\varepsilon, k, l, m, n)$ with:

(a) $\varepsilon > 0$, $m, n \in \mathbb{N}$, $k, l \in C_{oo}(G)$ with $\|k\|_1 = \|l\|_1 = 1$;
(b) supp$(k)$ and supp$(l)$ are neighborhoods of $e$ in $G$;
(c) $\left| \int_{G^{(m)}} k(xh)dm_{G^{(m)}}(h) - 1 \right| < \frac{\varepsilon}{2}$, for all $x \in$ supp$(k)$;
(d) $\left| \int_{G^{(m)}} l(xh)dm_{G^{(m)}}(h) - 1 \right| < \frac{\varepsilon}{2}$, for all $x \in$ supp$(l)$;
(e) $\left| \int_{G_n^{(m)}} k(xh)dm_{G_n^{(m)}}(h) - \int_{G^{(m)}} k(u)dm_{G^{(m)}}(u) \right| < \varepsilon$, for all $x \in$ supp$(k) \cap G^{(m)}$;
(f) $\left| \int_{G_n^{(m)}} l(xh)dm_{G_n^{(m)}}(h) - \int_{G^{(m)}} k(u)dm_{G^{(m)}}(u) \right| < \varepsilon$, for all $x \in$ supp$(l) \cap G^{(m)}$;
(g) $\left| \langle i_{m,n} \left(\Lambda_{m,n,k,l}(\lambda^p_{G^{(m)}}(f))\right) \varphi, \psi \rangle_{L^p(G^{(m)})} - \langle \lambda^p_{G^{(m)}}(f)\varphi, \psi \rangle_{L^{p'}(G^{(m)})} \right| \leq \varepsilon$.

Lemma [2] and Theorem [3] imply that $\mathcal{K}$ is non-empty. Moreover with the relation $(\varepsilon, k, l, m, n) \leq (\varepsilon', k', l', m', n')$ if and only if $\varepsilon' \leq \varepsilon$, supp$(k') \subset$ supp$(k)$, supp$(l') \subset$ supp$(l)$, $m \leq m'$ and $n \leq n'$, $\mathcal{K}$ is a directed set.

For each $\gamma \in \mathcal{K}$, we define

$$\mathcal{L}_\gamma = \bar{i}_{m,n} \circ \bar{\Lambda}_{m,n,k,l}.$$

The points (ii) and (iii) are straightforward from the proof of Theorem [3.1].

Let $\varepsilon > 0$ and $\gamma_0 = (\varepsilon_0, k_0, l_0, m_0, n_0) \in \mathcal{K}$.

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2See [2].
I) There are \(k, l \in C_{oo}(G)\) with \(\|k\|_1 = \|l\|_1 = 1\), \(\text{supp}(k) \subset \text{supp}(k_0)\), \(\text{supp}(l) \subset\text{supp}(l_0)\) and
\[
\|\tau_p \varphi - k \ast G^{(m)} \tau_p \varphi\|_p < \frac{\epsilon}{16(1 + \|f\|_1\|\varphi\|^{p'})},
\]
\[
\|\tau_p \psi - l \ast G^{(m)} \tau_p \psi\|_p < \frac{\epsilon}{16(1 + \|f\|_1\|\varphi\|^{p'})}.
\]

Therefore, for each \(m \in \mathbb{N}\),
\[
\left|\langle \lambda^{p}_{G^{(m)}}(f) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} - \langle f \ast G^{(m)} k \ast G^{(m)} \tau_p \varphi, l \ast G^{(m)} \tau_p \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| < \frac{\epsilon}{4}.
\]

Moreover, by an indirect use of Theorem 2.1, there is \(m_1 \geq m_0\) with, for each \(m \geq m_1\),
\[
\left|\langle f \ast G^{(m)} k \ast G^{(m)} \tau_p \varphi, l \ast G^{(m)} \tau_p \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} - \langle i_m \left(\Lambda_{m,k,l}(\lambda^{p}_{G^{(m)}}(f)) \varphi, \psi \right)_{L^p(G), L^{p'}(G)} \right| < \frac{\epsilon}{8}.
\]

In conclusion, for all \(m \geq m_1\), we have
\[
\left|\langle i_m \left(\Lambda_{m,k,l}(\lambda^{p}_{G^{(m)}}(f)) \varphi, \psi \right)_{L^p(G), L^{p'}(G)} - \langle \lambda^{p}_{G^{(m)}}(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| < \frac{\epsilon}{4}.
\]

With similar arguments, we can choose \(k, l \in C_{oo}(G)\) with \(\|k\|_1 = \|l\|_1 = 1\) and \(m_1 \in \mathbb{N}\) such that, for each \(m \geq m_1\) and \(n \in \mathbb{N}\), we also have
\[
\left|\langle i_{m,n} \left(\Lambda_{m,n,k,l}(\lambda^{p}_{G^{(m)}}(f)) \varphi, \psi \right)_{L^p(G), L^{p'}(G)} - \langle \lambda^{p}_{G^{(m)}}(f) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| < \frac{\epsilon}{4}.
\]

II) From Theorem 3.1 there exists \(m \in \mathbb{N}\) with \((\varepsilon_0, k, l, m, n) \geq (\varepsilon_0, k, l, m_1, n_0)\), such that
\[
\left|\langle i_m \left(\Lambda_{m,k,l}(\lambda^{p}_{G^{(m)}}(f)) \varphi, \psi \right)_{L^p(G), L^{p'}(G)} - \langle \lambda^{p}_{G^{(m)}}(f) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \right| < \frac{\epsilon}{4}.
\]

But, \((G^{(m)}_n)_{n=1}^{\infty}\) is a dense sequence of \(G^{(m)}\). Therefore, by Theorem 3.1 there exists \(n \geq n_0\) such that
\[
\left|\langle i_{m,n} \left(\Lambda_{m,n,k,l}(\lambda^{p}_{G^{(m)}}(f)) \varphi, \psi \right)_{L^p(G^{(m)}), L^{p'}(G^{(m)})} - \langle \lambda^{p}_{G^{(m)}}(f) \varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \right| < \frac{\epsilon}{4}.
III) Finally, let $\gamma = (\varepsilon_0, k, l, m, n)$. We have $\gamma \geq \gamma_0$ and
\[
\left| \langle i_{m,n} \left( \Lambda_{m,n,k,l} \left( \lambda_{G}^{p}(f) \right) \right) \varphi, \psi \rangle_{L^{p}(G),L^{p'}(G)} - \langle \lambda_{G}^{p}(f) \varphi, \psi \rangle_{L^{p}(G),L^{p'}(G)} \right| \\
\leq \left| \langle i_{m,n} \left( \Lambda_{m,n,k,l} \left( \lambda_{G}^{p}(f) \right) \right) \varphi, \psi \rangle_{L^{p}(G),L^{p'}(G)} - \langle i_{m,n} \left( \Lambda_{m,n,k,l} \left( \lambda_{G}^{p}(f) \right) \right) \varphi, \psi \rangle_{L^{p}(G^{(m)}),L^{p'}(G^{(m)})} \right| \\
+ \left| \langle i_{m,n} \left( \Lambda_{m,n,k,l} \left( \lambda_{G}^{p}(f) \right) \right) \varphi, \psi \rangle_{L^{p}(G^{(m)}),L^{p'}(G^{(m)})} - \langle \lambda_{G}^{p}(f) \varphi, \psi \rangle_{L^{p}(G^{(m)}),L^{p'}(G^{(m)})} \right| \\
+ \left| \langle \lambda_{G}^{p}(f) \varphi, \psi \rangle_{L^{p}(G),L^{p'}(G)} - \langle i_{m} \left( \Lambda_{m,k,l} \left( \lambda_{G}^{p}(f) \right) \right) \varphi, \psi \rangle_{L^{p}(G),L^{p'}(G)} \right| \\
+ \left| \langle i_{m} \left( \Lambda_{m,k,l} \left( \lambda_{G}^{p}(f) \right) \right) \varphi, \psi \rangle_{L^{p}(G),L^{p'}(G)} - \langle \lambda_{G}^{p}(f) \varphi, \psi \rangle_{L^{p}(G),L^{p'}(G)} \right| \leq \varepsilon.
\]

\[\square\]

**Example 4.2.** Consequently, we have an atomization process for abelian metrizable locally connected groups \[22\text{ Prop. 8.17}.\]

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**References**

8. **Algèbres $A_p(G)$ et convoluteurs de $L^p(G)$**, Doctorat d’état, Université Paris-Sud, Centre d’Orsay, 1971.


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