PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 11, November 2006, Pages 3231–3241
S 0002-9939(06)08345-6
Article electronically published on May 12, 2006

ATOMIZATION PROCESS FOR CONVOLUTION OPERATORS ON LOCALLY COMPACT GROUPS

CÉDRIC DELMONICO

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We develop for a large class of locally compact groups a method of approximation of convolution operators on G by finitely supported measures with control of the support and of the operator norm of the approximating measures.

1. INTRODUCTION

Let G be a locally compact group, let $1 < p < \infty$ and let $CV_p(G)$ be the Banach space of all p-convolution operators on $L^p(G)$. If μ is a bounded measure on G, the corresponding p-convolution operator is denoted $\lambda_G^p(f)(\varphi) = \varphi \star (\Delta^{-1/p}\check{f})$.

If G is amenable, given $T \in CV_p(G)$ and Ω any open subset containing $\operatorname{supp} T$, then the operator T is in the weak closure of the set of all operators $\lambda_G^p(f)$, where f is a continuous function with compact support ($f \in C_{oo}(G)$) with $\operatorname{supp} f \subset \Omega$ and $|||\lambda_G^p(f)|||_p \leq |||T|||_p$ (see [6, Thm. 5]). The following problem is discussed in the present paper: is it true that the operator T is in the weak closure of the set of all operators $\lambda_G^p(\mu)$ where μ is a finitely supported measure with $\operatorname{supp}\mu \subset \Omega$ and $|||\lambda_G^p(\mu)|||_p \leq |||T|||_p$?

In [7, 8, 9], N. Lohoué investigated a similar question for G abelian. He used in a strong way the dual of G, and his approach only applies to commutative groups. Our techniques are different and apply to a large class of non-commutative groups, such as the Heisenberg group. For a given $T \in CV_p(G)$, we construct in an explicit way an approximating net of finitely supported measures converging to T. In a preprint of [6] C. Herz announced a similar result for locally compact amenable groups. In the final version of the paper this was suppressed.

This article represents a significant portion of the author's doctoral thesis [4]. The author would like to express his deep gratitude to his supervisor, Antoine Derighetti, for support, guidance and encouragement throughout his tenure as a graduate student at the University of Lausanne and afterward as a postdoc at the EPFL of Lausanne. The author would also like to thank Noël Lohoué for the valuable suggestions he gives when he visits us in Lausanne.

Received by the editors February 3, 2005 and, in revised form, May 17, 2005.
2000 *Mathematics Subject Classification.* Primary 43A05; Secondary 43A22, 47B38, 47B48.
Key words and phrases. Convolution operators, discrete measures, approximations, dense nets.

2. Preliminaries

Let G be a locally compact group and $\mathcal{H}$ the set of the closed subgroups of G. For each K compact subset of G and $\mathcal{F}$ finite family of open subsets of G, we set $\mathcal{U}(K,\mathcal{F}) = \{H \in \mathcal{H} : H \cap K = \emptyset, \quad H \cap A \neq \emptyset \quad \forall a \in \mathcal{F}\}$. We provide $\mathcal{H}$ with the topology generated by $\mathcal{U}(K,\mathcal{F})$. A net of closed subgroups which is converging to G in this topology is called a dense net. We use the terminology of [12]. For example, an increasing sequence $(H_n)_{n\in\mathbb{N}}$ of closed subgroups with $\bigcup_{n=1}^{\infty} H_n$ dense in G is a dense net. Let m_G denote a left Haar measure of G.

Theorem 2.1. *Let $(H_\alpha)_{\alpha\in I}$ be a dense net of closed subgroups of G. For each $\alpha \in I$, there exists a left Haar measure m_α of H_α such that*

$$\lim_{\alpha\in I} \int_{H_\alpha} f(h)dm_\alpha(h) = \int_G f(x)dx, \text{ for all } f \in C_{oo}(G).$$

Moreover, if $f \in C_{oo}(G)$, then

$$\int_{H_\alpha} f(xh)dm_\alpha(h) \longrightarrow \int_G f(u)du$$

uniformly on the compact subsets of G.

This result is due to K. A. Ross & K. Stromberg [11, Thm. 2] for sequences and to J.-L. Rubio [12, Cor. 3] for nets.

The following lemma will be useful.

Lemma 2.2. *Let Φ be a continuous map of G into $\big(C_{oo}(G), \|.\|_\infty\big)$ and K a compact subset of G such that $\operatorname{supp}\big(\Phi(y)\big) \subset K$, for all $y \in G$. Then, for each $\varepsilon > 0$ and L, V compact subsets of G, there is $\alpha_0 \in I$ such that, for all $\alpha \geq \alpha_0$,*

$$\Big|\int_{H_\alpha} \Phi(y)(xh)dm_\alpha(h) - \int_G \Phi(y)(u)du\Big| < \varepsilon,$$

for all $x \in L$ and $y \in V$.

We also need to transfer convolution operators from groups to subgroups and from subgroups to groups.

If H is a closed subgroup of G, there is a isometry i of $CV_p(H)$ into $CV_p(G)$ (cf. [5, p. 76]) which satisfies, for all $T \in CV_p(H)$ and $r, s \in C_{oo}(G)$,

$$\langle i(T)r, s\rangle_{L^p(G),L^{p'}(G)} = \int_{G/H} \langle T\Big(\frac{r}{q^{1/p}}\Big)_{x,H}, \Big(\frac{s}{q^{1/p'}}\Big)_{x,H}\rangle_{L^p(H),L^{p'}(H)} d\dot{x},$$

where $r_{x,H}(h) = r(xh)$. The following properties are satisfied (cf. [1, 5]):

$$|||i(T)|||_p = |||T|||_p \tag{2.1}$$

and

$$\operatorname{supp}\big(i(T)\big) = \operatorname{supp}(T). \tag{2.2}$$

Let H be a closed subgroup of G, $f \in C_{oo}(G)$ and $\dot{x} \in G/H$. We define

$$T_{H,q}(f)(\dot{x}) = \int_H \frac{f(xh)}{q(xh)} dh,$$

where $q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1})$ (cf. [10]).

For each $T \in CV_p(G)$ and $k, l \in C_{oo}(G)$, J. Delaporte and A. Derighetti defined in [3]

$$\begin{aligned}(2.3)\qquad &\langle \Lambda_{k,l}^{(q)}(T)\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} \\ &= \langle T\Big(\tau_p\big(q^{1/p}(k \star_H \tau_p\varphi)\big)\Big), \tau_{p'}\big(q^{1/p'}(l \star_H \tau_{p'}\psi)\big)\rangle_{L^p(G),L^{p'}(G)},\end{aligned}$$

for all $\varphi, \psi \in C_{oo}(G)$. Then $\Lambda_{k,l}^{(q)}$ is a linear map of $CV_p(G)$ into $CV_p(H)$ with

$$|||\Lambda_{k,l}^{(q)}(T)||| \le \|T_H(|k|)\|_p \|T_H(|l|)\|_{p'}, \tag{2.4}$$

$$\Lambda_{k,l}^{(q)}(PM_p(G)) \subset PM_p(H), \tag{2.5}$$

$$\operatorname{supp}\big(\Lambda_{k,l}^{(q)}(T)\big) \subset H \cap \Big(\operatorname{supp}(k)^{-1} \operatorname{supp}(T) \operatorname{supp}(l)\Big). \tag{2.6}$$

In the present paper, we always suppose $q = 1$ (i.e. $\mathrm{Res}_H(\Delta_G) = \Delta_H$). This property is verified for normal closed subgroups or more generally for neutral closed subgroups of G. In this case, we set $\Lambda_{k,l}^{(q)} = \Lambda_{k,l}$ and $T_{H,q} = T_H$.

Lemma 2.3. *Let $1 < p < \infty$, G a locally compact group, H a closed subgroup of G, $f \in C_{oo}(G)$, $k, l \in C_{oo}^+(G)$, $\varphi \in L^p(G)$ and $\psi \in L^{p'}(G)$. Then, we have*

$$\begin{aligned}&\langle i\big(\Lambda_{k,l}(\lambda_G^p(f))\big)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)} - \langle \lambda_G^p(f)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)} \\ &= \int_G f(z)\bigg(\int_G\bigg(\int_G \Delta_G^{1/p}(u)\ k(z^{-1}yu)\ \varphi(u) \\ &\qquad \Big(\overline{T_H\big(\Delta_G^{1/p'}({}_y l)\psi\big)}\big(\pi(u)\big) - \overline{\int_G \Delta_G^{1/p'}(v)\ l(yv)\ \psi(v)\ dv}\Big)\ du\bigg)dy\bigg)dz \\ &\qquad + \langle f \star k \star \tau_p\varphi, l \star \tau_{p'}\psi - \tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)} \\ &\qquad\qquad + \langle f \star (k \star \tau_p\varphi - \tau_p\varphi), \tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)},\end{aligned}$$

where π is the canonical map of G onto G/H.

Proof. Let β be a Bruhat function of (G, H). On the one hand, we have

$$\begin{aligned}&\langle i\big(\Lambda_{k,l}(\lambda_G^p(f))\big)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)} \\ &= \int_{G/H} \langle \Lambda_{k,l}\big(\lambda_G^p(f)\big)\varphi_{x,H}, \psi_{x,H}\rangle_{L^p(H),L^{p'}(H)} d\dot{x} \\ &= \int_G \beta(x)\langle \lambda_G^p(f)\tau_p\big(k \star_H \tau_p(\varphi_{x,H})\big), \tau_{p'}\big(l \star_H \tau_{p'}(\psi_{x,H})\big)\rangle_{L^p(G),L^{p'}(G)} dx \\ &= \int_G f(z)\bigg(\int_G\bigg(\int_G \beta(x)\, T_H\Big(\Delta_G^{1/p}\ ({}_{z^{-1}y}k)\ \varphi\ \Big(\overline{T_H\big(\Delta_G^{1/p'}({}_y l)\psi\big)} \circ \pi\Big)\Big)\big(\pi(x)\big)\ dx\bigg)dy\bigg)dz \\ &= \int_G f(z)\bigg(\int_G\bigg(\int_G\Big(\Delta_G^{1/p}\ ({}_{z^{-1}y}k)\ \varphi\Big(\overline{T_H\big(\Delta_G^{1/p'}({}_y l)\psi\big)} \circ \pi\Big)\Big)(u)\ du\bigg)dy\bigg)dz \\ &= \int_G f(z)\bigg(\int_G\bigg(\int_G \Delta_G^{1/p}(u)\ k(z^{-1}yu)\ \varphi(u)\ \overline{T_H\big(\Delta_G^{1/p'}({}_y l)\psi\big)}\big(\pi(u)\big)\ du\bigg)dy\bigg)dz.\end{aligned}$$

On the other hand,

$$\langle \lambda_G^p(f)\varphi,\psi\rangle_{L^p(G),L^{p'}(G)} = \langle f\star\tau_p\varphi,\tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}$$

and

$$\begin{aligned}&\int_G f(z)\Big(\int_G\Big(\int_G \Delta_G^{1/p}(u)\ k(z^{-1}yu)\ \varphi(u)\ \Big(\overline{\int_G \Delta_G^{1/p'}(v)\ l(yv)\ \psi(v)\ dv}\Big)du\Big)dy\Big)dz\\ =&\langle f\star k\star\tau_p\varphi, l\star\tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}.\end{aligned}$$

Finally,

$$\begin{aligned}&\langle i\big(\Lambda_{k,l}(\lambda_G^p(f))\big)\varphi,\psi\rangle_{L^p(G),L^{p'}(G)} - \langle \lambda_G^p(f)\varphi,\psi\rangle_{L^p(G),L^{p'}(G)}\\ =&\langle i\big(\Lambda_{k,l}(\lambda_G^p(f))\big)\varphi,\psi\rangle_{L^p(G),L^{p'}(G)} - \langle f\star k\star\tau_p\varphi, l\star\tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}\\ &+\langle f\star k\star\tau_p\varphi, l\star\tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)} - \langle f\star k\star\tau_p\varphi, \tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}\\ &+\langle f\star k\star\tau_p\varphi, \tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)} - \langle \lambda_G^p(f)\varphi,\psi\rangle_{L^p(G),L^{p'}(G)}\\ =&\int_G f(z)\Big(\int_G\Big(\int_G \Delta_G^{1/p}(u)\ k(z^{-1}yu)\ \varphi(u)\\ &\Big(\overline{T_H\big(\Delta_G^{1/p'}({}_yl)\psi\big)}\big(\pi(u)\big) - \overline{\int_G \Delta_G^{1/p'}(v)\ l(yv)\ \psi(v)\ dv}\Big)\ du\Big)dy\Big)dz\\ &+\langle f\star k\star\tau_p\varphi, l\star\tau_{p'}\psi-\tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}\\ &+\langle f\star(k\star\tau_p\varphi-\tau_p\varphi), \tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}.\end{aligned}$$

□

3. Atomization process for an inductive limit of discrete groups

Let $1<p<\infty$, G a locally compact group and $\big(H_\alpha\big)_{\alpha\in I}$ a dense net of closed subgroups[1] of G.

Theorem 3.1. *Let $(H_\alpha)_{\alpha\in I}$ be a dense net of closed subgroups of G. Then, there is a net $(\mathcal{L}_\beta)_{\beta\in\mathcal{I}}$ of uniformly bounded endomorphisms of $CV_p(G)$ such that:*

- (i) *for every $f\in C_{oo}(G)$, $\lambda_G^p(f)$ is a cluster point of the net $\Big(\mathcal{L}_\beta\big(\lambda_G^p(f)\big)\Big)_{\beta\in\mathcal{I}}$ in the weak operator topology;*
- (ii) *for each $\beta=(\varepsilon,k,l,\alpha)\in\mathcal{I}$, $\|\mathcal{L}_\beta\|\le 1+\varepsilon$;*
- (iii) *for every neighborhood U of $\operatorname{supp} f$, there is $\beta_0\in\mathcal{I}$ with $\operatorname{supp}\mathcal{L}_\beta\big(\lambda_G^p(f)\big)\subset U\cap H_\alpha$, for $\beta=(\varepsilon,k,l,\alpha)\ge\beta_0$.*

Proof. Let

$$A=\Big\{k\in C_{oo}^+(G):\int_G k(u)du=1\Big\}$$

[1] We suppose $\operatorname{Res}_{H_\alpha}(\Delta_G)=\Delta_{H_\alpha}$.

and

$$\mathcal{I} = \Big\{(\varepsilon, k, l, \alpha) : 0 < \varepsilon < 1,\ k, l \in A,\ \operatorname{supp} k,\ \operatorname{supp} l \text{ neigh. of } e \text{ in } G,$$
$$\Big|\int_{H_\alpha} k(xh)dm_\alpha(h) - 1\Big| < \varepsilon \text{ for all } x \in \operatorname{supp} k,$$
$$\Big|\int_{H_\alpha} l(xh)dm_\alpha(h) - 1\Big| < \varepsilon \text{ for all } x \in \operatorname{supp} l\Big\}.$$

For $\beta = (\varepsilon, k, l, \alpha), \beta' = (\varepsilon', k', l', \alpha') \in \mathcal{I}$, we define the relation $\beta \le \beta'$ if and only if $\varepsilon' \le \varepsilon$, $\alpha \le \alpha'$, $\operatorname{supp} k' \subset \operatorname{supp} k$ and $\operatorname{supp} l' \subset \operatorname{supp} l$. Then, $(\mathcal{I}, \le)$ is a directed set.

For each $\beta = (\varepsilon, k, l, \alpha) \in \mathcal{I}$, we define the endomorphism of $CV_p(G)$,

$$\mathcal{L}_\beta = i_\alpha \circ \Lambda_{\alpha,k,l},$$

where i_α is the canonical isometry of $CV_p(H_\alpha)$ into $CV_p(G)$ and $\Lambda_{\alpha,k,l}$ the map of $CV_p(G)$ into $CV_p(H_\alpha)$ defined by (2.3).

Let $\beta = (\varepsilon, k, l, \alpha) \in I$, $\varphi \in L^p(G)$ and $\psi \in L^{p'}(G)$. From (2.1),

$$\Big|\langle i_\alpha(\Lambda_{\alpha,k,l}(T))\varphi, \psi\rangle_{L^p(G),L^{p'}(G)}\Big| \le |||T|||_p \big\|T_{H_\alpha}(k)\big\|_p \big\|T_{H_\alpha}(l)\big\|_{p'} \|\varphi\|_p \|\psi\|_{p'}.$$

But, the net $\big(T_{H_\alpha}(k)\big)_{\alpha\in I}$ is converging to $\int_G k(u)\, du = 1$ uniformly on $\pi_\alpha\big(\operatorname{supp} k\big)$, and $\big\|T_{H_\alpha}(k)\big\|_p^p = \int_{G/H_\alpha} \Big|T_{H_\alpha}(k)(\dot{x})\Big|^p d\dot{x} \le \big\|T_{H_\alpha}(k)\big\|_\infty^{p-1}$.

Moreover, for each $\dot{x} \in \pi_n\big(\operatorname{supp} k\big)$, $\big|T_{H_\alpha}(k)(\dot{x})\big| < \varepsilon + 1$. Then, $\big\|T_{H_\alpha}(k)\big\|_p \le (\varepsilon+1)^{1/p'}$ and $\big\|T_{H_\alpha}(l)\big\|_{p'} \le (\varepsilon+1)^{1/p}$. Therefore

$$\Big|\langle i_\alpha(\Lambda_{n,k,l}(T))\varphi, \psi\rangle_{L^p(G),L^{p'}(G)}\Big| \le |||T|||_p\ (\varepsilon+1)\ \|\varphi\|_p \|\psi\|_{p'},$$

and (ii) is verified.

From (2.2) and (2.6), for each $\beta \in \mathcal{I}$,

$$\begin{aligned}\operatorname{supp}\big(\mathcal{L}_\beta(T)\big) &= \operatorname{supp}\Big(i_\alpha\big(\Lambda_{\alpha,k,l}(T)\big)\Big) \subset i_\alpha\Big(\operatorname{supp}\big(\Lambda_{\alpha,k,l}(T)\big)\Big)\\ &= \operatorname{supp}\big(\Lambda_{\alpha,k,l}(T)\big) \subset H_\alpha \cap \big(\operatorname{supp}(k)^{-1}\operatorname{supp}(T)\operatorname{supp}(l)\big).\end{aligned}$$

Then, (iii) is verified.

Let $f \in C_{oo}(G)$, $W \subset G$ a neighborhood of $\operatorname{supp}(f)^{-1}$, $\varepsilon_1 > 0$, $\varphi, \psi \in C_{oo}(G)$ and $\beta = (\varepsilon, k, l, \alpha_0) \in \mathcal{I}$.

Let $0 < \varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$.

There is a compact neighborhood V of e with $V = V^{-1}$ and $V\operatorname{supp}(f)V \subset W$. There are $k', l' \in C_{oo}(G)$ with $\|k'\|_1 = \|l'\|_1 = 1$ such that $\operatorname{supp}(k')$ and $\operatorname{supp}(l')$ are neighborhoods of e, $\operatorname{supp}(k') \subset \operatorname{supp}(k) \cap V$, $\operatorname{supp}(l') \subset \operatorname{supp}(l) \cap V$,

$$\|k' \star \tau_p\varphi - \tau_p\varphi\|_p < \frac{\varepsilon}{1 + 3\|f\|_1\|\psi\|_{p'}}$$

and

$$\|l' \star \tau_{p'}\psi - \tau_{p'}\psi\|_{p'} < \frac{\varepsilon}{1 + 3\|f\|_1\|\varphi\|_p}.$$

There is $\alpha_1 \in I$ such that, for all $\alpha \ge \alpha_1$, $\big|T_{H_\alpha}(k')(\dot{x}) - 1\big| < \varepsilon$, for each $\dot{x} \in \pi_\alpha\big(\operatorname{supp}(k')\big)$ and $\big|T_{H_\alpha}(l')(\dot{x}) - 1\big| < \varepsilon$, for each $\dot{x} \in \pi_\alpha\big(\operatorname{supp}(l')\big)$.

By Lemma 2.2, there is $\alpha_2 \in I$ such that, for all $\alpha \geq \alpha_2$,

$$\Big|\int_{H_\alpha} \big(\Delta_G^{1/p'}({}_y l)\psi\big)(uh)dm_\alpha(h) - \int_G \big(\Delta_G^{1/p'}({}_y l)\psi\big)(v)\ dv\Big| < \frac{\varepsilon}{1+3\|f\|_1\|\tau_p\varphi\|_1},$$

for each $y \in \operatorname{supp}(f)V\operatorname{supp}(\varphi)^{-1}$ and for each $u \in \operatorname{supp}(\varphi)$.

Let $\alpha' \geq \alpha_0, \alpha_1, \alpha_2$. We define $\beta' = (\varepsilon, k', l', \alpha')$. We have $\beta' \in \mathcal{I}$ and $\beta' \geq \beta$.

We have $\Big\{y \in G : \exists\ u \in \operatorname{supp}(\varphi),\ \exists\ z \in \operatorname{supp}(f) \text{ with } k(z^{-1}yu) \neq 0\Big\} \subset \operatorname{supp}(f)V\operatorname{supp}(\varphi)^{-1}$. Let us make some estimations:

$$\begin{aligned}
&\int_G \big|f(z)\big|\Big(\int_G 1_A(y)\Big(\int_G \big|\Delta_G^{1/p}(u)k(z^{-1}yu)\varphi(u)\big|du\Big)dy\Big)dz\\
&\quad= \int_G \big|f(z)\big|\Big(\int_G \Big(\int_G \big|\Delta_G^{1/p}(u)k(yu)\varphi(u)\big|du\Big)dy\Big)dz\\
&\quad= \int_G \big|f(z)\big|\ dz \int_G \big|k\big| \star \big|\tau_p\varphi\big|(y)dy = \|f\|_1\|\varphi\|_1.
\end{aligned}$$

Then,

$$\begin{aligned}
&\Big|\int_G f(z)\Big(\int_G\Big(\int_G \Delta_G^{1/p}(u)\ k(z^{-1}yu)\ \varphi(u)\\
&\qquad\Big(\overline{T_{H_\alpha}\big(\Delta_G^{1/p'}({}_y l)\psi\big)}\big(\pi_\alpha(u)\big) - \overline{\int_G \Delta_G^{1/p'}(v)\ l(yv)\ \psi(v)\ dv}\Big)\ du\Big)dy\Big)dz\Big|\\
&\leq \int_G \big|f(z)\big|\Big(\int_G 1_A(y)\Big(\int_G \big|\Delta_G^{1/p}(u)k(z^{-1}yu)\varphi(u)\big|\\
&\qquad\Big|T_{H_\alpha}(\Delta_G^{1/p'}({}_y l)\psi)(\pi_\alpha(u)) - \int_G \Delta_G^{1/p'}(v)\ l(yv)\ \psi(v)\ dv\Big|\ du\Big)dy\Big)dz\\
&< \int_G \big|f(z)\big|\Big(\int_G 1_A(y)\Big(\int_G \big|\Delta_G^{1/p}(u)k(z^{-1}yu)\varphi(u)\big|du\Big)dy\Big)dz\ \frac{\varepsilon}{1+3\|f\|_1\|\varphi\|_1}\\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Big|\langle f\star k\star\tau_p\varphi, l\star\tau_{p'}\psi - \tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}\Big|\\
&\quad< \big\|f\star k\star\tau_p\varphi\big\|_p\big\|l\star\tau_{p'}\psi - \tau_{p'}\psi\big\|_{p'} < \|f\|_1\big\|k\star\tau_p\varphi\big\|_p\big\|l\star\tau_{p'}\psi - \tau_{p'}\psi\big\|_{p'}\\
&\quad< \|f\|_1\|k\|_1\|\tau_p\varphi\|_p\frac{\varepsilon}{1+3\|f\|_1\|\varphi\|_p} < \frac{\varepsilon}{3}
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\langle f\star(k\star\tau_p\varphi - \tau_p\varphi), \tau_{p'}\psi\rangle_{L^p(G),L^{p'}(G)}\Big|\\
&\quad< \big\|f\star(k\star\tau_p\varphi - \tau_p\varphi)\big\|_p\|\tau_{p'}\psi\|_{p'} < \|f\|_1\big\|k\star\tau_p\varphi - \tau_p\varphi\big\|_p\|\psi\|_{p'}\\
&\quad< \|f\|_1\|\psi\|_{p'}\frac{\varepsilon}{1+3\|f\|_1\|\psi\|_{p'}} < \frac{\varepsilon}{3}.
\end{aligned}$$

Finally, from Lemma 2.3, we have

$$\Big|\langle \mathcal{L}_\beta\big(\lambda_G^p(f)\big)\varphi,\psi\rangle_{L^p(G),L^{p'}(G)} - \langle \lambda_G^p(f)\varphi,\psi\rangle_{L^p(G),L^{p'}(G)}\Big| < \varepsilon.$$

□

Corollary 3.2. *Let $1 < p < \infty$, G a locally compact group, $\big(H_\alpha\big)_{\alpha\in I}$ a dense net of discrete subgroups of G. Then, for each $f \in C_{oo}(G)$, there is a net (μ_β) of finitely supported measures of G such that:*

(i) *for every $f \in C_{oo}(G)$, $\lambda_G^p(f)$ is a cluster point of the net $\big(\lambda_G^p(\mu_\beta)\big)_{\beta\in\mathcal{I}}$ in the weak operator topology;*
(ii) *for each $\beta = (\varepsilon,k,l,\alpha) \in \mathcal{I}$, $|||\lambda_G^p(\mu_\beta)||| \le |||\lambda_G^p(f)|||_p + \varepsilon$;*
(iii) *for every neighborhood U of $\operatorname{supp} f$, there is $\beta_0 \in \mathcal{I}$ with $\operatorname{supp}\mathcal{L}_\beta\big(\lambda_G^p(f)\big) \subset U$, for $\beta \ge \beta_0$.*

Proof. By Theorem 3.1, we have $\operatorname{supp}\mathcal{L}_\beta\big(\lambda_G^p(f)\big) \subset \operatorname{supp}\Omega_{k,l}\big(\lambda_G^p(f)\big)$, where $\beta = (\varepsilon,k,l,\alpha)$. But $\operatorname{supp}\Omega_{k,l}\big(\lambda_G^p(f)\big)$ is a compact subset of the discrete group H_α and therefore is a finite set. Finally, there is a finitely supported measure μ_β with $\mathcal{L}_\beta\big(\lambda_G^p(f)\big) = \lambda_G^p(\mu_\beta)$. □

Corollary 3.3. *Let $1 < p < \infty$, G a locally compact group with $\big(H_\alpha\big)_{\alpha\in I}$ a dense net of discrete subgroups of G and $T \in CV_p(G)$ with compact support. Then, there exists a net of finitely supported measures which converges to T with control of the support and the operator norms of the approximating measures.*

Proof. There exists a net $\big(f_\alpha\big)_{\alpha\in I}$ of positive compactly supported functions of G with $\|f_\alpha\|_1 = 1$ such that $\lambda_G^p\Big(\tau_p\big(T(\Delta_G^{1/p'}f_\alpha)\big)\Big)$ converges weakly to T with control of the support and of the operator norm (see [6, Prop. 9]). □

Remark 3.4.

(1) If G is amenable and has a dense net of discrete subgroups, every $T \in CV_p(G)$ can be approximated similarly by finitely supported measures with control of the support and of the operator norm. A similar result was formulated in a preprint of [6] but was suppressed in the published version.
(2) The amenability is not needed for the Corollaries 3.2 and 3.3.

Example 3.5. The atomization process is available for the following groups:

(1) $\mathbb{R}$ with the subgroups $H_n = 2^{-n}\mathbb{Z}$.
(2) $\mathbb{T}$ with the subgroups $H_n = \mathbb{Z}/2^{-n}\mathbb{Z}$.
(3) The group of the symmetries of the plane $0(2)$ with the subgroups H_n generated by the symmetries of the regular polygons with 2^n sides.
(4) The Heisenberg groups with the subgroups H_n of the matrices
$\begin{pmatrix} 1 & a2^{-n} & b2^{-2n} \\ 0 & 1 & c2^{-n} \\ 0 & 0 & 1 \end{pmatrix}$, where $a,b,c \in \mathbb{Z}$.

This also applies to finite products of groups admitting a dense net of discrete subgroups:

(5) Elementary groups, $\mathbb{R}^l \times \mathbb{T}^m \times \mathbb{Z}^n \times F$ with $l,m,n \in \mathbb{N}$, where F is finite.
(6) Locally compact abelian groups with no small subgroups [2, Prop. 7.9].

4. Inductive limits

Let G be a locally compact group and $\left(G^{(m)}\right)_{m=1}^{\infty}$ a dense sequence of closed subgroups of G. We suppose that for each $m \in \mathbb{N}$, there is a dense sequence $\left(G_n^{(m)}\right)_{n=1}^{\infty}$ of discrete subgroups of $G^{(m)}$.

With this hypothesis, we have:

Theorem 4.1. *There is a net $(\mathcal{L}_\gamma)_{\gamma\in\mathcal{K}}$ of uniformly bounded endomorphisms of $CV_p(G)$ such that:*

(i) *for every $f \in C_{oo}(G)$, $\lambda_G^p(f)$ is a cluster point of the net $\Big(\mathcal{L}_\gamma\big(\lambda_G^p(f)\big)\Big)_{\gamma\in\mathcal{K}}$ in the weak operator topology;*

(ii) *for each $\gamma = (\varepsilon, k, l, m, n) \in \mathcal{K}$, $\|\mathcal{L}_\gamma\| \leq 1 + \varepsilon$;*

(iii) *for every neighborhood U of $\operatorname{supp} f$, there is $\gamma_0 \in \mathcal{K}$ with $\operatorname{supp} \mathcal{L}_\gamma\big(\lambda_G^p(f)\big) \subset U \cap G_n^{(m)}$, for every $\gamma = (\varepsilon, k, l, m, n) \geq \gamma_0$.*

Proof. For $m, n \in \mathbb{N}$, the maps $i_{m,n} : G_n^m \to G^{(m)}$, $i_m : G^{(m)} \to G$ and $\breve{\imath}_{m,n} = i_n \circ i_{m,n} : G_n^m \to G$ denote the canonical inclusions; $\pi_m : G \to G/G^{(m)}$ and $\pi_{m,n} : G^{(m)} \to G^{(m)}/G_n^{(m)}$ denote the projections.

For $k, l \in C_{oo}(G)$, $\Lambda_{m,n,k,l}$, $\Lambda_{m,k,l}$ and $\breve{\Lambda}_{m,n,k,l}$ denote the associated[2] maps of these inclusions.

Let $f \in C_{oo}(G)$, $\varphi \in L^p(G)$ and $\psi \in L^{p'}(G)$.

We consider the set $\mathcal{K}$ of all $\gamma = (\varepsilon, k, l, m, n)$ with:

(a) $\varepsilon > 0$, $m, n \in \mathbb{N}$, $k, l \in C_{oo}(G)$ with $\|k\|_1 = \|l\|_1 = 1$;

(b) $\operatorname{supp}(k)$ and $\operatorname{supp}(l)$ are neighborhoods of e in G;

(c) $\displaystyle\Big|\int_{G^{(m)}} k(xh)dm_{G^{(m)}}(h) - 1\Big| < \frac{\varepsilon}{2}$, for all $x \in \operatorname{supp}(k)$;

(d) $\displaystyle\Big|\int_{G^{(m)}} l(xh)dm_{G^{(m)}}(h) - 1\Big| < \frac{\varepsilon}{2}$, for all $x \in \operatorname{supp}(l)$;

(e) $\displaystyle\Big|\int_{G_n^{(m)}} k(xh)dm_{G_n^{(m)}}(h) - \int_{G^{(m)}} k(u)dm_{G^{(m)}}(u)\Big| < \varepsilon$, for all $x \in \operatorname{supp}(k) \cap G^{(m)}$;

(f) $\displaystyle\Big|\int_{G_n^{(m)}} l(xh)dm_{G_n^{(m)}}(h) - \int_{G^{(m)}} k(u)dm_{G^{(m)}}(u)\Big| < \varepsilon$, for all $x \in \operatorname{supp}(l) \cap G^{(m)}$;

(g) $$\Big|\langle i_{m,n}\Big(\Lambda_{m,n,k,l}\big(\lambda^p_{G^{(m)}}(f)\big)\Big)\varphi, \psi\rangle_{L^p(G^{(m)}),L^{p'}(G^{(m)})} - \langle \lambda^p_{G^{(m)}}(f)\varphi, \psi\rangle_{L^p(G^{(m)}),L^{p'}(G^{(m)})}\Big| \leq \varepsilon.$$

Lemma 2.2 and Theorem 3.1 imply that $\mathcal{K}$ is non-empty. Moreover with the relation $(\varepsilon, k, l, m, n) \leq (\varepsilon', k', l', m', n')$ if and only if $\varepsilon' \leq \varepsilon$, $\operatorname{supp}(k') \subset \operatorname{supp}(k)$, $\operatorname{supp}(l') \subset \operatorname{supp}(l)$, $m \leq m'$ and $n \leq n'$, $\mathcal{K}$ is a directed set.

For each $\gamma \in \mathcal{K}$, we define

$$\mathcal{L}_\gamma = \breve{\imath}_{m,n} \circ \breve{\Lambda}_{m,n,k,l}.$$

The points (ii) and (iii) are straightforward from the proof of Theorem 3.1.

Let $\epsilon > 0$ and $\gamma_0 = (\varepsilon_0, k_0, l_0, m_0, n_0) \in \mathcal{K}$.

---

[2] See (2.3).

I) There are $k, l \in C_{oo}(G)$ with $\|k\|_1 = \|l\|_1 = 1$, $\operatorname{supp}(k) \subset \operatorname{supp}(k_0)$, $\operatorname{supp}(l) \subset \operatorname{supp}(l_0)$ and

$$\|\tau_p \varphi - k \star_{G^{(m)}} \tau_p \varphi\|_p < \frac{\epsilon}{16(1 + \|f\|_1 \|\psi\|_{p'})},$$
$$\|\tau_{p'} \psi - l \star_{G^{(m)}} \tau_{p'} \psi\|_p < \frac{\epsilon}{16(1 + \|f\|_1 \|\varphi\|_p)}.$$

Therefore, for each $m \in \mathbb{N}$,

$$\Big| \langle \lambda^p_{G^{(m)}}(f)\varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} - \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \Big| < \frac{\epsilon}{8}.$$

Moreover, by an indirect use of Theorem 2.1, there is $m_1 \geq m_0$ with, for each $m \geq m_1$,

$$\Big| \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} - \langle i_m\Big(\Lambda_{m,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \Big| < \frac{\epsilon}{8}.$$

In conclusion, for all $m \geq m_1$, we have

$$\begin{aligned}
&\Big| \langle i_m\Big(\Lambda_{m,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda^p_{G^{(m)}}(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \Big| \\
&\leq \Big| \langle i_m\Big(\Lambda_{m,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \\
&\qquad - \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \Big| \\
&+ \Big| \langle f \star_{G^{(m)}} k \star_{G^{(m)}} \tau_p \varphi, l \star_{G^{(m)}} \tau_{p'} \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \\
&\qquad - \langle \lambda^p_{G^{(m)}}(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \Big| < \frac{\epsilon}{4}.
\end{aligned}$$

With similar arguments, we can choose $k, l \in C_{oo}(G)$ with $\|k\|_1 = \|l\|_1 = 1$ and $m_1 \in \mathbb{N}$ such that, for each $m \geq m_1$ and $n \in \mathbb{N}$, we also have

$$\Big| \langle \breve{i}_{m,n}\Big(\breve{\Lambda}_{m,n,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle i_{m,n}\Big(\Lambda_{m,n,k,l}\big(\lambda^p_{G^{(m)}}(f)\big)\Big)\varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \Big| < \frac{\epsilon}{4}.$$

II) From Theorem 3.1, there exists $m \in \mathbb{N}$ with $(\varepsilon_0, k, l, m, n) \geq (\varepsilon_0, k, l, m_1, n_0)$, such that

$$\Big| \langle i_m\Big(\Lambda_{m,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} - \langle \lambda^p_G(f)\varphi, \psi \rangle_{L^p(G), L^{p'}(G)} \Big| < \frac{\epsilon}{4}.$$

But, $\big(G_n^{(m)}\big)_{n=1}^{\infty}$ is a dense sequence of $G^{(m)}$. Therefore, by Theorem 3.1, there exists $n \geq n_0$ such that

$$\Big| \langle i_{m,n}\Big(\Lambda_{m,n,k,l}\big(\lambda^p_{G^{(m)}}(f)\big)\Big)\varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} - \langle \lambda^p_{G^{(m)}}(f)\varphi, \psi \rangle_{L^p(G^{(m)}), L^{p'}(G^{(m)})} \Big| < \frac{\epsilon}{4}.$$

III) Finally, let $\gamma = (\varepsilon_0, k, l, m, n)$. We have $\gamma \geq \gamma_0$ and

$$\begin{aligned}
&\Big|\langle \breve{\imath}_{m,n}\Big(\breve{\Lambda}_{m,n,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)} - \langle \lambda^p_G(f)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)}\Big| \\
&\quad\leq \Big|\langle \breve{\imath}_{m,n}\Big(\breve{\Lambda}_{m,n,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)} \\
&\qquad\qquad - \langle i_{m,n}\Big(\Lambda_{m,n,k,l}\big(\lambda^p_{G^{(m)}}(f)\big)\Big)\varphi, \psi\rangle_{L^p(G^{(m)}),L^{p'}(G^{(m)})}\Big| \\
&\qquad + \Big|\langle i_{m,n}\Big(\Lambda_{m,n,k,l}\big(\lambda^p_{G^{(m)}}(f)\big)\Big)\varphi, \psi\rangle_{L^p(G^{(m)}),L^{p'}(G^{(m)})} \\
&\qquad\qquad - \langle \lambda^p_{G^{(m)}}(f)\varphi, \psi\rangle_{L^p(G^{(m)}),L^{p'}(G^{(m)})}\Big| \\
&\qquad + \Big|\langle \lambda^p_{G^{(m)}}(f)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)} \\
&\qquad\qquad - \langle i_{m}\Big(\Lambda_{m,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)}\Big| \\
&\qquad + \Big|\langle i_{m}\Big(\Lambda_{m,k,l}\big(\lambda^p_G(f)\big)\Big)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)} \\
&\qquad\qquad - \langle \lambda^p_G(f)\varphi, \psi\rangle_{L^p(G),L^{p'}(G)}\Big| < \epsilon.
\end{aligned}$$

□

Example 4.2. Consequently, we have an atomization process for abelian metrizable locally connected groups [2, Prop. 8.17].

References

1. Jean-Philippe Anker, *Applications de la p-induction en analyse harmonique*, Comment. Math. Helv. **58** (1983), no. 4, 622–645. MR0728457 (85c:22006)
2. David L. Armacost, *The structure of locally compact abelian groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 68, Marcel Dekker, Inc., New York, 1981. MR0637201 (83h:22010)
3. Jacques Delaporte and Antoine Derighetti, *p-pseudomeasures and closed subgroups*, Monatsh. Math. **119** (1995), no. 1-2, 37–47. MR1315682 (97a:43001)
4. Cédric Delmonico, *Opérateurs de convolution et homomorphismes de groupes localement compacts.*, Ph.D. thesis, Université de Lausanne, 2003.
5. Antoine Derighetti, *Relations entre les convoluteurs d'un groupe localement compact et ceux d'un sous-groupe fermé*, Bull. Sci. Math. (2) **106** (1982), no. 1, 69–84. MR0654425 (83j:43008)
6. Carl Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) **23** (1973), no. 3, 91–123. MR0355482 (50:7956)
7. Noël Lohoué, *Sur le critère de S. Bochner dans les algèbres $B_p(R^n)$ et l'approximation bornée des convoluteurs*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A247–A250. MR0273300 (42:8180)
8. ———, *Algèbres $A_p(G)$ et convoluteurs de $L^p(G)$*, Doctorat d'état, Université Paris-Sud, Centre d'Orsay, 1971.
9. ———, *La synthèse des convoluteurs sur un groupe abélien localement compact*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A27–A29. MR0275062 (43:820)
10. Hans Reiter and Jan D. Stegeman, *Classical harmonic analysis and locally compact groups*, second ed., The Clarendon Press, Oxford University Press, New York, 2000. MR1802924 (2002d:43005)

11. Kenneth A. Ross and Karl Stromberg, *Jessen's theorem on Riemann sums for locally compact groups*, Pacific J. Math. **20** (1967), 135–147. MR0207889 (34:7702)
12. Jose L. Rubio, *Nets of subgroups in locally compact groups*, Comment. Math. Prace Mat. **20** (1977/78), no. 2, 453–466. MR0519384 (80c:43002)

School of Basic Sciences, École Polytechnique Fédérale de Lausanne, Station 8, CH-1005 Lausanne, Switzerland

E-mail address: `cedric.delmonico@a3.epfl.ch`