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ON CARVALHO'S K-THEORETIC FORMULATION OF THE COBORDISM INVARIANCE OF THE INDEX

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Abstract. We give an analytic proof of the fact that the index of an elliptic operator on the boundary of a compact manifold vanishes when the principal symbol comes from the restriction of a K-theory class from the interior. The proof uses non-commutative residues inside the calculus of cusp pseudodifferential operators of Melrose.

Introduction

A classical result of Thom states that the topological signature of the boundary of a compact manifold with boundary vanishes. Regarding the signature as the index of an elliptic operator, Atiyah and Singer [2] generalized this vanishing to the so-called twisted signatures. The cobordism invariance of the index, as this vanishing is known, was the essential step in their first proof of the index formula on closed manifolds. Conversely, cobordism invariance follows from the index theorem of [2].

On open manifolds a satisfactory index formula is not available, and probably not reasonable to expect in full generality. Such formulae in various particular cases are given, e.g., in [1], [12] for manifolds with boundary, in [20], [9] for manifolds with fibered boundary, and in [11], [8] for manifolds with corners in the sense of Melrose. To advance in this direction, we believe it is important to understand conditions which ensure the vanishing of the index, in particular cobordism invariance, without using any index formula.

Direct proofs of the cobordism invariance of the index for first-order differential operators on closed manifolds were given, e.g., in [4], [7], [10], [19], and also [18, Theorem 1]. We have proposed in [18] an extension of cobordism invariance to manifolds with corners. The result states that the sum of the indices on the hyperfaces is null, under suitable hypothesis.

All these results are partial, in that they only apply to differential operators of a special type. A well-known fact states that the index of "geometrically defined" operators is cobordism-invariant; but besides being vague, this is also not true (look at the Gauß-Bonnet operator). Only very recently, Carvalho [5, 6] found a

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remarkable K-theoretic statement of cobordism invariance of the index on open manifolds, using the topological approach of [3]. Here is a reformulation of the main result of [5] specialized to closed manifolds:

Theorem 1. *Let M be the boundary of the compact manifold X and D an elliptic pseudodifferential operator on M. The principal symbol of D defines a vector bundle over the sphere bundle inside $T^*M \oplus \underline{\mathbb{R}}$. If the class in $K^0(S(T^*M \oplus \underline{\mathbb{R}}))$ of this bundle is the restriction of a class from $K^0(S^*X)$ modulo $K^0(M)$, then* $\operatorname{index}(D) = 0$.

The missing details appear in Theorem 3. The aim of this note is to reprove Theorem 1 with analytic methods. In order to make the proof likely to generalize to open manifolds, we have made a point of avoiding using results from K-theory, e.g., Bott periodicity and the index theorem. Our approach is based on Theorem 2, a statement about the cusp calculus of pseudodifferential operators of Melrose on the manifold with boundary X, in the spirit of [18].

Although they do not appear explicitly in the literature, Carvalho's statement (in the closed manifold case) and its present variant could be recovered from known results in K-theory and K-homology. We would like to mention here only [16, Prop. 3], whose arrow-theoretic proof could be extended to pseudodifferential operators. For completeness, we show in Section 4 how to also retrieve Theorem 1 from the Atiyah-Singer index formula.

1. Review of Melrose's cusp algebra

In this section we recall the facts about the cusp algebra needed in the sequel. For a full treatment of the cusp algebra we refer to [15] and [8].

Let X be a compact manifold with boundary M, and $x : X \to \mathbb{R}_+$ a boundary-defining function. Choose a product decomposition $M \times [0, \epsilon) \hookrightarrow X$. A vector field V on X is called *cusp* if $dx(V) \in x^2\mathcal{C}^\infty(X)$. The space of cusp vector fields forms a Lie subalgebra ${}^c\mathcal{V}(X) \hookrightarrow \mathcal{V}(X)$ which is a finitely generated projective $\mathcal{C}^\infty(X)$-module; indeed, a local basis of ${}^c\mathcal{V}(X)$ is given by $\{x^2\partial_x, \partial_{y_j}\}$, where y_j are local coordinates on M. Thus there exists a vector bundle ${}^cTX \to X$ such that ${}^c\mathcal{V}(X) = \mathcal{C}^\infty(X, {}^cTX)$. Fix a Riemannian metric g on $X \setminus M$ of the form $dx^2/x^4 + h^M$ near $x = 0$; it extends to a metric on the fibers of ${}^cTX$ over X and is called a *cusp metric* on X (such g is traditionally called an *exact* cusp metric).

The algebra $\mathcal{D}_c(X)$ of (scalar) cusp differential operators is defined as the universal enveloping algebra of ${}^c\mathcal{V}(X)$ over $\mathcal{C}^\infty(X)$. In a product decomposition as above, an operator in $\mathcal{D}_c(X)$ of order m takes the form

$$P = \sum_{j=0}^{m} P_{m-j}(x)(x^2\partial_x)^j, \tag{1}$$

where $P_{m-j}(x)$ is a smooth family of differential operators of order $m - j$ on M.

1.1. **Cusp pseudodifferential operators.** The operators in $\mathcal{D}_c(X)$ can be described alternately (see [15]) in terms of their Schwartz kernels. Namely, there exists a manifold with corners X_c^2 obtained by blow-up from $X \times X$, and a submanifold Δ_c, such that $\mathcal{D}_c(X)$ corresponds to the space of distributions on X_c^2 which are classical conormal to Δ_c, supported on Δ_c and smooth at the boundary face of X_c^2 which meets Δ_c. It is then a showcase application of Melrose's program [13] to construct a calculus of pseudodifferential operators $\Psi_c^\lambda(X)$, $\lambda \in \mathbb{C}$, in which $\mathcal{D}_c(X)$ sits as the subalgebra of differential operators (the symbols used in the definition are classical of order λ). No extra difficulty appears in defining cusp operators acting between sections of vector bundles over X. By adjoining the multiplication operators by x^z, $z \in \mathbb{C}$, we get a pseudodifferential calculus with two complex indices

$$\Psi_c^{\lambda,z}(X, \mathcal{F}, \mathcal{G}) := x^{-z}\Psi_c^{\lambda}(X, \mathcal{F}, \mathcal{G})$$

such that $\Psi_c^{\lambda,z}(X, \mathcal{E}, \mathcal{F}) \subset \Psi_c^{\lambda',z'}(X, \mathcal{E}, \mathcal{F})$ if and only if $\lambda' - \lambda \in \mathbb{N}$ and $z' - z \in \mathbb{N}$ (since we work with classical symbols). Also,

$$\Psi_c^{\lambda,z}(X, \mathcal{G}, \mathcal{H}) \circ \Psi_c^{\lambda',z'}(X, \mathcal{F}, \mathcal{G}) \subset \Psi_c^{\lambda+\lambda',z+z'}(X, \mathcal{F}, \mathcal{H}).$$

The fixed cusp metric and a metric on $\mathcal{F}$ allow one to define the space of cusp square-integrable sections $L_c^2(X, \mathcal{F})$. By closure, cusp operators act on a scale of weighted Sobolev spaces $x^\alpha H_c^\beta$:

$$\Psi_c^{\lambda,z}(X, \mathcal{F}, \mathcal{G}) \times x^\alpha H_c^\beta(X, \mathcal{F}) \to x^{\alpha-\Re(z)} H_c^{\beta-\Re(\lambda)}(X, \mathcal{G}).$$

1.2. **Symbol maps.** There exists a natural surjective *cusp principal symbol* map from Ψ_c^λ onto the space of homogeneous functions on ${}^cT^*X \setminus \{0\}$ of homogeneity λ, which extends the usual principal symbol map over the interior of X:

$$\sigma : \Psi_c^\lambda(X, \mathcal{E}, \mathcal{F}) \to \mathcal{C}^\infty_{[\lambda]}({}^cT^*X, \mathcal{E}, \mathcal{F}).$$

In the sequel we refer to σ as the principal symbol map. A cusp operator is called *elliptic* if its (cusp) principal symbol is invertible on ${}^cT^*X \setminus \{0\}$.

Definition ([14])**.** Let $\Psi^\lambda_{\mathrm{sus}}(M, \mathcal{E}, \mathcal{F})$ be the space of classical pseudo-differential operators P of order $\lambda \in \mathbb{C}$ from $\mathcal{E}$ to $\mathcal{F}$ which are translation invariant, and such that the convolution kernel $\kappa_P(x, y_1, y_2)$ (which is smooth for $x \neq 0$) decays rapidly as $|x|$ tends to infinity.

Under partial Fourier transform in the variable x, $\Psi_{\mathrm{sus}}(M, \mathcal{E}, \mathcal{F})$ is identified with the space of families of operators on M with one real parameter ξ, with joint symbolic behavior in ξ and in the cotangent variables of T^*M.

The second symbol map is a surjection $I_M : \Psi_c^\lambda(X, \mathcal{E}, \mathcal{F}) \to \Psi^\lambda_{\mathrm{sus}}(M, \mathcal{E}, \mathcal{F})$, called the *indicial family* map [14]. If P is given by (1) near $x = 0$, then

$$I_M(P)(\xi) = \sum_{j=0}^{m} P_{m-j}(x)(i\xi)^j.$$

The principal symbol map and the indicial family are star-morphisms, i.e., they are multiplicative and commute with taking adjoints. Elliptic cusp operators whose indicial family is invertible for each $\xi \in \mathbb{R}$ are called fully elliptic. Being fully elliptic is equivalent to being Fredholm (see [12]).

Let $L^\lambda := \{(U, \alpha) \in \Psi^\lambda_{\mathrm{sus}}(M, \mathcal{E}, \mathcal{F}) \times \mathcal{C}^\infty_{[\lambda]}({}^cT^*X, \mathcal{E}, \mathcal{F}); \sigma(U) = \alpha_{|x=0}\}$. It is proved in [15] that the joint symbol map

$$(\sigma_\lambda, I_M) : \Psi_c^\lambda(X, \mathcal{E}, \mathcal{F}) \to L^\lambda \tag{2}$$

is surjective.

1.3. **Analytic families of cusp operators.** Let $Q \in \Psi_c^{1,0}(X,\mathcal{E})$ be a positive fully elliptic cusp operator of order 1. Then the complex powers Q^λ form an analytic family of cusp operators of order λ.

Let $\mathbb{C}^2 \ni (\lambda, z) \mapsto P(\lambda, z) \in \Psi_c^{\lambda,z}(X,\mathcal{E})$ be an analytic family in two complex variables. Then $P(\lambda, z)$ is trace-class on $L^2_c(M,\mathcal{E})$ for $\Re(\lambda) < -\dim(X), \Re(z) < -1$. Moreover, $(\lambda, z) \mapsto \mathrm{Tr}(P(\lambda, z))$ is analytic, extends to $\mathbb{C}^2$ meromorphically with at most simple poles in each variable at $\lambda \in \mathbb{N} - \dim(X)$, $z \in \mathbb{N} - 1$, and

$$\mathrm{Res}_{z=-1}\mathrm{Tr}(P(\lambda, z)) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}(I_M(x^{-1}P(\lambda,-1)))d\xi. \tag{3}$$

This identity is the content of [18, Prop. 3].

2. Cobordism invariance for cusp operators

This section extends a result from [18] to pseudodifferential operators, in a form which can be applied to K-theory. We use the same line of proof, with some extra technical difficulties. A similar extension from the differential to the pseudodifferential case appears in [17] when computing the K-theory of the algebra $\Psi^0_{\mathrm{sus}}(M)$.

Theorem 2. *Let X be a compact manifold with boundary $\partial X = M$, and*

$$D : \mathcal{C}^\infty(M,\mathcal{E}^+) \to \mathcal{C}^\infty(M,\mathcal{E}^-)$$

a classical pseudodifferential operator of order 1 on M. Assume that there exist hermitian vector bundles $V^+, V^- \to M$, $\mathcal{G} \to X$ with $\mathcal{G}|_M = \mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+ \oplus V^-$, and an elliptic symmetric cusp pseudodifferential operator $A \in \Psi_c^{1,0}(X,\mathcal{G})$ such that

$$I_M(A)(\xi) = \begin{bmatrix} \xi & \tilde{D}^*(\xi) & & \\ \tilde{D}(\xi) & -\xi & & \\ & & (1+\xi^2+\Delta^+)^{\frac{1}{2}} & \\ & & & -(1+\xi^2+\Delta^-)^{\frac{1}{2}} \end{bmatrix}, \tag{4}$$

where Δ^+, Δ^- are connection Laplacians on V^+, V^-, $\tilde{D} \in \Psi^1_{\mathrm{sus}}(M,\mathcal{E}^+,\mathcal{E}^-)$, and $\tilde{D}(0) = D$. Then $\mathrm{index}(D) = 0$.

Proof. We first show that we can assume without loss of generality that D is either injective or surjective. Assuming this, we construct from A a positive cusp operator Q of order 1. The complex powers of Q are used in defining a complex number N as a non-commutative residue. The proof will be finished by computing N in two ways; first we get $N = 0$, then N is shown to be essentially $\mathrm{index}(D)$.

Reduction to the case where D is injective or surjective. Fix an operator $T \in \Psi^{-\infty}(M,\mathcal{E}^+,\mathcal{E}^-)$ such that $D + T$ is either injective or surjective (or both). Choose $\tilde{T} \in \Psi^{-\infty}_{\mathrm{sus}}(X,\mathcal{E}^+,\mathcal{E}^-)$ with $\tilde{T}(0) = T$. Choose $S \in \Psi_c^{-\infty,0}(X,\mathcal{G})$ such that

$$I_M(S)(\xi) = \begin{bmatrix} & \tilde{T}^*(\xi) & & \\ \tilde{T}(\xi) & & & \\ & & 0 & \\ & & & 0 \end{bmatrix}.$$

We can assume that S is symmetric (if not, replace S by $(S + S^*)/2$). Replace D by $D+T$ and A by $A+S$. Note that $\mathrm{index}(D) = \mathrm{index}(D+T)$, since $T : H^1_c \to L^2_c$ is compact. The hypothesis of the theorem (in particular (4)) still holds for $D+T$ instead of D and with $A+S$ instead of A. So we can additionally assume that D is surjective or injective.

Construction of a positive cusp operator Q. For $\xi \in \mathbb{R}$ we have $\sigma_1(\tilde{D}(\xi)) = \sigma_1(D)$, so $\tilde{D}(\xi)$ is elliptic as an operator on M and $\operatorname{index}(\tilde{D}(\xi)) = \operatorname{index}(D)$. If D is surjective or injective, then 0 does not belong to the spectrum of DD^* (respectively D^*D), so by continuity $\tilde{D}(\xi)$ will have the same property for small enough $|\xi|$. Thus there exists $\epsilon > 0$ such that the kernel and the cokernel of $\tilde{D}(\xi)$ have constant dimension (hence they vary smoothly) for all $|\xi| < \epsilon$. Choose a smooth real function ϕ supported in $[-\epsilon, \epsilon]$ such that $\phi(0) = 1$. By [18, Lemma 2] and the choice of ϕ, the families $\phi(\xi)P_{\ker \tilde{D}(\xi)}$ and $\phi(\xi)P_{\ker \tilde{D}(\xi)}$ define suspended operators in $\Psi^{-\infty}_{\mathrm{sus}}(M)$. Let $R \in \Psi_c^{-\infty,0}(X, \mathcal{G})$ be such that

$$
I_M(R)(\xi) = \begin{bmatrix} \phi(\xi)P_{\ker \tilde{D}(\xi)} & & & \\ & \phi(\xi)P_{\operatorname{coker}\tilde{D}(\xi)} & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \tag{5}
$$
$$
\in \Psi^{-\infty}_{\mathrm{sus}}(M, \mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+ \oplus V^-).
$$

It follows that $I_M(A^2 + R^*R)(\xi)$ is invertible for all $\xi \in \mathbb{R}$, so the cusp operator $A^2 + R^*R$ is fully elliptic; this implies that it is Fredholm, and moreover its kernel is made of smooth sections vanishing rapidly towards ∂X. Let $P_{\ker(A^2+R^*R)}$ be the orthogonal projection on the finite-dimensional nullspace of $A^2 + R^*R$. Clearly $A^2 + R^*R \geq 0$, thus $A^2 + R^*R + P_{\ker(A^2+R^*R)}$ is strictly positive. Set

$$
Q := (A^2 + R^*R + P_{\ker(A^2+R^*R)})^{1/2}
$$

and let Q^λ be the complex powers of Q. Since $Q^2 - A^2 \in \Psi_c^{-\infty,0}(X, \mathcal{G})$ and A is self-adjoint, we deduce that for all $\lambda \in \mathbb{C}$,

$$
[A, Q^\lambda] \in \Psi_c^{-\infty,0}(X, \mathcal{G}). \tag{6}
$$

A non-commutative residue. Let $P(\lambda, z) \in \Psi_c^{-\lambda-1,-z-1}(X, \mathcal{G})$ be the analytic family of cusp operators

$$
P(\lambda, z) := [x^z, A]Q^{-\lambda-1}.
$$

From (3), $\operatorname{Tr}(P(\lambda, z))$ is holomorphic in $\{(\lambda, z) \in \mathbb{C}^2; \Re(\lambda) > \dim(X)-1, \Re(z) > 0\}$ and extends meromorphically to $\mathbb{C}^2$. Following the scheme of [18, Theorem 1], our proof of Theorem 3 will consist of computing in two different ways the complex number

$$
N := \operatorname{Res}_{\lambda=0}\left(\operatorname{Tr}(P(\lambda, z))|_{z=0}\right),
$$

i.e., N is the coefficient of $\lambda^{-1}z^0$ in the Laurent expansion of $\operatorname{Tr}(P(\lambda, z))$ around $(0, 0)$. The idea is to evaluate at $z = 0$ *before* and then *after* taking the residue at $\lambda = 0$, noting that the final answer is independent of this order.

Vanishing of N. On one hand,

$$
P(\lambda, z) = x^z[A, Q^{-\lambda-1}] + [A, Q^{-\lambda-1}x^z].
$$

The meromorphic function $\operatorname{Tr}[A, Q^{-\lambda-1}x^z]$ is identically zero since it vanishes on the open set $\{(\lambda, z) \in \mathbb{C}^2; \Re(\lambda) > \dim(X) - 1, \Re(z) > 0\}$ by the trace property. By (6), the function $\operatorname{Tr}(x^z[A, Q^{-\lambda-1}])$ is regular in $\lambda \in \mathbb{C}$, so in particular the meromorphic function

$$
z \mapsto \operatorname{Res}_{\lambda=0}\operatorname{Tr}(x^z[A, Q^{-\lambda-1}])
$$

vanishes. We conclude that $N = 0$.

Second computation of N. On the other hand, $P(\lambda,0)=0$ so

$$U(\lambda,z):=z^{-1}P(\lambda,z)\in\Psi_c^{-\lambda-1,-z-1}(X,\mathcal{G})$$

is an analytic family in $\Psi_c(X,\mathcal{G})$. Set $[\log x,A]:=(z^{-1}[x^z,A])|_{z=0}\in\Psi_c^{0,1}(X,\mathcal{G})$. Then $U(\lambda,0)=[\log x,A]Q^{-\lambda-1}$. By multiplicativity of the indicial family,

$$I_M(x^{-1}U(\lambda,0))=I_M(x^{-1}[\log x,A])I_M(Q^{-\lambda-1}).$$

By (4) and [8, Lemma 3.4], we see that $I_M(x^{-1}[\log x,A])$ is the 4×4 diagonal matrix

$$\begin{bmatrix} i &&& \\ & -i && \\ && i\xi(1+\xi^2+\Delta^+)^{-\frac{1}{2}} & \\ &&& -i\xi(1+\xi^2+\Delta^-)^{-\frac{1}{2}}\end{bmatrix}$$

and

$$I_M(Q^{-\lambda-1})=I_M(A^2+R^*R)^{-\frac{\lambda+1}{2}}.$$

Also, using (5), we deduce that $I_M(A^2+R^*R)$ is the 4×4 diagonal matrix with entries

$$\begin{aligned} a_{11}&=\xi^2+\tilde{D}(\xi)^*\tilde{D}(\xi)+\phi^2(\xi)P_{\ker\tilde{D}(\xi)}, && a_{33}=1+\xi^2+\Delta^+,\\ a_{22}&=\xi^2+\tilde{D}(\xi)\tilde{D}(\xi)^*+\phi^2(\xi)P_{\operatorname{coker}\tilde{D}(\xi)}, && a_{44}=1+\xi^2+\Delta^-.\end{aligned}$$

By (3),

$$\begin{aligned}\operatorname{Tr}(P(\lambda,z))|_{z=0}&=\frac{1}{2\pi}\int_{\mathbb{R}}\operatorname{Tr}(I_M(x^{-1}U(\lambda,0)))d\xi\\ &=\frac{i}{2\pi}\int_{\mathbb{R}}\Big(\operatorname{Tr}(\xi^2+\tilde{D}(\xi)^*\tilde{D}(\xi)+\phi^2(\xi)P_{\ker\tilde{D}(\xi)})^{-\frac{\lambda+1}{2}}\\ &\quad-\operatorname{Tr}(\xi^2+\tilde{D}(\xi)\tilde{D}(\xi)^*+\phi^2(\xi)P_{\operatorname{coker}\tilde{D}(\xi)})^{-\frac{\lambda+1}{2}}\\ &\quad+\xi\operatorname{Tr}(1+\xi^2+\Delta^+)^{-\frac{\lambda}{2}-1}-\xi\operatorname{Tr}(1+\xi^2+\Delta^-)^{-\frac{\lambda}{2}-1}\Big)d\xi.\end{aligned}$$

The third and fourth terms in this sum are odd in ξ so their integral vanishes. For each fixed ξ we compute the trace of the first two terms by using the orthonormal basis of $L^2_c(M,\mathcal{E}^+)$, $L^2_c(M,\mathcal{E}^-)$ given by eigensections of $\tilde{D}(\xi)^*\tilde{D}(\xi)$, respectively $\tilde{D}(\xi)\tilde{D}(\xi)^*$. The non-zero parts of the spectrum of $\tilde{D}(\xi)^*\tilde{D}(\xi)$ and $\tilde{D}(\xi)\tilde{D}(\xi)^*$ coincide, so what is left is

$$\int_{\mathbb{R}}\operatorname{index}(\tilde{D}(\xi))(\xi^2+\phi^2(\xi))^{-\frac{\lambda+1}{2}}d\xi.$$

The subtle point here is that the kernel and cokernel of $\tilde{D}(\xi)$ may have jumps when $|\xi|>\epsilon$, but our formula involves only the index, which is homotopy invariant and equals $\operatorname{index}(D)$ for all $\xi\in\mathbb{R}$. Thus the index comes out of the integral; the residue

$$\operatorname{Res}_{\lambda=0}\int_{\mathbb{R}}(\xi^2+\phi^2(\xi))^{-\frac{\lambda+1}{2}}d\xi$$

is independent of the compactly supported function ϕ and equals 2, so

$$0=N=\operatorname{Res}_{\lambda=0}\operatorname{Tr}(P(\lambda,z))|_{z=0}=\frac{i}{\pi}\operatorname{index}(D).\qquad\square$$

3. The K-theoretic characterization of cobordism invariance

We now interpret Theorem 2 in topological terms. Let

$$p : S^*_{\mathrm{sus}}(M) \to M$$

be the sphere bundle inside $T^*_{\mathrm{sus}}M := T^*M \oplus \underline{\mathbb{R}}$. We also denote by p the bundle projections $T^*M \to M$, $T^*X \to X$, $S^*X \to X$. The total space of $S^*_{\mathrm{sus}}(M)$ is the boundary of ${}^cS^*X$. By fixing a product decomposition of X near M, we get non-canonical vector bundle isomorphisms making the diagram

$$\begin{tikzcd} {}^cT^*X \arrow[r, "r"] \arrow[d, "\cong"] & T^*_{\mathrm{sus}}M \arrow[d, "\cong"] \\ T^*X \arrow[r, "r"] & T^*X|_M \end{tikzcd}$$

commutative, so we can replace ${}^cS^*X$ with the more familiar space S^*X in all the topological considerations of this section.

The interior unit normal vector inclusion $\imath : M \to S^*_{\mathrm{sus}}(M)$ and the bundle projection $p : S^*_{\mathrm{sus}}(M) \to M$ induce a splitting

$$K^0(S^*_{\mathrm{sus}}(M)) = \ker(\imath) \oplus p^*(K^0(M)).$$

Let

$$r : K^0(S^*X) \to K^0(S^*_{\mathrm{sus}}(M))$$

be the map of restriction to the boundary, and

$$d : K^0(T^*M) \to K^0(S^*_{\mathrm{sus}}(M))/p^*(K^0(M))$$

the isomorphism defined as follows: if $(\mathcal{E}^+, \mathcal{E}^-, \sigma)$ is a triple defining a class in $K^0(T^*M)$ with $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ an isomorphism outside the open unit ball, then

$$d(\mathcal{E}^+, \mathcal{E}^-, \sigma) = \begin{cases} \mathcal{E}^+ & \text{on } S^*_{\mathrm{sus}}(M) \cap \{\xi \geq 0\}, \\ \mathcal{E}^- & \text{on } S^*_{\mathrm{sus}}(M) \cap \{\xi \leq 0\} \end{cases}$$

with the two bundles identified via σ over $S^*_{\mathrm{sus}}(M) \cap \{\xi = 0\} = S^*M$. We can now reformulate Theorem 1 as follows:

Theorem 3. *Let X be a compact manifold with closed boundary M, $\mathcal{E}^\pm \to M$ hermitian vector bundles, and $D \in \Psi(M, \mathcal{E}^+, \mathcal{E}^-)$ an elliptic pseudodifferential operator with symbol class*

$$[\sigma(D)] := (p^*\mathcal{E}^+, p^*\mathcal{E}^-, \sigma(D)) \in K^0(T^*M).$$

Assume that $d[\sigma(D)] \in p^*(K^0(M)) + r(K^0(S^*X))$. Then $\mathrm{index}(D) = 0$.

Proof. The idea is to construct an operator A as in Theorem 2. We must first construct the vector bundles $V^\pm$, and then extend the principal symbol of (4) to an elliptic symbol in the interior of X. Note that none of the bundles $\mathcal{E}^\pm, V^\pm$ has any reason to extend to X.

We can assume that D is of order 1. Extend $\sigma(D)_{|S^*M}$ arbitrarily to a homomorphism $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ (not necessarily invertible) over $S^*_{\mathrm{sus}}(M)$. Let $\mathcal{F}^\pm \to S^*_{\mathrm{sus}}(M)$ be the vector bundles defined as the span of the eigenvectors of positive, respectively negative, eigenvalues of the symmetric automorphism of $p^*(\mathcal{E}^+ \oplus \mathcal{E}^-)$:

$$a := \begin{bmatrix} \xi & \sigma^* \\ \sigma & -\xi \end{bmatrix} : p^*(\mathcal{E}^+ \oplus \mathcal{E}^-) \to p^*(\mathcal{E}^+ \oplus \mathcal{E}^-).$$

Lemma. *The K-theory class of the vector bundle $\mathcal{F}^+$ is $d[\sigma(D)]$.*

Proof. $\mathcal{F}^+$ is the image of the projector $\frac{1+a(a^2)^{-\frac{1}{2}}}{2}$ inside $p^*(\mathcal{E}^+ \oplus \mathcal{E}^-)$, or equivalently the image of the endomorphism $(a^2)^{\frac{1}{2}} + a$:

$$\begin{aligned}\mathcal{F}^+ = \{((\xi + (\xi^2+\sigma^*\sigma)^{\frac{1}{2}})v, \sigma v); v \in \mathcal{E}^+\} \\ + \{(\sigma^* w, (-\xi + (\xi^2 + \sigma\sigma^*)^{\frac{1}{2}})w); w \in \mathcal{E}^-\}.\end{aligned}$$

Now $\xi + (\xi^2+\sigma^*\sigma)^{\frac{1}{2}}$ is invertible when $\xi \geq 0$, and $-\xi + (\xi^2+\sigma\sigma^*)^{\frac{1}{2}}$ is invertible when $\xi \leq 0$. Thus the projection from $\mathcal{F}^+$ on $p^*\mathcal{E}^+$, respectively on $p^*\mathcal{E}^-$, are isomorphisms for $\xi \geq 0$, respectively for $\xi \leq 0$. Over $\{\xi = 0\}$ these isomorphisms differ by $\sigma(\sigma^*\sigma)^{-\frac{1}{2}}$, which is homotopic to σ by varying the exponent from $-\frac{1}{2}$ to 0. $\square$

The hypothesis therefore says that

$$[\mathcal{F}^+] \in p^*(K^0(M)) + r(K^0(S^*X)). \tag{7}$$

Lemma. *There exist vector bundles $G^{\pm} \to S^*X$, $V^{\pm} \to M$ such that*

$$\mathcal{F}^{\pm} \oplus p^*V^{\pm} = G^{\pm}_{|S^*_{\mathrm{sus}}(M)}, \tag{8}$$

and moreover there exists $N \in \mathbb{N}$ with

$$\mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+ \oplus V^- \cong \underline{\mathbb{C}}^N, \qquad\qquad G^+ \oplus G^- \cong \underline{\mathbb{C}}^N. \tag{9}$$

Proof. From (7), there exist $V_0^+ \to M$, $G_0^+ \to X$ and $k \in \mathbb{N}$ with $\mathcal{F}^+ \oplus \underline{\mathbb{C}}^k = p^*V_0^+ \oplus G^+_{0\,|S^*_{\mathrm{sus}}(M)} \oplus \underline{\mathbb{C}}^k$. Let V_1^+ be a complement of V_0^+ inside some $\underline{\mathbb{C}}^h$. Then $V^+ := \underline{\mathbb{C}}^k \oplus V_1^+$ and $G^+ := \underline{\mathbb{C}}^{h+k} \oplus G_0^+$ satisfy (8). This implies

$$[\mathcal{F}^-] = [p^*(\mathcal{E}^+ \oplus \mathcal{E}^-)] - [\mathcal{F}^+] = p^*[\mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+] - r[G^+].$$

Let G_0^- be a complement (inside $\underline{\mathbb{C}}^{N_0}$) of G^+, and V_0^- a complement of $\mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+$ inside $\underline{\mathbb{C}}^{N_1}$. Then

$$[\mathcal{F}^-] + p^*[V_0^-] + \underline{\mathbb{C}}^{N_0} = \underline{\mathbb{C}}^{N_1} + r[G_0^-]$$

which amounts to saying that there exists $N_2 \in \mathbb{N}$ with

$$\mathcal{F}^- \oplus p^*V_0^- \oplus \underline{\mathbb{C}}^{N_0+N_2} \cong \underline{\mathbb{C}}^{N_1+N_2} \oplus G_0^-|_{S^*_{\mathrm{sus}}(M)}.$$

Thus $V^- := V_0^- \oplus \underline{\mathbb{C}}^{N_0+N_2}$ and $G^- := \underline{\mathbb{C}}^{N_1+N_2} \oplus G_0^-$ satisfy (8). From the construction of V^- and G^-, (9) holds for $N := N_0 + N_1 + N_2$. $\square$

Let $\mathcal{G} := \underline{\mathbb{C}}^N \to X$ be the trivial bundle. From (9), $\mathcal{G}_{|M} \cong \mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+ \oplus V^-$ (as bundles over M) and $p^*\mathcal{G} \cong G^+ \oplus G^-$ (as bundles over S^*X). Define $\tilde{a} : p^*\mathcal{G} \to p^*\mathcal{G}$ to be the automorphism of $p^*\mathcal{G}$ over S^*X that equals ± 1 on $G^{\pm}$. From the definition of $\mathcal{F}^{\pm}$ and (8) it follows that $\tilde{a}_{|S^*_{\mathrm{sus}}(M)}$ and the automorphism $\begin{bmatrix} a & & \\ & 1 & \\ & & -1 \end{bmatrix}$ (written in the decomposition $p^*\mathcal{G}_{|S^*_{\mathrm{sus}}(M)} = p^*(\mathcal{E}^+ \oplus \mathcal{E}^-) \oplus p^*V^+ \oplus p^*V^-$) have the same spaces of eigenvectors of positive, respectively negative eigenvalues. Thus we can deform $\tilde{a}$ inside self-adjoint automorphisms to an automorphism α with

$$\alpha_{|S^*_{\mathrm{sus}}(M)} = \begin{bmatrix} a & & \\ & 1 & \\ & & -1 \end{bmatrix}. \tag{10}$$

We extend α to $T^*X \setminus 0$ with homogeneity 1.

As noted at the beginning of this section, we replace S^*X by ${}^{\mathrm{c}}S^*X$. By (10) and the definition of a, $\alpha|_{S^*_{\mathrm{sus}}(M)}$ coincides with the principal symbol of the right-hand side of (4). Therefore using (2), there exists an elliptic cusp operator $A \in \Psi_c^1(X, \mathcal{G})$ with $\sigma_1(A) = \alpha$ and with indicial family given by the symmetric suspended operator (4). By replacing A with $(A + A^*)/2$ we can assume A to be symmetric. The hypothesis of Theorem 2 is fulfilled, so we conclude that $\operatorname{index}(D) = 0$. □

4. Variants of Theorem 3

4.1. **Carvalho's theorem.** Carvalho [5] obtained a slightly different statement of cobordism invariance (her result holds for non-compact manifolds as well). Namely, in the context of Theorem 3 she proved that $\operatorname{index}(D) = 0$ provided that $[\sigma(D)]$ lies in the image of the composite map

$$K^1(T^*X) \xrightarrow{r} K^1(T^*M \oplus \mathbb{R}) \xrightarrow{\beta^{-1}} K^0(T^*M)$$

defined by restriction and by Bott periodicity. Consider the relative pairs

$$S^*X \hookrightarrow B^*X, \qquad S^*_{\mathrm{sus}}(M) \hookrightarrow B^*_{\mathrm{sus}}M,$$

the inclusion map between them and the induced boundary maps in the long exact sequences in K-theory. We claim that we get the following commutative diagram:

$$\begin{tikzcd}
K^0(S^*X) \arrow[r, two heads] \arrow[d, "r"] & K^1(T^*X) \arrow[d, "r"] \\
K^0(S^*_{\mathrm{sus}}(M)) \arrow[r, two heads] \arrow[d, "q"] & K^1(T^*_{\mathrm{sus}}M) \\
K^0(S^*_{\mathrm{sus}}(M))/p^*K^0(M) \arrow[r, "d^{-1}"] & K^0(T^*M) \arrow[u, "\beta"]
\end{tikzcd}$$

Indeed, the upper square commutes by naturality and the lower one by checking the definitions. Moreover, the existence[1] of non-zero sections in $T^*X \to X$ and $T^*_{\mathrm{sus}}M \to M$ shows that the rows are surjective. Also β, d are isomorphisms, so $d[\sigma(D)]$ lies in the image of $q \circ r$ if and only if $[\sigma(D)]$ lies in the image of $\beta^{-1} \circ r$. Thus Theorem 3 is equivalent to Carvalho's statement applied to closed manifolds. Our formulation is marginally simpler because it does not involve the Bott isomorphism.

4.2. **An indirect proof of Theorem 3.** As mentioned in the Introduction, Theorem 3 follows from the Atiyah-Singer formula:

$$\operatorname{index}(D) = \langle M, \operatorname{Td}(TM) \cup p_* \operatorname{ch}([\sigma(D)]) \rangle,$$

where p_* denotes integration along the fibers of $p : T^*M \to M$, taking values in the cohomology of M twisted with the orientation bundle. Indeed, the normal bundle to M in X is trivial so $\operatorname{Td}(TM) = \operatorname{Td}(TX)_{|M}$. We can embed T^*M into $S^*X_{|M}$ via the central projection from the interior pole of each sphere; the pull-back through this map of $d[\sigma(D)])$ coincides with $[\sigma(D)]$ modulo $p^*K^0(M)$, in particular the push-forward on M of $\operatorname{ch}(d[\sigma(D)])$ and of $\operatorname{ch}([\sigma(D)])$ are equal. So the hypothesis that $d[\sigma(D)]$ is the restriction of a class on S^*X modulo $p^*K^0(M)$ implies, by the functoriality of the Chern character, that $p_*\operatorname{ch}([\sigma(D)]) \in H^*(M, \mathcal{O})$ is the restriction of a (twisted) cohomology class from X. Finally Stokes formula shows that $\operatorname{index}(D) = 0$.

[1] The obstruction lives in $H^n(X)$ which is 0 when X has non-empty boundary; I am grateful to Gustavo Granja for this argument.

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