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AN INVERSE PROBLEM OF HAMILTONIAN DYNAMICS

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Abstract. We study the question of whether for a natural Hamiltonian system on a two-dimensional compact configuration manifold, a single trajectory of sufficiently high energy is almost surely enough to reconstruct a real analytic potential.

Consider a compact configuration manifold M^n equipped with a finite Borel measure (essentially we deal with the dimension $n = 2$) and a natural Hamiltonian system thereon, with the Hamiltonian

$$H(p, q) = \langle p, p\rangle_q + U(q), \quad (p, q) \in T^*M^n.$$

Above, $\langle\cdot,\cdot\rangle_q$ is a Riemannian metric on M^n and U a potential. The direct problem of dynamics on M^n is finding the trajectory $q(t) \subset M^n$, with initial conditions $q(0) = q_0$ and $\dot{q}(0) = v_0$, moving in the known force field $f(q) = -\nabla_q U(q)$ on M^n, where the gradient ∇_q has been associated with the metric $\langle\cdot,\cdot\rangle_q$.

Let us call the inverse problem of dynamics the problem of reconstruction of the potential by observing the system's trajectories $q(t)$. The first problem of this type was explored in Newton's Principia, in a quest for a physical law determining the planetary motion compatible with observational data.[1] In the general case, knowledge of infinitely many trajectories is required to completely solve the problem. In this note we show that in the special case when M^n is two dimensional, compact and topologically non-trivial, a single trajectory with sufficiently large energy would almost surely suffice to reconstruct the potential.

In the sequel, we assume that M^n as well as all the quantities involved are real-analytic. Also suppose, there is an a-priori estimate $|U(q)| < C_0$, $\forall q \in M^n$, and we consider only the trajectories $q(t)$ with total energy $E \geq C_0$.

Theorem. *Let $n = 2$ and suppose M^2 is not diffeomorphic to S^2 or $\mathbb{R}P^2$. Almost every trajectory $q(t)$, $t \geq 0$, with energy $E \geq C_0$, suffices to reconstruct the potential U as a real-analytic function on M^2.*

Let us recall the definition of a *key set*, or set of uniqueness; see e.g. [3].

Definition. Let D be a domain in $\mathbb{R}^n$ and $C^\omega(D)$ the class of real-analytic functions in D. A set $K \subset D$ is a key set if any $f \in C^\omega(D)$ vanishing identically on K vanishes identically on D.

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[1]Note that the term "inverse problem of mechanics" has also been used to address the problem of deciding whether a given system of second order ODEs on M^n has a Lagrangian; see e.g. [5].

The theorem will follow from the following Lemma.

Lemma. *If a real-analytic dynamical system $A: \ \dot{x} = F(x)$ on a compact phase space P is non-singular (i.e. for no $x \in P$, $F(x) = 0$) and the set of its closed orbits has positive measure, then all its orbits are closed.*

Proof of the Lemma. Since the manifold P is compact and the vector field F is non-singular, there exists a Riemannian metric g on P, such that in this metric the vector field F has unit length at every $x \in P$. Furthermore, by compactness of P, the curvature (associated with g) of integral trajectories of A is bounded from above, and therefore there exists some $T_m > 0$, such that any periodic orbit of A has period not smaller than T_m. This represents a particular case of the general result of Yorke ([7]).

Let us partition the range $[T_m, \infty)$ of possible periods (henceforth periods stand for minimum periods) for closed orbits into intervals of some small length δ_1 to be specified. Let $I_k = [T_m + k\delta_1, T_m + (k+1)\delta_1) \equiv [T_k, T_{k+1})$, for $k = 0, 1, \ldots$. Let Γ_k be the set of all closed orbits, whose periods lie in I_k. Then for some $k = k_*$ the set $\Gamma_* = \Gamma_{k_*}$, considered as a subset of P, has positive measure. (We refer to the sets Γ either as point sets or sets of orbits, depending on the context. As there are only measure-theoretical considerations involved, this should not cause confusion.)

Let $D_{\delta_2}(x)$ be a codimension one disk in P, centered at some $x \in P$, with radius δ_2 and perpendicular (in the sense of metric g) to the vector field $F(x)$ at the point x. Let δ_2 be small enough, so that the vector field is transversal to $D_{\delta_2}(x)$ at every point of the disk. Clearly, δ_2 can be taken as a universal constant independent of x.

Let x_0 be a Lebesgue point of the set Γ_*. Recall that at a Lebesgue point, the density of the set is one. Let γ_0 be the closed trajectory passing through x_0, so $\gamma_0 \in \Gamma_*$. Take a disk $D_{\delta_2}(x_0)$.

Then there is a well-defined analytic Poincaré map S from a disk $D_\epsilon(x_0)$ contained in $D_{\delta_2}(x_0)$, where $\epsilon < \frac{1}{C_1 T_{k_*}}\delta_2$, for some $C_1 = C_1(\delta_1)$, and x_0 is a fixed point of S. On the disk $D_\epsilon(x_0)$, points that are initial conditions for orbits from Γ_* form a set of positive measure, as x_0 is a Lebesgue point. Besides, the quantity $\delta_1 < T_m$ can be chosen small enough to ensure that any $x \in D_\epsilon(x_0) \cap \Gamma_*$ is also a fixed point of the map S.

The union of all $x \in D_\epsilon(x_0) \cap \Gamma_*$ is a positive measure subset of $D_\epsilon(x_0)$, and hence is a key set (see [6] for the proof that every set of positive measure is a key set). Therefore, every point of $D_\epsilon(x_0)$ is an equilibrium of the map S, and hence an initial condition for a periodic orbit of A. Let Γ_ϵ be the union of all such orbits, with initial conditions in $D_\epsilon(x_0)$. Let $\Gamma \subseteq P$ be the maximum connected open set, which contains Γ_ϵ and is a union of periodic orbits.

To complete the proof, let us show that the set Γ does not have a boundary, i.e. $\Gamma = P$. To show it, we use the following Gronwall-type estimate.

Proposition. *Let $\phi(s) \subset \Gamma$ be a curve of length L, where s is a natural parameter with respect to the metric g. Then for all $s \in [0, L]$ and some absolute constant C_2,*

$$T(\phi(s)) \leq T(\phi(0))e^{C_2 L},$$

where $T(\phi(s))$ is the period of the closed orbit passing through the point $\phi(s)$.

Indeed, the Proposition follows immediately from the following infinitesimal estimate: for some C_2,

$$\left|\frac{d}{ds}T(\phi(s))\right| \leq C_2 T(\phi(s)).$$

Returning to the proof of the lemma, suppose the boundary $\partial\Gamma$ is non-empty. As $\partial\Gamma$ is a compact set, the distance (in the sense of metric g) between $\partial\Gamma$ and the above-mentioned point x_0 attains its minimum at some point $y \in \partial\Gamma$. Connect x_0 and y by a geodesic segment. Let the latter segment have length L; clearly all its points, except y, belong to Γ. Let γ_y be the trajectory of A with initial condition y.

For any point $x_1 \neq y$ on the above geodesic segment, there is a uniform bound for the period of the corresponding closed orbit, by the Proposition. Hence, for any such x_1, there exists a uniform ε (one can take $\varepsilon = \epsilon e^{-C_2 L}$, where ϵ has been defined earlier) such that an analytic Poincaré map can be defined in exactly the same way as S above, but now with the domain $D_\varepsilon(x_1)$. Choosing x_1 such that the intersection $\gamma_y \cap D_\varepsilon(x_1)$ is not empty and repeating the key set argument leads to contradiction: all orbits in some tubular neighborhood of γ_y, including γ_y itself, are periodic. □

Proof of the Theorem. Consider a randomly chosen trajectory γ on some energy level $H^{-1}(E)$, $E \geq C_0$, which is obviously a non-critical level.

According to the Lemma, either (i) all the trajectories on $H^{-1}(E)$ are periodic, or (ii) a randomly chosen initial condition $(p_0, q_0) \in H^{-1}(E)$ results in a phase trajectory of infinite length almost surely.

The former case (i) may occur only if M^2 is a so-called P-manifold. Indeed, according to the Maupertuis principle, the phase trajectories of motions with total energy E project onto M^2 as geodesic lines of the corresponding (Riemannian) Jacobi metric. P-manifolds are Riemannian manifolds, all of whose geodesics are closed; see [1]. Topological properties of P-manifolds are characterized in great detail in various dimensions within the framework of the Bott-Samelson theorem. In our (simplest possible) case, it is easy to see that M^2 can only be diffeomorphic to either S^2 or $\mathbb{R}P^2$.

Indeed, the proof of the lemma implies that all the (closed) phase trajectories on the energy level E are homotopic to one another. Then their images on M^2 (under natural projection) are also homotopic. On the other hand, if M^2 is different from S^2 or $\mathbb{R}P^2$, the number of generators for its fundamental group equals at least two. As for any Riemannian metric there exist closed geodesics in each free homotopy class, we would have a contradiction, unless M^2 is S^2 or $\mathbb{R}P^2$.

In the case (ii), let $q(t) = (q_1(t), q_2(t))$ be a randomly chosen trajectory: it almost surely has infinite length. Clearly, we can easily derive the gradient of U at every point of the trajectory from the Euler-Lagrange equations.

Thus to complete the proof of the Theorem, let us show that any non-closed trajectory is a key set. Consider orbit segments $\{q(t),\, t \in [k, k+1)\}$, $k = 0, 1, \ldots$, in M^2. (Note that time can always be scaled to ensure that each segment is a simple curve in M^2, or shorter time intervals can be considered.) As M^2 is compact, there is a limit point q_* of the point sequence $\{q(k + 1/2)\}$. Consider a sufficiently small circle centered at q_*. There are two options. Either the circle intersects the trajectory $q(t)$ at infinitely many points, or at some point on the circle the trajectory $q(t)$ intersects itself (transversely as it is a geodesic) infinitely many times. In the

latter case, take the point of infinite self-intersection for q_*, otherwise leave q_* as it is. In either case, there exists a point q_*, with the property that any sufficiently small circle centered at q_* is intersected by the trajectory $q(t)$ infinitely many times.

Therefore, the force f and the potential U can be uniquely reconstructed on any sufficiently small circle centered at q_* (an infinite point set on a circle is a key set for the circle), and therefore in some neighborhood of q_*, and hence on the whole configuration manifold M^2. □

In conclusion, let us make several remarks.

1. The Theorem can be stated in terms of analytic geodesic flows on compact Riemannian 2-manifolds. Namely, if M^2 is not diffeomorphic to a sphere or real projective plane, a randomly chosen geodesic suffices to reconstruct the metric, almost surely. Indeed, any Riemannian metric is locally conformal to the Euclidean one, i.e. can be locally associated with the Hamiltonian $H(p,q) = e^{\rho(q_1,q_2)}(p_1^2+p_2^2)$. Our Theorem enables one to reconstruct the real-analytic function $\rho(q_1,q_2)$ locally near q_* from the Hamilton equations, with subsequent analytic continuation to get the metric globally on M^2.

2. The Theorem is essentially two dimensional, as for $n \geq 3$ the fact that a random trajectory has infinite length does not suffice to reconstruct the potential. Consider for instance the Euler top, where $M^n = SO(3)$. In this case, the phase space is foliated by invariant two-tori, where the trajectories are in general conditionally periodic. Clearly, a projection of a single invariant two-torus onto the three-dimensional configuration space is not a key set.

3. The lemma does not apply to the case of invariant tori of dimension higher than one. Indeed, a particular case of the KAM theorem states that a sufficiently small perturbation of a non-degenerate Liouville-integrable Hamiltonian system in T^*M^n yields a positive measure set of invariant n-tori that do not fill the whole energy surface.

4. In the special case $M^2 = \mathbb{R}P^2$, by the theorem of L. Green ([4]), the only metric for which all geodesics are closed is the standard metric. The case $M^2 = S^2$ has been a subject of extensive research for over 100 years, arguably beginning with the doctoral thesis of O. Zoll ([8]). The reader is referred to the excellent book by L. Besse ([1]), which gives the issue a thorough treatment.

5. The condition $E \geq C_0$ seems unavoidable. For small energies the domain of possible motions can be a disk, with the Jacobi metric degenerate on the boundary, in which case one may expect a scenario similar to the case $M^2 = S^2$.

6. Observe that in the exceptional case when M^2 is a P-manifold, the geodesic flow thereon is completely integrable (see e.g. [2] for the proof of this fact). Hence our theorem implies that if $n = 2$, it is sufficient for restoration of the potential almost surely from a single trajectory that the system possess no other analytic integrals of motion but energy. It seems likely that in the latter weaker formulation the theorem should extend to the case $n > 2$, however we do not know how to prove it.

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