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AUTOMATIC CONTINUITY OF σ-DERIVATIONS ON C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$, let $\sigma : \mathcal{A} \to B(\mathcal{H})$ be a linear mapping and let $d : \mathcal{A} \to B(\mathcal{H})$ be a σ-derivation. Generalizing the celebrated theorem of Sakai, we prove that if σ is a continuous $*$-mapping, then d is automatically continuous. In addition, we show the converse is true in the sense that if d is a continuous $*$-σ-derivation, then there exists a continuous linear mapping $\Sigma : \mathcal{A} \to B(\mathcal{H})$ such that d is a $*$-Σ-derivation. The continuity of the so-called $*$-(σ, τ)-derivations is also discussed.

1. Introduction

Let $\mathcal{A}$ be a subalgebra of an algebra $\mathcal{B}$, let $\mathcal{X}$ be a $\mathcal{B}$-bimodule and let $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$ (see [3] and [4] and the references therein). If $\mathcal{B} = \mathcal{X} = \mathcal{A}$ and σ is the identity map on $\mathcal{A}$, then we reach to the usual notion of a derivation on the algebra $\mathcal{A}$.

In 1958, I. Kaplansky [2] conjectured that every derivation on a C^*-algebra is continuous. H. Sakai [8] proved this conjecture and then J. R. Ringrose [7] showed that every derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is continuous.

In this paper, we investigate the continuity of σ-derivations on C^*-algebras. Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$. We prove that if $\sigma : \mathcal{A} \to B(\mathcal{H})$ is a continuous $*$-linear mapping, then every σ-derivation from $\mathcal{A}$ into $B(\mathcal{H})$ is automatically continuous, and so Sakai's theorem [8] is generalized. In addition, we establish the converse in the sense that if $d : \mathcal{A} \to B(\mathcal{H})$ is a continuous $*$-σ-derivation, then there exists a continuous mapping $\Sigma : \mathcal{A} \to B(\mathcal{H})$ such that d is a $*$-Σ-derivation. In the last section we discuss the continuity of the so-called $*$-(σ, τ)-derivations.

The importance of our approach is that σ is a linear mapping in general, not necessarily an algebra homomorphism. There are some applications of σ-derivations used to develop an approach to deformations of Lie algebras which have many applications in models of quantum phenomena and in analysis of complex systems; cf. [1].

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For the definition and elementary properties of C^*-algebras we refer the reader to [5] and [6].

2. Elementary properties of σ-derivations

Throughout this section, $\mathcal{A}$ is a subalgebra of an algebra $\mathcal{B}$, $\mathcal{X}$ is a $\mathcal{B}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{B}$ is a linear mapping.

A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation if

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$$

for all $a, b \in \mathcal{A}$. Familiar examples are:

(i) every ordinary derivation δ of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{X}$ is an ι-derivation (here ι denotes the identity map on $\mathcal{A}$);

(ii) every endomorphism φ on $\mathcal{A}$ is a $\frac{\varphi}{2}$-derivation;

(iii) for a given homomorphism σ on $\mathcal{A}$ and a fixed arbitrary element x in an $\mathcal{A}$-bimodule $\mathcal{X}$, the so-called σ-inner derivation is defined to be $d_x(a) = x\sigma(a) - \sigma(a)x$.

An interesting link between σ-derivations and algebra homomorphisms follows.

Theorem 2.1. *Let $\sigma : \mathcal{A} \to \mathcal{B}$ be a homomorphism and let $d : \mathcal{A} \to \mathcal{X}$ be a σ-derivation. Then the following hold:*

(i) *$\mathcal{X}$ equipped with the module multiplications $a \cdot x = \sigma(a)x$ and $x \cdot a = x\sigma(a)$ is an $\mathcal{A}$-bimodule denoted by $\widetilde{\mathcal{X}}$.*

(ii) *$d : \mathcal{A} \to \widetilde{\mathcal{X}}$ is an ordinary derivation.*

(iii) *$\mathcal{C} = \mathcal{A} \oplus \widetilde{\mathcal{X}}$ equipped with the multiplication $(a, x)(b, y) = (ab, x \cdot b + a \cdot y)$ is an algebra, and $\varphi_d : \mathcal{A} \to \mathcal{C}$ defined by $\varphi_d(a) = (a, d(a))$ is an injective homomorphism.*

(iv) *If $\mathcal{A}, \mathcal{B}$ and $\mathcal{X}$ are normed, σ is continuous, and $\mathcal{C}$ is equipped with the norm $\|(a, x)\| = \|a\| + \sup\{\|x\|, \|a_1 \cdot x\|, \|x \cdot a_2\|, \|a_1 \cdot x \cdot a_2\| : a_1, a_2 \in \mathcal{A}, \|a_1\| \leq 1, \|a_2\| \leq 1\}$, then φ_d is continuous if and only if d is continuous. Thus if every injective homomorphism of $\mathcal{A}$ into a Banach algebra is continuous, then every σ-derivation of $\mathcal{A}$ into a Banach $\mathcal{B}$-bimodule is continuous.*

Proof. Straightforward (see [6]). □

Recall that if $\mathcal{Y}$ and $\mathcal{Z}$ are normed spaces and $T : \mathcal{Y} \to \mathcal{Z}$ is a linear mapping, then the set of all z such that there is a sequence $\{y_n\}$ in $\mathcal{Y}$ with $y_n \to 0$ and $Ty_n \to z$ is called the separating space $\mathcal{S}(T)$ of T. Clearly,

$$\mathcal{S}(T) = \bigcap_{n=1}^{\infty} \overline{\{T(y) : \|y\| < 1/n\}}$$

is a closed linear space. If $\mathcal{Y}$ and $\mathcal{Z}$ are Banach spaces, by the closed graph theorem, T is continuous if and only if $\mathcal{S}(T) = \{0\}$.

Lemma 2.2. *Let $d : \mathcal{A} \to \mathcal{X}$ be a σ-derivation. Then*

$$d(c)(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ca) - \sigma(c)\sigma(a))d(b)$$

for all $a, b, c \in \mathcal{A}$.

Proof.

$$\begin{aligned}
d(cab) &= d(c)\sigma(ab) + \sigma(c)d(ab) \\
d(c)\sigma(ab) &= d(cab) - \sigma(c)d(ab) \\
&= (d(ca)\sigma(b) + \sigma(ca)d(b)) - \sigma(c)d(ab) \\
&= (d(c)\sigma(a) + \sigma(c)d(a))\sigma(b) + \sigma(ca)d(b) - \sigma(c)d(ab) \\
&= d(c)\sigma(a)\sigma(b) + \sigma(c)d(a)\sigma(b) + \sigma(ca)d(b) - \sigma(c)d(ab) \\
&= d(c)\sigma(a)\sigma(b) + \sigma(c)d(a)\sigma(b) + \sigma(ca)d(b) \\
&\quad -\sigma(c)(d(a)\sigma(b) + \sigma(a)d(b)),
\end{aligned}$$

whence

$$d(c)(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ca) - \sigma(c)\sigma(a))d(b).$$

□

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be normed algebras, let $\sigma : \mathcal{A} \to \mathcal{B}$ be a continuous mapping and let $d : \mathcal{A} \to \mathcal{B}$ be a σ-derivation. Then for each $a \in \mathcal{S}(d)$ and $b, c \in \mathcal{A}$ we have $a(\sigma(bc) - \sigma(b)\sigma(c)) = 0$.*

Proof. For each $a \in \mathcal{S}(d)$ there exists a sequence $\{a_n\}$ such that $a_n \to 0$ and $d(a_n) \to a$. By Lemma 2.2 we have

$$d(a_n)(\sigma(bc) - \sigma(b)\sigma(c)) = (\sigma(a_n b) - \sigma(a_n)\sigma(b))d(c) \to 0.$$

Thus $a(\sigma(bc) - \sigma(b)\sigma(c)) = 0$. □

Remark 2.4. Recall that if E is a subset of an algebra $\mathcal{B}$, the right annihilator $ran(E)$ of E (resp. the left annihilator $lan(E)$ of E) is defined to be $\{b \in \mathcal{B} : Eb = \{0\}\}$ (resp. $\{b \in \mathcal{B} : bE = \{0\}\}$). The set $ann(E) := ran(E) \cap lan(E)$ is called the annihilator of E. The previous lemma shows that if $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, $\sigma : \mathcal{A} \to \mathcal{B}$ is a continuous linear mapping, $d : \mathcal{A} \to \mathcal{B}$ is a σ-derivation and $ran(\mathcal{S}(d)) = \{0\}$, then σ is an endomorphism, and if

$$lan(\{\sigma(bc) - \sigma(b)\sigma(c) : b, c \in \mathcal{A}\}) = \{0\},$$

then d is continuous.

Proposition 2.5. *Suppose that $\mathcal{A}$ is a Banach algebra, $\mathcal{B}$ is a simple Banach algebra, $\sigma : \mathcal{A} \to \mathcal{B}$ is a surjective continuous linear mapping and $d : \mathcal{A} \to \mathcal{B}$ is a σ-derivation. Then d is continuous or σ is an endomorphism.*

Proof. At first we show that $\mathcal{S}(d)$ is a closed bi-ideal of $\mathcal{B}$. To see this let $b \in \mathcal{B}$ and $a \in \mathcal{S}(d)$. Then there is a sequence $\{a_n\}$ such that $a_n \to 0$ and $d(a_n) \to a$. Since σ is surjective, there is an element $c \in \mathcal{A}$ such that $b = \sigma(c)$. Now we have $ca_n \to 0$ and $d(ca_n) = d(c)\sigma(a_n) + \sigma(c)d(a_n) \to 0 + ba = ba$. This shows that $ba \in \mathcal{S}(d)$. By the same way $ab \in \mathcal{S}(d)$. Thus $\mathcal{S}(d)$ is a bi-ideal.

$\mathcal{S}(d)$ is $\{0\}$ or $\mathcal{B}$. If $\mathcal{S}(d) = \{0\}$, then d is continuous, and if $\mathcal{S}(d) = \mathcal{B}$, then $\mathcal{S}(d)$ has zero right annihilator and so, by Remark 2.4, σ is an endomorphism. □

The following example yields a continuous σ-derivation with a non-continuous linear mapping σ.

Example 2.6. Let $\mathcal{A} = C[0,2]$, the C^*-algebra of all complex-valued continuous functions defined on the interval $[0,2]$. Define the continuous function $h : [0,2] \to \mathbf{C}$ by $h(t) = 0$ on $[0,1]$ and $h(t) = t-1$ on $[1,2]$, define the linear mapping $\sigma : \mathcal{A} \to \mathcal{A}$ by

$$\sigma(f)(t) = \begin{cases} \alpha(f|_{[0,1/2]})(t) & \text{if } 0 \leq \text{t} \leq \frac{1}{2}, \\ 2(1-t)\alpha(f|_{[0,1/2]})(\frac{1}{2}) + (t-\frac{1}{2})f(1) & \text{if } \frac{1}{2} \leq \text{t} \leq 1, \\ \frac{1}{2}f(t) & \text{if } 1 \leq \text{t} \leq 2, \end{cases}$$

where α is a discontinuous linear mapping on $C[0,1/2]$, and define the linear mapping $d : \mathcal{A} \to \mathcal{A}$ by $d(f) = fh$. For all $t \in [0,1]$, we have

$$\begin{aligned} d(fg)(t) &= f(t)g(t)h(t) \\ &= 0 \\ &= f(t)h(t)\sigma(g)(t) + \sigma(f)(t)g(t)h(t) \\ &= d(f)(t)\sigma(g)(t) + \sigma(f)(t)d(g)(t) \\ &= (d(f)\sigma(g) + \sigma(f)d(g))(t), \end{aligned}$$

and for all $t \in [1,2]$,

$$\begin{aligned} d(fg)(t) &= f(t)g(t)h(t) \\ &= f(t)h(t)(\frac{g}{2})(t) + (\frac{f}{2})(t)g(t)h(t) \\ &= d(f)(t)\sigma(g)(t) + \sigma(g)(t)d(g)(t) \\ &= (d(f)\sigma(g) + \sigma(f)d(g))(t). \end{aligned}$$

In this example if we define Σ on $\mathcal{A}$ by $\Sigma(f) = \frac{f}{2}$, then d is a Σ-derivation and Σ is continuous. In the next section we will show that this is true in general, i.e., if d is a continuous $*$-σ-derivation, then we can find a continuous linear mapping Σ such that d is a $*$-Σ-derivation.

3. σ-derivations on C^*-algebras

In this section we establish several significant theorems concerning the continuity of σ-derivations on C^*-algebras. Throughout this section, $\mathcal{A}$ denotes a C^*-algebra acting on a Hilbert space $\mathcal{H}$, i.e., a closed $*$-subalgebra of the algebra $B(\mathcal{H})$ consisting of all bounded linear mappings on $\mathcal{H}$. In addition, we assume that σ and d are linear mappings of $\mathcal{A}$ into $B(\mathcal{H})$.

Our first result states that when we deal with a continuous σ-derivation we may assume that σ is continuous.

Lemma 3.1. *Let $d : \mathcal{A} \to \mathcal{B}$ be a continuous σ-derivation. Then $\mathcal{S}(\sigma) \subseteq ann(d(\mathcal{A}))$.*

Proof. Assume $A \in \mathcal{S}(\sigma)$. Then there is a sequence $\{A_n\}$ in $\mathcal{A}$ such that $A_n \to 0$ and $\sigma(A_n) \to A$. For each $B \in \mathcal{A}$ we have

$$d(A_nB) = d(A_n)\sigma(B) + \sigma(A_n)d(B) \to d(0)\sigma(B) + Ad(B).$$

Since $d(A_nB) \to d(0)$ and $d(0) = 0$, we obtain $Ad(B) = 0$. Similarly, we can show that $d(B)A = 0$. □

Theorem 3.2. *Let σ be a linear mapping and let d be a continuous $*$-σ-derivation. Then there is a continuous linear mapping $\Sigma : \mathcal{A} \to B(\mathcal{H})$ such that d is a $*$-Σ-derivation.*

Proof. Let $\mathcal{L}_0 = \bigcup_{A\in\mathcal{A}} d(A)(\mathcal{H})$ and let $\mathcal{L}$ be the closed linear span of $\mathcal{L}_0$. Then we can write $\mathcal{H} = \mathcal{L} \oplus \mathcal{K}$, where $\mathcal{K} = \mathcal{L}^\perp$. Thus $0 = \langle d(A^*)(h), k\rangle = \langle h, d(A)k\rangle$ for all $h \in \mathcal{H}, k \in \mathcal{K}, A \in \mathcal{A}$ shows that $\mathcal{K} = \bigcap_{A\in\mathcal{A}} \ker d(A)$. Now define Σ on $\mathcal{A}$ by $\Sigma(A) = \sigma(A)P$, where P denotes the corresponding projection to $\mathcal{L}$.

We show that d is a Σ-derivation. Let $A, B \in \mathcal{A}$ and $h = \ell + k \in \mathcal{L} \oplus \mathcal{K} = \mathcal{H}$. Then

$$\begin{aligned}
d(AB)(h) &= d(AB)(\ell + k)\\
&= d(AB)(\ell) + d(AB)(k)\\
&= d(AB)(\ell) + 0\\
&= d(A)\sigma(B)(\ell) + \sigma(A)d(B)(\ell)\\
&= d(A)\Sigma(B)(\ell) + \Sigma(A)d(B)(\ell)\\
&= d(A)\Sigma(B)(\ell) + \Sigma(A)d(B)(\ell) + d(A)\Sigma(B)(k) + \Sigma(A)d(B)(k)\\
&= d(A)\Sigma(B)(\ell + k) + \Sigma(A)d(B)(\ell + k)\\
&= (d(A)\Sigma(B) + \Sigma(A)d(B))(h).
\end{aligned}$$

Moreover, Σ is continuous on $\mathcal{A}$. Let $A_n \in \mathcal{A}$ with $A_n \to 0$ and $\Sigma(A_n) \to A$. Then for each $\ell \in \mathcal{L}_0$ there is a $B \in \mathcal{A}$ and there is an $h \in \mathcal{H}$ such that $\ell = d(B)(h)$. Thus we can write $A(\ell) = A(d(B)(h)) = (Ad(B))(h) = 0$, since $A \in ann(d(\mathcal{A}))$. We therefore have $A = 0$ on $\mathcal{L}_0$ and so $A = 0$ on $\mathcal{L}$, since A is continuous. On the other hand, for each $k \in \mathcal{K}$ we have $0 = \Sigma(A_n)(k) \to A(k)$ and so $A = 0$ on $\mathcal{K}$. Thus we have proved that $A = 0$ on $\mathcal{H}$. This shows that $\mathcal{S}(\Sigma) = \{0\}$ and so Σ is continuous on $\mathcal{A}$. □

Theorem 3.3. *Let σ be a $*$-linear mapping and let d be a continuous σ-derivation. Then there is a continuous linear mapping $\Sigma : \mathcal{A} \to B(\mathcal{H})$ such that d is a Σ-derivation.*

Proof. Define the σ-derivation $d^* : \mathcal{A} \to B(\mathcal{H})$ by $d^*(A) = d(A^*)^*$. Let $\mathcal{L}_0 = (\bigcup_{A\in\mathcal{A}} d(A)(\mathcal{H})) \cup (\bigcup_{A\in\mathcal{A}} d^*(A)(\mathcal{H}))$ and let $\mathcal{L}$ be the closed linear span of $\mathcal{L}_0$. Then we can write $\mathcal{H} = \mathcal{L} \oplus \mathcal{K}$, where $\mathcal{K} = \mathcal{L}^\perp$. Thus $0 = \langle d(A^*)(h), k\rangle = \langle h, d^*(A)k\rangle$ and $0 = \langle d^*(A^*)(h), k\rangle = \langle h, d(A)k\rangle$ for all $h \in \mathcal{H}, k \in \mathcal{K}, A \in \mathcal{A}$ show that $\mathcal{K} = (\bigcap_{A\in\mathcal{A}} \ker d(A)) \cap (\bigcap_{A\in\mathcal{A}} \ker d^*(A))$. Now define Σ on $\mathcal{A}$ by $\Sigma(A) = \sigma(A)P$ where P denotes the corresponding projection to $\mathcal{L}$.

Using the same argument in the proof of Theorem 3.2 one can show that both d and d^* are Σ-derivations (note that d^* is a σ-derivation).

Σ is continuous on $\mathcal{A}$. To see this, assume that $A_n \in \mathcal{A}$, $A_n \to 0$ and $\Sigma(A_n) \to A$. Then for each $\ell \in \mathcal{L}_0$ there is a $B \in \mathcal{A}$ and there is an $h \in \mathcal{H}$ such that $\ell = d(B)(h)$ or $\ell = d^*(B)(h)$. Thus we can write $A(\ell) = A(d(B)(h)) = (Ad(B))(h) = 0$ or $A(\ell) = A(d^*(B)(h)) = (Ad^*(B))(h) = 0$, since $A \in ann(d(\mathcal{A})) \cap ann(d^*(\mathcal{A}))$. We therefore have $A = 0$ on $\mathcal{L}_0$ and so $A = 0$ on $\mathcal{L}$, since A is continuous. On the other hand, for each $k \in \mathcal{K}$ we have $0 = \Sigma(A_n)(k) \to A(k)$ and so $A = 0$ on $\mathcal{K}$. Thus we have proved that $A = 0$ on $\mathcal{H}$. This shows that $\mathcal{S}(\Sigma) = \{0\}$ and so Σ is continuous on $\mathcal{A}$. □

The next two propositions allow us to assume that σ is a homomorphism when we discuss the continuity of σ-derivations.

Proposition 3.4. *Let σ be a continuous $*$-linear mapping and let d be a σ-derivation. Then there is a continuous $*$-homomorphism $\Sigma : \mathcal{A} \to B(\mathcal{H})$ and a*

Σ-derivation $D : \mathcal{A} \to B(\mathcal{H})$ such that D is continuous if and only if so is d. Moreover, if d preserves $*$, then so does D.

Proof. By Lemma 2.3 for each $A \in \mathcal{S}(d)$ and $B, C \in \mathcal{A}$ we have

$$(*) \qquad A(\sigma(BC) - \sigma(B)\sigma(C)) = 0$$

Now let $\mathcal{L}_0 = \bigcup_{B,C\in\mathcal{A}}(\sigma(BC) - \sigma(B)\sigma(C))(\mathcal{H})$ and let $\mathcal{L}$ be the closed linear span of $\mathcal{L}_0$. Then $(*)$ implies that $A(\mathcal{L}) = 0$ for each $A \in \mathcal{S}(d)$.

We can write $\mathcal{H} = \mathcal{L} \oplus \mathcal{K}$, where $\mathcal{K} = \mathcal{L}^\perp$. For each $B, C \in \mathcal{A}, h \in \mathcal{H}$ and $k \in \mathcal{K}$ we have

$$\begin{aligned} 0 &= \langle(\sigma(BC) - \sigma(B)\sigma(C))(h), k\rangle \\ &= \langle h, (\sigma(BC) - \sigma(B)\sigma(C))^*(k)\rangle \\ &= \langle h, (\sigma(C^*B^*) - \sigma(C^*)\sigma(B^*))(k)\rangle. \end{aligned}$$

Since $\mathcal{A}$ is a $*$-subalgebra of $B(\mathcal{H})$, we infer that $(\sigma(BC) - \sigma(B)\sigma(C))(k) = 0$ for each $B, C \in \mathcal{A}$ and $k \in \mathcal{K}$. This shows that $\mathcal{K} = \bigcap_{B,C\in\mathcal{A}} \ker(\sigma(BC) - \sigma(B)\sigma(C))$.

Now let $P = P_{\mathcal{K}}$ be the projection corresponding to $\mathcal{K}$. At first we show that $\sigma(A)P = P\sigma(A)$ for all $A \in \mathcal{A}$. For each $A, B, C \in \mathcal{A}$ and $k \in \mathcal{K}$ we have

$$\begin{aligned} (\sigma(BC) - \sigma(B)\sigma(C))\sigma(A)(k) &= (\sigma(BC)\sigma(A) - \sigma(B)\sigma(C)\sigma(A))(k) \\ &= (\sigma(BCA) - \sigma(BCA))(k) \\ &= 0. \end{aligned}$$

This shows that $\sigma(A)(\mathcal{K}) \subseteq \mathcal{K}$ and so $\sigma(A)P = P\sigma(A)$.

By using Lemma 2.2 we get

$$0 = d(B)(\sigma(CA) - \sigma(C)\sigma(A))(k) = (\sigma(BC) - \sigma(B)\sigma(C))d(A)(k)$$

for all $k \in \mathcal{K}$. This implies that $d(A)(k) \in \mathcal{K}$ for all $k \in \mathcal{K}$. Hence $d(A)(\mathcal{K}) \subseteq \mathcal{K}$ and so $d(A)P = Pd(A)$.

Now put $\Sigma(A) = \sigma(A)P$ and $D(A) = d(A)P$ for all $A \in \mathcal{A}$. First, Σ is a $*$-homomorphism. For $k \in \mathcal{K}$ we have

$$\begin{aligned} \Sigma(AB)(k) &= \sigma(AB)P(k) \\ &= \sigma(AB)(k) \\ &= \sigma(A)\sigma(B)(k) \\ &= \sigma(A)\sigma(B)P^2(k) \\ &= \sigma(A)P\sigma(B)P(k) \\ &= \Sigma(A)\Sigma(B)(k). \end{aligned}$$

Also, for $\ell \in \mathcal{L}$,

$$\Sigma(AB)(\ell) = \sigma(AB)P(\ell) = 0 = \sigma(A)P\sigma(B)P(\ell) = \Sigma(A)\Sigma(B)(\ell).$$

Moreover, $\Sigma(A^*) = \sigma(A^*)P = \sigma(A)^*P = (P\sigma(A))^* = (\sigma(A)P)^* = \Sigma(A)^*(A \in \mathcal{A})$.

Second, D is a Σ-derivation, since for $A, B \in \mathcal{A}$ and $k \in \mathcal{K}$ we have

$$\begin{aligned} D(AB)(k) &= d(AB)P(k) \\ &= d(AB)(k) \\ &= d(A)\sigma(B)(k) + \sigma(A)d(B)(k) \\ &= d(A)\sigma(B)P^2(k) + \sigma(A)d(B)P^2(k) \\ &= d(A)P\sigma(B)P(k) + \sigma(A)Pd(B)P(k) \\ &= (D(A)\Sigma(B) + \Sigma(A)d(B))(k). \end{aligned}$$

Also, for $\ell \in \mathcal{L}$,

$$\begin{aligned} D(AB)(\ell) &= d(AB)P(\ell) \\ &= 0 \\ &= d(A)P\sigma(B)P(\ell) + \sigma(A)Pd(B)P(\ell) \\ &= (D(A)\Sigma(B) + \Sigma(A)d(B))(\ell). \end{aligned}$$

Moreover, D is continuous if and only if d is also. To show this let D be continuous and $A \in \mathcal{S}(d)$. Then there is a sequence $\{A_n\}$ in $\mathcal{A}$ such that $A_n \to 0$ and $d(A_n) \to A$. By the first paragraph of the proof we know that $A(\mathcal{L}) = 0$, and for $k \in \mathcal{K}$ we have

$$A(k) = \lim_{n\to\infty} d(A_n)(k) = \lim_{n\to\infty} d(A_n)P(k) = \lim_{n\to\infty} D(A_n)(k) = 0,$$

since D is continuous.

On the other hand, if d is continuous and $A \in \mathcal{S}(D)$, then there is a sequence $\{A_n\}$ in $\mathcal{A}$ such that $A_n \to 0$ and $D(A_n) \to A$. Obviously for $\ell \in \mathcal{L}$ we have

$$A(\ell) = \lim_{n\to\infty} D(A_n)(\ell) = \lim_{n\to\infty} d(A_n)P(\ell) = 0,$$

and for $k \in \mathcal{K}$,

$$A(k) = \lim_{n\to\infty} D(A_n)(k) = \lim_{n\to\infty} d(A_n)P(k) = \lim_{n\to\infty} d(A_n)(k) = 0,$$

since d is continuous.

Similarly, one can show that Σ is continuous. If d is a $*$-σ-derivation, then the relation $d(A)P = Pd(A)$ implies that D is also a $*$-Σ-derivation. □

Remark 3.5. If $\mathcal{A}$ is a von Neumann algebra and both σ and d are mappings of $\mathcal{A}$ into $\mathcal{A}$, then, due to the fact that the range projection of every element of $\mathcal{A}$ is in $\mathcal{A}$, we conclude that the projection P constructed in the proof of Proposition 3.4 belongs to $\mathcal{A}$ and so the ranges of the derivation D and the linear mapping Σ are contained in $\mathcal{A}$.

Proposition 3.6. *Let σ be a continuous linear mapping and let d be a $*$-σ-derivation. Then there is a continuous $*$-homomorphism $\Sigma : \mathcal{A} \to B(\mathcal{H})$ and a $*$-Σ-derivation $D : \mathcal{A} \to B(\mathcal{H})$ such that D is continuous if and only if d is also.*

Proof. Define $\sigma^* : \mathcal{A} \to B(\mathcal{H})$ by $\sigma^*(A) = \sigma(A^*)^*$. Then d is a σ^*-derivation. Clearly d is a τ-derivation where $\tau = \frac{\sigma+\sigma^*}{2}$ is a continuous $*$-linear mapping. By Proposition 3.4 there exist a continuous $*$-homomorphism Σ and a $*$-Σ-derivation D such that D is continuous if and only if d is also. □

The following theorem is an extension of Sakai's theorem [8] to σ-derivations.

Theorem 3.7. *Let σ be a continuous $*$-linear mapping. Then every σ-derivation d is automatically continuous.*

Proof. By Proposition 3.4 we may assume that σ is a continuous $*$-homomorphism. Theorem 2.1 implies that d is an ordinary derivation from $\mathcal{A}$ into $\widetilde{B(\mathcal{H})}$. By the main theorem of [7] we conclude that d is continuous. □

Using the same argument as in the proof of Proposition 3.6 we can establish the following theorem:

Theorem 3.8. *Let σ be a continuous linear mapping. Then every $*$-σ-derivation is automatically continuous.*

4. (σ,τ)-derivations on C^*-algebras

In [3] the authors considered the notion of (σ,τ)-derivation. Assume that $\mathcal{A}$ is a $*$-subalgebra of a $*$-algebra $\mathcal{B}$ and $\sigma,\tau:\mathcal{A}\to\mathcal{B}$ are $*$-linear mappings. A linear mapping $d:\mathcal{A}\to\mathcal{B}$ is called a $*$-(σ,τ)-derivation if d preserves $*$ and $d(ab)=d(a)\sigma(b)+\tau(a)d(b)$ for all $a,b\in\mathcal{A}$. Obviously, a $*$-σ-derivation is a $*$-(σ,σ)-derivation.

Theorem 4.1. *Let $\mathcal{A}$ be a $*$-subalgebra of a $*$-algebra $\mathcal{B}$ and $\sigma,\tau:\mathcal{A}\to\mathcal{B}$ are $*$-linear mappings. Then every $*$-(σ,τ)-derivation $d:\mathcal{A}\to\mathcal{B}$ is a $*$-$(\frac{\sigma+\tau}{2},\frac{\sigma+\tau}{2})$-derivation.*

Proof. First we show that each $*$-(σ,τ)-derivation is a $*$-(τ,σ)-derivation. We have $d(ab)=d(b^*a^*)^*=(d(b)^*\sigma(a)^*+\tau(b)^*d(a)^*)^*=d(a)\tau(b)+\sigma(a)d(b)$.

Now we conclude that $d(ab)=\frac{1}{2}d(ab)+\frac{1}{2}d(ab)=\frac{1}{2}(d(a)\sigma(b)+\tau(a)d(b))+\frac{1}{2}(d(a)\tau(b)+\sigma(a)d(b))=d(a)\frac{\tau+\sigma}{2}(b)+\frac{\tau+\sigma}{2}(a)d(b)$. □

The previous theorem enables us to focus on σ-derivations while we deal with $*$-algebras. In particular, by using Theorem 4.1 we obtain the following generalizations of Theorem 3.2 and Theorem 3.7.

Theorem 4.2. *Let σ and τ be continuous $*$-linear mappings from a C^*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ into $B(\mathcal{H})$ and let $d:\mathcal{A}\to B(\mathcal{H})$ be a continuous $*$-(σ,τ)-derivation. Then there is a continuous linear mapping $\Sigma:\mathcal{A}\to B(\mathcal{H})$ such that d is a $*$-Σ-derivation.*

Theorem 4.3. *If σ and τ are continuous $*$-linear mappings from a C^*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ into $B(\mathcal{H})$, then every $*$-(σ,τ)-derivation $d:\mathcal{A}\to B(\mathcal{H})$ is automatically continuous.*

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