NONLINEAR CAUCHY PROBLEMS
WITH SMALL ANALYTIC DATA

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Abstract. We study the lifespan of solutions to fully nonlinear Cauchy problems with small real- or complex-analytic data. Our proofs are based on the method of majorants and the fixed point theorem for a contraction mapping.

1. Introduction

The Cauchy problem for nonlinear wave equations with small data has been studied by many authors in the $C^\infty$-category. They usually consider initial data with compact support and estimate the lifespan of a solution from below by using Fourier analysis. In particular, much attention has been paid to semilinear wave equations. Some monographs ([2], [4] and [5]) are available on this subject and detailed lists of references are found in them.

On the other hand, some results have been obtained about the Kirchhoff equation in the real-analytic category ([1] and [3]).

In the present paper, we consider fully nonlinear problems in the real- or complex-analytic category without hyperbolicity assumption. Our main tool is a combination of the fixed point technique and the method of majorants. We basically follow [6] and [7] with somewhat different notation.

Now we state our result.

Let $\Omega$ be an open set of $\mathbb{R}^n_x$, $x = (x_1, \ldots, x_n)$. A $C^\infty$-function $\varphi(x)$ on $\Omega$ is said to be uniformly analytic on $\Omega$ if it has the uniform bound below:

$$\exists C > 0, \forall \alpha \in \mathbb{N}^n, \sup_{x \in \Omega} |D^\alpha \varphi(x)| \leq C^{||\alpha||+1} |\alpha|!.$$ 

Note that the right-hand side is equivalent to the Cauchy-type bound $C^{||\alpha||+1} |\alpha|!$ up to the choice of $C$.

We define the function space $A(\Omega)$ to be the totality of uniformly analytic functions on $\Omega$. It is trivial that $A(\Omega)$ is closed under differentiation.

Let $t$ be a point of $\mathbb{R}$. For $T > 0$, the open interval $]-T, T[$ is denoted by $I_T$.

We set $\Omega_T = I_T \times \Omega$.

For $k \in \mathbb{N}$, a continuous function $u(t, x)$ on $\Omega_T = I_T \times \Omega$ is said to belong to $C^k(T; A(\Omega))$ if

(i) $\forall j \in \{0, \ldots, k\}, \forall \alpha \in \mathbb{N}^n, \partial^j_t \partial^\alpha u \in C(\Omega_T),$

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(ii) \( \forall T' \in [0, T], \exists C = C_{T'} > 0, \forall j \in \{0, \ldots, k\}, \forall \alpha \in \mathbb{N}^n, \)
\[
\sup_{|t| \leq T', x \in \Omega} |\partial_t^j \partial^\alpha u(t, x)| \leq C^{[\alpha]+1} |\alpha|!. \]

Let \( P = P(\partial) = \sum_{k=1}^n \sum_{j=1}^k p_{jk} \partial_j \partial_k \) be a second-order linear partial differential operator with constant coefficients, where \( \partial_j = \partial/\partial x_j \) and \( p_{jk} \in \mathbb{C} \). We consider the following Cauchy problem for a fully nonlinear equation:

\[
\begin{cases}
(\partial_t^2 - P(\partial)) u(t, x) = f(\nabla u(t, x), \nabla^2 u(t, x)), \\
u(0, x) = \varphi(x), \quad \partial_t u(0, x) = \psi(x),
\end{cases}
\]

where \( \nabla u(t, x) = (\partial_j u)_{1 \leq j \leq n} \) and \( \nabla^2 u(t, x) = (\partial_j \partial_k u)_{1 \leq j \leq k \leq n} \). Here \( \varphi(x) \) and \( \psi(x) \) are uniformly analytic in an open subset \( \Omega \) of \( \mathbb{R}^n_+ \). We assume that \( f(X) \) is real-analytic near \( X = 0 \in \mathbb{R}^N, N = (n^2 + 3n)/2 \), and vanishes of second order at \( X = 0 \). The right-hand side in (CP) does not depend on the variables \( t, x \), or on the unknown function \( u \) and its time derivative \( \partial_t u \). This condition makes it small in the sense described below (See Proposition 2.5 and the proof of Theorem 1.1 in [4]).

We shall study the lifespan of a solution when the data are small in some sense.

**Theorem 1.1.** There exist \( \mu > 0 \) and \( \varepsilon_0 > 0 \) such that the following holds for all \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \):

If \( \sup_{x \in \Omega} |\partial^\alpha \varphi| \leq \varepsilon^{[\alpha]+1}|\alpha|! \) and \( \sup_{x \in \Omega} |\partial^\alpha \psi| \leq \varepsilon^{[\alpha]+1}|\alpha|! \) for all \( \alpha \in \mathbb{N}^n \), then (CP) has a solution \( u(t, x) \in C^2(T; A(\Omega)) \) for \( T = \mu/\varepsilon \).

Roughly speaking, the bound on \( \varphi \) and \( \psi \) is equivalent to saying that they can be analytically continued to a large open set in \( \mathbb{C}^n \) and have small modulus there.

Note that [7] shows that a hyperbolicity condition is necessary for a nonlinear Cauchy problem to be well posed.

Next, we shall state a complex-analytic version, in which \( t \) and \( x \) are both complex variables.

Let \( U \) be an open set of \( \mathbb{C}^n_+ \) (not \( \mathbb{R}^n \)). We consider (CP) again, but now \( \varphi(x) \) and \( \psi(x) \) are both assumed to be complex-analytic functions on \( U \). Naturally we try to find a solution which is complex-analytic in \((t, x)\).

For \( T > 0 \), set \( B_T = \{t \in \mathbb{C}; |t| < T\} \).

**Theorem 1.2.** There exist \( \mu > 0 \) and \( \varepsilon_0 > 0 \) such that the following holds for all \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \):

If \( \sup_{x \in U} |\partial^\alpha \varphi| \leq \varepsilon^{[\alpha]+1}|\alpha|! \) and \( \sup_{x \in U} |\partial^\alpha \psi| \leq \varepsilon^{[\alpha]+1}|\alpha|! \) for all \( \alpha \in \mathbb{N}^n \), then (CP) has a unique solution \( u(t, x) \) which is complex-analytic on \( B_T \times U \) for \( T = \mu/\varepsilon \) and satisfies the following estimate: for all \( T' > 0 \) with \( 0 < T' < T = \mu/\varepsilon \), there exists \( C = C_{T'} > 0 \) such that

\[
\sup_{|t| \leq T', x \in U} |\partial^\alpha u(t, x)| \leq C^{[\alpha]+1} |\alpha|!
\]

for \( \alpha \in \mathbb{N}^n \). (An estimate on \( \partial_t^j \partial^\alpha u(t, x) \) can be obtained by using Cauchy’s inequality.)
2. The Banach algebra $\mathcal{G}_{T,ζ}(Ω)$

Some material in this and the next sections has already appeared in [9] or [3], possibly in a different formulation. We present proofs here for the reader’s convenience.

Let $f(X) = \sum_{k=0}^{\infty} a_k X^k$ and $g(X) = \sum_{k=0}^{\infty} b_k X^k$ be two formal series with $a_k \in \mathbb{R}, b_k \in \mathbb{R}_+$. Here $\mathbb{R}_+$ is the totality of nonnegative real numbers. We write $f(X) \ll g(X)$ if $|a_k| \leq b_k$ for all $k \geq 0$.

If $0 \ll f(X) \ll g(X)$ and $p \geq 0$ is smaller than the radius of convergence of $g$, then we have $0 \ll f(X+p) \ll g(X+p)$. In fact the assumption $0 \leq |a_k| \leq b_k$ implies $0 \leq \sum a_k p^k \leq \sum b_k p^k$, $0 \leq \sum k a_k p^{k-1} \leq \sum k b_k p^{k-1}$, $0 \leq \sum (k-1) a_k p^{k-2} \leq \sum k(k-1) b_k p^{k-2}$, $\ldots$.

A combination of (1) and (2) shows that $\sum a_k X^k \leq \sum b_k X^k$, $0 \leq \sum k a_k X^{k-1} \leq \sum k b_k X^{k-1}$, $0 \leq \sum k(k-1) a_k X^{k-2} \leq \sum k(k-1) b_k X^{k-2}$, $\ldots$.

For a formal power series $f(X) = \sum_{k=0}^{\infty} a_k X^k$, set

\[ Df(X) = \sum_{k=1}^{\infty} k a_k X^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} X^k, \]
\[ D^{-1}f(X) = \sum_{k=0}^{\infty} a_k (k+1)^{-1} X^{k+1} = \sum_{k=1}^{\infty} \frac{a_{k-1}}{k} X^k. \]

We have $DD^{-1}f(X) = f(X)$, but $D^{-1}Df(X) = \sum_{k=1}^{\infty} a_k X^k \neq f(X)$.

Set $\varphi(X) = 1/k \sum_{k=0}^{\infty} a_k/k X^k$, $K = 4\pi^2/3$. It is a series due to Lax. It can be proved that $\varphi^2(X) \ll \varphi(X)$. Hence we have $0 \ll \varphi^2(X+p) \ll \varphi(X+p)$ if $p \geq 0$.

Assume that $k \leq 0$, $a \geq 0$, $b \geq 0$, $k+a \leq 0$. Then $D^k D^{a+b} \varphi(X)$ is obtained by cutting off the terms of degree $\leq -k$ from $D^b D^{k+a} \varphi(X)$. We have

\[ D^k D^{a+b} \varphi(X) \ll D^b D^{k+a} \varphi(X). \]

Since $k+a \leq 0$, Lemma 2.5 of [9] implies that $D^{k+a} \varphi(X) \ll c^{k+a} \varphi(X)$ with $c = 2/9$ (our $c$ is the reciprocal of Wagschal’s). Hence

\[ D^b D^{k+a} \varphi(X) \ll c^{k+a} D^b \varphi(X). \]

A combination of (1) and (2) shows that

\[ D^b D^{a+b} \varphi(X) \ll c^{k+a} D^b \varphi(X). \]

Passing from an indeterminate $X$ to a definite value $X_0$, we get

**Proposition 2.1.** Assume $a \geq 0, b \geq 0, k + a \leq 0$ and $0 \leq X_0 < 1$; we have

\[ D^k D^{a+b} \varphi(X_0) \leq c^{k+a} D^b \varphi(X_0). \]

If $ζ > 0$, then a continuous function $u(t, x)$ on $Ω_T$ is said to be an element of $\mathcal{G}_{T,ζ}(Ω)$ if it is infinitely differentiable in $x$ and there exists a constant $C > 0$ such that

\[ \forall α \in \mathbb{N}^n, \forall t \in I_T, \sup_{x \in Ω} |\partial^α u(t, x)| \leq Cζ^{|α|} |D^{|α|} \varphi|(t/T), \]

where $\partial^α = ∂^{α_1} \ldots ∂^{α_n} / ∂x_1^{α_1} \ldots ∂x_n^{α_n}$.

We set

\[ \varphi_{T,ζ}(t, x) = \varphi \left( \frac{t}{T} + ζ \sum_{j=1}^{n} x_j \right). \]
Then we have \( \partial^{\alpha} \varphi_{T,\zeta}(t,0) = \zeta^{[\alpha]} D^{[\alpha]} \varphi(|t|/T) \), which is a factor of the right-hand side of (4). We denote (4) by
\[
u(t, x) \leq C \varphi_{T,\zeta}(t, x).
\]
We define the norm \( \|u\| \) to be the infimum of such \( C \)'s.

**Proposition 2.2.** \( \mathcal{G}_{T,\zeta}(\Omega) \) becomes a Banach space.

**Proof.** Let \( C^{0,\infty}(\Omega_T) \) be the space of continuous functions on \( \Omega_T \) which are infinitely differentiable in \( x \). For a compact subset \( K \) of \( \Omega_T \) and \( \alpha \in \mathbb{N}^n \), set \( p_{\alpha,K}(u(t, x)) = \sup_K |\partial^{\alpha} u(t, x)| \). Then \( C^{0,\infty}(\Omega_T) \) becomes a Fréchet space with these norms.

Obviously we have \( \mathcal{G}_{T,\zeta}(\Omega) \subset C^{0,\infty}(\Omega_T) \), and the canonical injection is continuous, because
\[
\sup_{|t| \leq T', x \in \Omega} |\partial^{\alpha} u(t, x)| \leq \|u\| \zeta^{[\alpha]} D^{[\alpha]} \varphi(|t|/T), \quad 0 < T' < T.
\]
If \( (u_k) \) is a Cauchy sequence in \( \mathcal{G}_{T,\zeta}(\Omega) \), it converges to a limit \( u \) in the Fréchet space \( C^{0,\infty}(\Omega_T) \). We have only to prove that \( u \in \mathcal{G}_{T,\zeta}(\Omega) \) and that \( (u_k) \) converges to \( u \) in \( \mathcal{G}_{T,\zeta}(\Omega) \).

For all \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that \( \|u_p - u_q\| \leq \varepsilon \) if \( p, q \geq k \). In other words, for all \( \alpha \in \mathbb{N}^n \) we have
\[
\sup_{x \in \Omega} |\partial^{\alpha} u_p - \partial^{\alpha} u_q| \leq \varepsilon \zeta^{[\alpha]} D^{[\alpha]} \varphi(|t|/T).
\]
Since \( \partial^{\alpha} u_p(t, x) \to \partial^{\alpha} u(t, x) \) for all \( (t, x) \in \Omega_T \), we get
\[
\sup_{x \in \Omega} |\partial^{\alpha} u_p - \partial^{\alpha} u| \leq \varepsilon \zeta^{[\alpha]} D^{[\alpha]} \varphi(|t|/T).
\]
This means that \( u \in \mathcal{G}_{T,\zeta}(\Omega) \) and that \( (u_k) \) converges to \( u \) in \( \mathcal{G}_{T,\zeta}(\Omega) \). \( \square \)

Note that our \( \varphi_{T,\zeta}(t, x) \) and \( \mathcal{G}_{T,\zeta}(\Omega) \) are different from \( \Phi_{T,\zeta}(t, x) \) and \( \mathcal{G}_{T,\zeta}(\Omega) \) of [3]. It affects the formulation of Proposition 2.3. In the present paper we employ the unfamiliar symbol \( \ll \) in dealing with estimates global in \( x \), in contrast to \( \ll \) which only gives local information.

**Proposition 2.3.** \( \mathcal{G}_{T,\zeta}(\Omega) \) is a Banach algebra.

**Proof.** Since \( 0 < \varphi^2(X + p) \ll \varphi(X + p) \) for \( p \in [0, 1] \), we have for \( t \in I_T \),
\[
\partial^{\alpha} (\varphi^2_{T,\zeta})(t,0) \leq \partial^{\alpha} \varphi_{T,\zeta}(t,0).
\]
If \( u(t, x) \ll C_1 \varphi_{T,\zeta}(t, x), v(t, x) \ll C_2 \varphi_{T,\zeta}(t, x) \), then
\[
\begin{align*}
\sup_{x \in \Omega} |\partial^{\alpha} u(t, x)| & \leq C_1 \partial^{\alpha} \varphi_{T,\zeta}(t,0), \\
\sup_{x \in \Omega} |\partial^{\alpha} v(t, x)| & \leq C_2 \partial^{\alpha} \varphi_{T,\zeta}(t,0).
\end{align*}
\]
Therefore
\[
\sup_{x \in \Omega} |\partial^{\alpha} (uv)(t, x)| \leq \sum_{\beta \leq \alpha} \sup_{x \in \Omega} \left| \binom{\alpha}{\beta} (\partial^{\alpha-\beta} u \cdot \partial^{\beta} v)(t, x) \right|
\leq C_1 C_2 \sum_{\beta \leq \alpha} \left( \binom{\alpha}{\beta} \right) \partial^{\alpha-\beta} \varphi_{T,\zeta}(t,0) \cdot \partial^{\beta} \varphi_{T,\zeta}(t,0)
= C_1 C_2 \partial^{\alpha} (\varphi^2_{T,\zeta})(t,0)
\leq C_1 C_2 \partial^{\alpha} \varphi_{T,\zeta}(t,0).
\]
Hence \( uv \ll C_1 C_2 \varphi_{T,\zeta} \). This implies that \( \|uv\| \leq \|u\| \|v\| \). \( \square \)
We equip the direct sum \( \bigoplus_{\alpha \geq 2} \mathcal{G}_{T,\zeta}(\Omega) \) with the norm \( \| \cdot \|_N \) defined by

\[
\| \varphi(t, x) \|_N = \max_{j=1, \ldots, N} |\tau_j(t, x)|,
\]

\[
\varphi(t, x) = (\tau_1(t, x), \ldots, \tau_N(t, x)) \in \bigoplus_{\alpha \geq 2} \mathcal{G}_{T,\zeta}(\Omega).
\]

**Proposition 2.4.** Let \( f(X) = f(X_1, \ldots, X_N) = \sum_{|\alpha| \geq 2} a_\alpha X^\alpha \) be a convergent power series which vanishes of second order at \( X = 0 \). If \( \varphi(t, x), \varphi(t, x) \in \bigoplus_{\alpha \geq 2} \mathcal{G}_{T,\zeta}(\Omega) \) have sufficiently small norms, then \( f(\varphi(t, x)) \) and \( f(\varphi(t, x)) \) are well defined as elements of \( \mathcal{G}_{T,\zeta}(\Omega) \). Moreover, there exist positive constants \( C_f \) and \( C_f' \) depending only on \( f \) and independent of \( \varphi, \varphi, T, \zeta \) and \( \Omega \) such that

\[
\| f(\varphi(t, x)) \| \leq C_f \| \varphi \|_N^2,
\]

\[
\| f(\varphi(t, x)) - f(\varphi(t, x)) \| \leq C_f' \| \varphi - \varphi \|_N (\| \varphi \|_N + \| \varphi \|_N).
\]

**Proof.** By Proposition 2.3 we have

\[
f(\varphi) = \sum_{|\alpha| \geq 2} a_\alpha \varphi^\alpha \leq \sum_{|\alpha| \geq 2} |a_\alpha| |\tau_1|^{\alpha_1} \cdots |\tau_N|^{\alpha_N} \varphi_{T,\zeta}.
\]

We find that \( \| f(\varphi(t, x)) \| \leq C_f \| \varphi \|_N^2 \) for some \( C_f \) if \( \| \varphi \|_N \) is sufficiently small.

We have \( f(Y) - f(X) = (Y - X) \cdot g(X, Y) \) for a vector-valued real-analytic function \( g(X, Y) = \int_0^1 \nabla f((1 - t)X + tY)dt \). Since \( g(0, 0) = 0 \), the inequality

\[
\| f(\varphi(t, x)) - f(\varphi(t, x)) \| \leq C_f' \| \varphi - \varphi \|_N (\| \varphi \|_N + \| \varphi \|_N)
\]

follows. \( \square \)

Set \( \partial_t^{-1} u(t, x) = \int_0^t u(s, x)ds \).

**Proposition 2.5.** For all \((k, \alpha) \in (-\mathbb{N}) \times \mathbb{N}^n \) with \( k + |\alpha| \leq 0 \), there exists a constant \( C_{k,|\alpha|} > 0 \) such that \( \partial_t^k \partial^\alpha \) is an endomorphism of the Banach space \( \mathcal{G}_{T,\zeta}(\Omega) \) and its norm is not larger than \( C_{k,|\alpha|} T^{-k} \zeta^{k|\alpha|} \).

**Proof.** We fix \( \alpha \). If \( u \in \mathcal{G}_{T,\zeta}(\Omega) \), we have for all \( \beta \in \mathbb{N}^n \),

\[
\sup_{x \in \Omega} |\partial_t^k \partial^\alpha u(t, x)| \leq \| u \| \zeta^{k|\alpha|} D^{k|\alpha|} \varphi(|t|/T).
\]

Then by Proposition 2.1 we obtain the following estimate, in which we choose \( \pm \partial_t \) if \( \pm t \geq 0 \):

\[
\sup_{x \in \Omega} |\partial_t^k \partial^\alpha u(t, x)| \leq \| u \| \zeta^{k|\alpha|} (\pm \partial_t)^k \{ D^{k|\alpha|} \varphi(|t|/T) \}
\]

\[
\leq \| u \| T^{-k} \zeta^{k|\alpha|} D^{k|\alpha|} \varphi(|t|/T)
\]

\[
\leq \| u \| c^{k|\alpha|} T^{-k} \zeta^{k|\alpha|} D^{k|\alpha|} \varphi(|t|/T).
\]

We have shown that

\[
\partial_t^k \partial^\alpha u(t, x) \leq \| u \| c^{k|\alpha|} T^{-k} \zeta^{k|\alpha|} \varphi_{T,\zeta}(t, x),
\]

because \( \partial_t \) and \( \partial^\beta \) commute. \( \square \)
3. **Uniformly Analytic Functions**

The spaces $A(\Omega)$ and $C^k(T; A(\Omega))$ have been defined in the first section. Recall condition (ii) in the definition of the latter, which is only locally uniform in $t$. This condition has been chosen so that the following proposition may hold. Note that $D^k\varphi(1)$ diverges if $k \geq 1$.

**Proposition 3.1.** $\forall T > 0, \forall \zeta > 0$, $G_{T, \zeta}(\Omega) \subset C(T; A(\Omega))$.

To formulate an almost converse inclusion, we introduce the following notation.

If $\varphi(x) \in A(\Omega)$, there exist positive constants $p(\varphi) > 0$ and $q(\varphi) > 0$ such that

$$\forall \alpha \in \mathbb{N}^n, \sup_{x \in \Omega} |\partial^\alpha \varphi(x)| \leq p(\varphi)q(\varphi)^{|\alpha|} |\alpha|!.$$ 

They are not unique.

**Remark 3.2.** If $\Omega$ is star-shaped and $\varphi(x) \in A(\Omega)$, we set $\varphi_\varepsilon(x) = \varepsilon \varphi(\varepsilon x)$. Then we can take $p(\varphi_\varepsilon) = \varepsilon p(\varphi), q(\varphi_\varepsilon) = \varepsilon q(\varphi)$.

**Proposition 3.3.** Assume that $\varphi(x) \in A(\Omega)$ satisfies $\sup_{x \in \Omega} |\partial^\alpha \varphi(x)| \leq \varepsilon^{2|\alpha|+1} |\alpha|!$. (We can take $p(\varphi) = q(\varphi) = \varepsilon$.) Then we can take

$$p(\partial_j \varphi) = \varepsilon^2, \quad p(\partial_j \partial_k \varphi) = 3\varepsilon^3, \quad p(\partial_j \partial_k \partial_l \varphi) = 15\varepsilon^4,$$

$$q(\partial_j \varphi) = q(\partial_j \partial_k \varphi) = q(\partial_j \partial_k \partial_l \varphi) = 2\varepsilon.$$

**Proof.** We have

$$\sup_{x \in \Omega} |\partial^\alpha (\partial_j \varphi)| \leq \varepsilon^{2j+1} (|\alpha| + 1)! = \frac{|\alpha| + 1}{2^{|\alpha|}} \varepsilon^2 \cdot (2\varepsilon)^{|\alpha|} |\alpha|!.$$ 

Then we employ the fact that $j/2^{j-1} < 1$ for $j \geq 1$.

Next we have

$$\sup_{x \in \Omega} |\partial^\alpha (\partial_j \partial_k \varphi)| \leq \varepsilon^{2j+3} (|\alpha| + 2)! \leq \frac{(|\alpha| + 2)(|\alpha| + 1)}{2^{|\alpha|}} \varepsilon^3 \cdot (2\varepsilon)^{|\alpha|} |\alpha|!.$$ 

Then we employ the fact that $j(j+1)/2^{j-1} < 3$ for $j \geq 1$.

The assertion about $\partial_j \partial_k \partial_l \varphi$ can be proved in a similar way. \hfill $\square$

**Proposition 3.4.** If $\psi(x) \in A(\Omega)$, then for all $T > 0$ and for all $\zeta \geq e^2 q(\psi)$, we have $\psi \in G_{T, \zeta}(\Omega)$ and $|\psi| \leq K p(\psi)$.

**Proof.** For all $\alpha \in \mathbb{N}^n$, we have

$$(|\alpha| + 1)^2 D^{[\alpha]} \varphi(0) \geq (|\alpha| + 1)^2 D^{[\alpha]} \varphi(0) = K^{-1} |\alpha|!$$

and $(|\alpha| + 1)^2 \leq e^{2|\alpha|}$. Hence we obtain

$$|\alpha|! \leq K e^{2|\alpha|} D^{[\alpha]} \varphi(|t|/T).$$

On the other hand, $\psi(x) \in A(\Omega)$ satisfies

$$\sup_{x \in \Omega} |\partial^\alpha \psi(x)| \leq p(\psi)q(\psi)^{|\alpha|} |\alpha|!.$$ 

By (5) and (6), we find that

$$\sup_{x \in \Omega} |\partial^\alpha \psi(x)| \leq \{K p(\psi)\} \cdot \{e^2 q(\psi)\} |\alpha|!.$$ 

This completes the proof. \hfill $\square$
4. Proofs of the theorems

Proof of Theorem 1.1. Set $v(t, x) = u(t, x) - \varphi(x) - t\psi(x)$. Then $v(0, x) = \partial_t v(0, x) = 0$ and

$$\partial_t^2 v = P(v + \varphi + t\psi) + f(\nabla^{1.2}(v + \varphi + t\psi)),$$

where we set $\nabla^{1.2} u = (\nabla u, \nabla^2 u)$ for simplicity.

Next we set $w = \partial_t^2 v$. Then $v = \partial_t^{-2} w$ and (CP) is reduced to $w = L(w)$, where we define the mapping $L$ by

$$L(w) = P(\partial_t^{-2} w + \varphi + t\psi) + f(\nabla^{1.2}(\partial_t^{-2} w + \varphi + t\psi)).$$

We shall find a fixed point $w$ of $L$ in a suitable complete metric space by showing that $L$ is a contraction.

We assume that $w \in \mathcal{G}_{T, \zeta}(\Omega)$, where $T$ and $\zeta$ are to be specified later.

By Propositions 2.3, we have

$$|P\partial_t^{-2} w| \leq A|w|, \quad A := C_p C_{-2,2} T^2 \zeta^2,$$

where $C_p = \sum |p_{jk}|$. By Propositions 3.3 and 3.4 if $\zeta \geq 2\varepsilon^2 \varepsilon$, we have

$$|P(\varphi + t\psi)| \leq B, \quad B := 3C_p K(1 + T) \varepsilon^3.$$

The nonlinear term is estimated by using Propositions 2.3, 3.3 and 3.4. If $\zeta \geq 2\varepsilon^2 \varepsilon$ we have

$$|f(\nabla^{1.2}(\partial_t^{-2} w + \varphi + t\psi))| \leq C_f |\nabla^{1.2} \partial_t^{-2} w|_N + |\nabla^{1.2}(\varphi + t\psi)|_N^2 \leq C_f \left( \max(C_{-2,1} T^2 \zeta, C_{-2,2} T^2 \zeta^2) |w| + K(1 + T) \varepsilon^2 \right)^2 = (A'|w| + B')^2,$$

where $A' := \sqrt{C_f} \max(C_{-2,1} T^2 \zeta, C_{-2,2} T^2 \zeta^2)$, $B' := \sqrt{C_f} K(1 + T) \varepsilon^2$. The terms caused by $\nabla^2(\varphi + t\psi)$ can be estimated by $3K(1 + T) \varepsilon^3$, which is much smaller than $K(1 + T) \varepsilon^2$ if $\varepsilon$ is sufficiently small. These cubic terms have been neglected in the above estimate.

To sum up, we have $|Lw| \leq A|w| + B + (A'|w| + B')^2$.

We fix $(\zeta, T)$ and introduce a number $r$ as in the following (\zeta satisfying the condition indicated above), where $\mu > 0$ is a small parameter:

$$\zeta = 2\varepsilon^2 \varepsilon, \quad T = \frac{\mu}{\varepsilon}, \quad r = \frac{2B}{1 - 2A'}, \quad \zeta = 2\varepsilon^2 \varepsilon, \quad T = \frac{\mu}{\varepsilon}, \quad r = \frac{2B}{1 - 2A'},$$

We have $0 < A < 1/3$ and $r > 6B > 0$ if $\mu$ is sufficiently small. If $0 < \varepsilon < 1$, there exist positive constants $C_A, C_B, C_r, C_N$ and $C_{B'}$ such that

$$A = C_A \mu^2, \quad B = C_B (\varepsilon^3 + \mu \varepsilon^2),$$

$$Ar + B = r/2, \quad 0 < r \leq C_r (\varepsilon^3 + \mu \varepsilon^2),$$

$$A' \leq C_A \mu^2 \varepsilon^{-1}, \quad B' = C_B (\varepsilon^2 + \mu \varepsilon).$$

Note that $T^2 \zeta^2$ is much smaller than $T^2 \zeta$. It means that the terms related to $\nabla^2$ are much smaller than those related to $\nabla$.

There exists a positive constant $C_1$ such that

$$(A'r + B')^2 \leq C_1 \varepsilon^2 (\varepsilon + \mu).$$
On the other hand, \( r \) can be estimated from below, and there exists a positive constant \( C_2 \) such that
\[
C_2 \varepsilon^2 (\varepsilon + \mu) \leq r.
\]
Therefore if \( \varepsilon + \mu \) is sufficiently small, we have
\[
(7) \quad Ar + B + (A'r + B')^2 = \frac{T}{2} + (A'r + B')^2 \leq r.
\]

When \( \zeta, T \) and \( r \) are as in (\( \ast \)), let \( B(r; T, \zeta) \subset \mathcal{G}_{T, \zeta}(\Omega) \) be the closed ball of radius \( r \) centered at 0. The above calculation shows that \( \mathcal{L} \) is a mapping from \( B(r; T, \zeta) \) to itself if \( \varepsilon + \mu \) is sufficiently small.

Next we shall show that \( \mathcal{L} \) is a contraction mapping. Take \( w_1, w_2 \in B(r; T, \zeta) \) with \( r, T, \zeta \) as in (\( \ast \)). We have
\[
\mathcal{L}(w_1) - \mathcal{L}(w_2) = P \partial_t^{-2}(w_1 - w_2) + f(\tau_1) - f(\tau_2),
\]
where \( \tau_j = \nabla^{1,2}(\partial_t^{-2} w_j + \varphi + t\psi) \).

Then by Propositions 2.4 and 2.5, we have
\[
(8) \quad \|\mathcal{L}(w_1) - \mathcal{L}(w_2)\| \leq A\|w_1 - w_2\| + C_1' \|\tau_1 - \tau_2\|_N(\|\tau_1\|_N + \|\tau_2\|_N).
\]

Since \( T^2 \zeta^2 \) is much smaller than \( T^2 \zeta \) and \( T^2 \zeta \leq C_2 \mu^2 \varepsilon^{-1} \) for some \( C_2 \), we have
\[
\|\tau_1 - \tau_2\|_N \leq \max(C_{-2,1} T^2 \zeta, C_{-2,2} T^2 \zeta^2)\|w_1 - w_2\| = C_2 \mu^2 \varepsilon^{-1} \|w_1 - w_2\|.
\]

On the other hand, since \( \|w_j\| \leq r \leq C_1(\varepsilon^3 + \mu \varepsilon^2) \), there exists \( C_3 > 0 \) such that
\[
\|\tau_j\| \leq C_{-2,1} T^2 \zeta \|w_j\| + K(1 + T) \varepsilon^2 \leq C_3(\varepsilon^2 + \mu \varepsilon).
\]

Hence for some \( C_4 > 0 \),
\[
\frac{|\mathcal{L}(w_1) - \mathcal{L}(w_2)|}{|w_1 - w_2|} \leq C_A \mu^2 + 2C_4 \mu^2 (\varepsilon + \mu).
\]

We find that \( \mathcal{L} \) is a contraction mapping if \( \mu + \varepsilon \) is sufficiently small. Its fixed point
\[
w \in \mathcal{G}_{T, \zeta}(\Omega) \subset C(T; A(\Omega))
\]
gives us a solution \( u(t, x) = \partial_t^{-2} w(t, x) + \varphi(x) + t\psi(x) \in C^2(T; A(\Omega)). \)

\( \square \)

\textbf{Proof of Theorem 1.2} \quad \text{Local uniqueness follows from the Cauchy-Kovalevskaya theorem, and we can extend it by analytic continuation.}

Now we sketch the proof of existence. A complex-analytic function on \( B_T \times U \) is said to be an element of \( \mathcal{G}_{T, \zeta}^C(U) \) if there exists a constant \( C > 0 \) such that
\[
(9) \quad \forall \alpha \in \mathbb{N}^n, \forall t \in B_T, \quad \sup_{x \in U} |\partial^\alpha u(t, x)| \leq C \zeta^{[\alpha]} |D^{[\alpha]} \varphi(|t|/T)|.
\]

It can be proved that \( \mathcal{G}_{T, \zeta}^C(U) \) is a Banach algebra. The theorem can be proved in the same way as in the real case. \( \square \)
References


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