ON THE BOUNDARIES OF SELF-SIMILAR TILES IN \( \mathbb{R}^1 \)

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Abstract. The aim of this note is to study the construction of the boundary of a self-similar tile, which is generated by an iterated function system \( \{ \phi_i(x) = \frac{1}{N}(x + d_i) \}_{i=1}^N \). We will show that the boundary has complicated structure (no simple points) in general; however, it is a regular fractal set.

1. Introduction

Let \( N \) be a positive integer and let \( D = \{ d_1, d_2, \ldots, d_q \} \subset \mathbb{R} \) be a set of real numbers. In this note we consider an iterated function system (IFS) \( \{ \phi_i(x) \}_{i=1}^q \) defined as
\[
\phi_i(x) = \frac{1}{N}(x + d_i), \quad i = 1, 2, \ldots, q.
\]
It is well known that there exists a unique nonempty compact set \( T \) satisfying
\[
T = \bigcup_{i=1}^q \phi_i(T)
\]
(see, e.g., [F]). We call \( T \) a self-similar set. If \( T \), written as \( T(N, D) \), has nonempty interior and \( q = N \), \( T \) is termed a self-similar tile. It was proved by Kenyon [K] and Lagarias and Wang [LW] that, if \( T \) is a self-similar tile, the set \( D \) can be rationalized, that is, there exist real numbers \( a \) and \( c \) such that \( D = cD' + a \) and \( D' \subset \mathbb{Z} \). We will mainly study the geometric properties of self-similar tiles, so we can assume that \( D \) lies in \( \mathbb{Z} \).

Without loss of generality we can assume that \( d_1 = 0 < d_2 < \cdots < d_q \) throughout this paper.

Now we introduce the concept product form defined by Odlyzko [O] and Lagarias and Wang [LW2]. Denote \( E + F := \{ x + y : x \in E, \ y \in F \} \) for any two sets \( E \) and \( F \). For the given \( N, D \) is said to have the product form if there is a residue system \( \mathcal{E} \) (mod \( N \)) with \( 0 \in \mathcal{E} \) so that
\[
\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_k
\]
with all sums distinct and
\[
D = N^{l_1}\mathcal{E}_1 + N^{l_2}\mathcal{E}_2 + \cdots + N^{l_k}\mathcal{E}_k,
\]
where \( N \) is the scaling factor.
where \( 0 \in E_i \) for \( 1 \leq i \leq k \), \(#E_i\), the cardinality of \( E_i \), is larger than one and all \( l_i \), are integers with \( 0 \leq l_1 \leq l_2 \leq l_3 \leq \cdots \leq l_k \). \( D \) is said to have the strict product form if \( E = \{0, 1, \ldots, N-1\} \). For example, for \( N = 4 \), \( D_1 = \{0, 1\} + 4\{0, 2\} = \{0, 1, 8, 9\} \) is a strict product form and \( D_2 = \{0, 5\} + 4\{0, 2\} = \{0, 5, 8, 13\} \) is a (not strict) product form.

**Theorem 1.1 ([1], [LW2]).** If \( D \) has the product form, then the self-similar set \( T(N, D) \) is a self-similar tile. Moreover, \( T(N, D) \) is a union of finite closed intervals if and only if \( D \) has the strict product form.

The sufficient condition (\( D \) is a product form) in Theorem 1.1 is far from being necessary. In general it is difficult to characterize the set \( D \) for an integer \( N \) so that \( T(N, D) \) is a tile; only the case \( N = p^n \) and \( N = pr \), where \( p \) and \( r \) are prime, was solved by Lagarias and Wang [LW2], Lau and Rao [LR] respectively. Theorem 1.1 implies that the interior of a tile \( T(N, D) \) contains an infinite number of disjoint open intervals if \( D \) is not a strict product form. What can we say about the construction of the tile in this case?

Some notions and initial ideas come from Xu [X]. Let \( x \in \partial T \) be a point on the boundary of \( T \); \( x \) is called a simple point if there exists \( \epsilon > 0 \) such that either \((x - \epsilon, x) \cap T = \emptyset \) and \((x, x + \epsilon) \cap T = (x, x + \epsilon) \) or \((x - \epsilon, x) \cap T = (x - \epsilon, x) \) and \((x, x + \epsilon) \cap T = \emptyset \).

**Theorem 1.2.** If \( T = T(N, D) \) is a self-similar tile but \( D \subset \mathbb{Z} \) is not a strict product form, then the boundary \( \partial T \) of \( T \) contains no simple points.

Theorem 1.2 implies that \( \partial T \) is a nonempty compact set which has no isolated points. Then it contains infinite members. Moreover we have the following result:

**Theorem 1.3.** If \( T = T(N, D) \) is a self-similar tile but \( D \subset \mathbb{Z} \) is not a strict product form, then the boundary \( \partial T \) of \( T \) is a fractal, that is, \( 0 < \text{dim}_H \partial T < 1 \).

We remark that \( \text{dim}_H \partial T < 1 \) was already established in Strichartz and Wang [SW]. Also the above two theorems no longer hold in the nonuniform dilations setting. For example, let \( f_1(x) = \frac{2}{3}x \), \( f_2(x) = \frac{2}{3}x + \frac{1}{3} \) and \( f_3(x) = \frac{2}{3}x + \frac{8}{3} \). Then it can be checked that the self-similar set is

\[
T = \bigcup_{k=0}^{\infty} \left[ 1 - \frac{1}{3^{2k}}, 1 - \frac{1}{3^{2k+1}} \right] \cup \{1\}.
\]

Then \( \partial T \) has simple points and \( \text{dim}_H \partial T = 0 \). For the definitions of Hausdorff and box dimensions we refer to [E]. In general we have

**Theorem 1.4.** Let \( T = T(N, D) \) be a self-similar set with \( D \subset \mathbb{Z} \) and \( \#D = q \geq N \). Then \( \partial T \) is a regular set, i.e., \( \text{dim}_H \partial T = \text{dim}_B \partial T \).

### 2. Proof of Theorem 1.2

Throughout this section we shall assume that the digit set \( D \) for the self-affine tile \( T(N, D) \) has the form \( D = \{0,d_2,\ldots,d_N\} \) with \( d_j \in \mathbb{Z} \) and \( 0 < d_2 < \cdots < d_N \).

Let \( \Sigma_q = \{1, 2, \ldots, q\} \), \( \Sigma_q^* = \{(i_1, i_2, \ldots, i_n) : \text{all } i_j \in \Sigma_q \} \) and \( \Sigma_q^* = \bigcup_{n=1}^{\infty} \Sigma_q^n \). For the IFS \( \phi_i = \frac{1}{d_i}(x + d_i) \), \( i = 1, 2, \ldots, q \), and \( \sigma = (i_1, i_2, \ldots, i_n) \in \Sigma_q^n \), as usual we define \( \phi_\sigma(x) = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n}(x) \), the composition of \( \phi_{i_s} \) (\( 1 \leq s \leq n \)).
say that an IFS \( \{ \phi_i(x) \}_{i=1}^q \) satisfies the open set condition (OSC) if there exists a nonempty open set \( O \) (bounded) such that
\[
\bigcup_{i=1}^q \phi_i(O) \subseteq O \quad \text{and} \quad \phi_i(O) \cap \phi_j(O) = \emptyset \quad \forall i \neq j.
\]

We remark that an IFS which generates a self-similar tile satisfies the OSC. This property will be used in the following. We say that a sequence of open intervals \( \{(a_n, b_n)\}_{n=1}^\infty \) is monotonically decreasing to \( a \) if \( b_{n+1} < a_n \) for all \( n \geq 1 \) and \( \lim_{n \to \infty} a_n = a \); Similarly a sequence of open intervals \( \{(a_n, b_n)\}_{n=1}^\infty \) is monotonically increasing to \( b \) if \( b_n < a_{n+1} \) for all \( n \geq 1 \) and \( \lim_{n \to \infty} b_n = b \). Since \( T \) can be expressed explicitly as
\[
T = \left\{ \sum_{k=1}^\infty N^{-k}d_k : d_k \in \mathcal{D} \right\},
\]
clearly \( T \subset [0, d_\eta/(N - 1)] \).

**Lemma 2.1.** Let \( T(N, \mathcal{D}) \) be a self-similar tile and let \( a_1 = 0 \). Suppose that \( T \cap [0, b_1] = \bigcup_{i=1}^k [a_i, b_i] \) with \( b_1 < a_{i+1}, \ i = 1, 2, \ldots, k - 1 \), and \( b_k \in \partial T \). Then \( b_k \) is a simple point.

**Proof.** Suppose that \( b_k \) is not a simple point. Then there exists a sequence of open intervals \( \{(\alpha_n, \beta_n)\}_{n=1}^\infty \) monotonically decreasing to \( b_k \) satisfying \( (\alpha_n, \beta_n) \subseteq T \) and for each \( n \geq 1 \) there is an open interval \( (\alpha_n', \beta_n') \subseteq (\beta_{n+1}, \alpha_n) \) such that \( (\alpha_n', \beta_n') \cap T = \emptyset \). Note that \( T = \bigcup_{i=1}^k \phi_i(T) \) and \( \phi_1(b_k) = b_k / N < b_k \). Then there exists \( i \in \{1, \ldots, k\} \) such that \( \phi_i(b_k) \in [a_i, b_i] \). We claim that \( \phi_1(b_k) \neq b_k \). In fact, if \( \phi_1(b_k) = b_k \), since the sequence \( \{(\alpha_n, \beta_n)\}_{n=1}^\infty \) is monotonically decreasing to \( b_k \), then \( \phi_1(\alpha_n, \beta_n) \subseteq (b_k, a_{i+1}) \) for large enough, which contradicts \( \phi_1(T) \subseteq T \). Hence \( \phi_1(b_k) \in [a_i, b_i] \).

Now we show that \( \phi_1(b_k) < \phi_2(a_k) \). Suppose otherwise, that is, \( \phi_1(b_k) \in \phi_2((a_k, b_k)) \). Then for \( n \) large enough, we have \( \phi_1((\alpha_n, \beta_n)) \subseteq \phi_2((a_k, b_k)) \), which contradicts the OSC. Hence \( \phi_1(b_k) < \phi_2(a_k) \). In this case there is an integer \( m \) such that \( \phi_1((\alpha_n', \beta_n')) \subseteq [a_i, b_i] \) and \( \phi_1(\beta_n') < \phi_2(a_k) \) for all \( n \geq m \). This implies that \( \bigcup_{n=m}^\infty \phi_1((\alpha_n', \beta_n')) \) can be covered by \( \bigcup_{n=2}^{m+1} \bigcup_{j=1}^k \phi_1([a_i, b_j]) \). Thus one of the intervals of the latter contains two adjacent intervals of the former, which contradicts the OSC again by the definitions of the sequences. So the result follows.

**Lemma 2.2.** Let \( T(N, \mathcal{D}) \) be a self-similar tile and let \( a_1 = 0 \). Suppose that \( T \cap [0, a_{k+1}] = a_{k+1} + \bigcup_{i=1}^k [a_i, b_i] \) with \( b_1 < a_{i+1}, \ i = 1, 2, \ldots, k \). Then \( a_{k+1} \) is a simple point.

**Proof.** Let \( \eta = a_{k+1} - b_k > 0 \) and choose \( n \) such that, for any \( \sigma \in \Sigma_N^\eta \), the diameter of \( \phi_\sigma(T) \) is less than \( \eta \). Since \( T = \bigcup_{\sigma \in \Sigma_N} \phi_\sigma(T) \), there exists \( \sigma \in \Sigma_N \) such that \( a_{k+1} \in \phi_\sigma(T) \). Observing \( (b_k, a_{k+1}) \cap T = \emptyset \) and the choice of \( n \), we have \( a_{k+1} = \phi_\sigma(0) \), and thus \( a_{k+1} \) is the left end point of the closed interval \( \phi_\sigma([0, b_k]) \) which is contained in \( T \). This implies that \( a_{k+1} \) is a simple point.

**Lemma 2.3.** Suppose that \( T = T(N, \mathcal{D}) \) is a self-similar tile but \( \mathcal{D} \) is not of the strict product form. Then both \( 0 \) and \( d_\eta/(N - 1) \) are not simple points.

**Proof.** Suppose that \( 0 \) is a simple point of \( \partial T \). Since \( \mathcal{D} \) is not of the strict product form, the interior \( T^0 \) of \( T \) consists of countable disjoint open intervals without
common end points. Let \( b_1 = \max \{ x : [0, x] \subseteq T \} \); then by Lemma 2.1 \( b_1 \) is a simple point. Denote by \( a_2 \) the smallest point in \( T \) which is larger than \( b_1 \). By Lemma 2.2 the point \( a_2 \) is simple too. With the same idea it is easy to see that there exists a point \( a \in T \) and a monotonically increasing sequence \( \{(a_n, b_n)\}_{n=1}^{\infty} \) which converges to \( a \) such that \( a_1 = 0 \) and

\[
T \cap [0, a] = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup \{a\}.
\]

It is obvious that there exists an \( i \) such that \( \phi_1(a) \in (a_i, \phi_1(a)) \) and \( m_1 \) such that \( \phi_1([a_n, b_n]) \subset (a_i, \phi_1(a)) \) for \( n \geq m_1 \). Since \( \phi_j(a) > \phi_1(a) \) for \( j \geq 2 \), there exists \( m_2 \) such that \( \phi_j(a_n) > \phi_1(a) \) for \( n \geq m_2 \) and \( 2 \leq j \leq N \). Let \( m = \max \{m_1, m_2\} \). Hence \( \bigcup_{s=m}^{\infty} \phi_1((b_s, a_{s+1})) \subset (a_i, \phi_1(a)) \) can be covered by \( \bigcup_{j=1}^{N} \bigcup_{n=1}^{m} \phi_j([a_n, b_n]) \). This is impossible by the proof of Lemma 2.4. Hence \( 0 \) is not a simple point. The proof of the result about the point \( d_N/(N-1) \) is similar (symmetric).

**Proof of Theorem 1.2** Suppose that \( a \in \partial T \) is a simple point. By the definition of a simple point, without loss of generality we assume that \( (a - \epsilon, a) \cap T = \emptyset \) and \( (a, a + \epsilon) \cap T = (a, a + \epsilon) \) for some \( \epsilon > 0 \). Similar to the proof of Lemma 2.2 we choose \( n \) so that the diameter of \( \phi_\sigma(T) \) is less than \( \epsilon \) for all \( \sigma \in \Sigma_N^+ \). Then there exists a \( \sigma \in \Sigma_N^+ \) satisfying \( a = \phi_\sigma(0) \). Note that, for different \( \sigma' \in \Sigma_N^+ \), either \( \max_{x \in T} \phi_{\sigma'}(x) \leq a - \epsilon \) or \( \phi_{\sigma'}(0) \geq \phi_{\sigma}(0) + N^{-n} \) by OSC. This implies that \( [a, a + N^{-n}] \cap T = [a, a + N^{-n}] \cap \phi_\sigma(T) \), which leads to \( a = \phi_\sigma(0) \) is not a simple point by Lemma 2.3. The result follows from this contradiction.

3. **Proofs of Theorems 1.3 and 1.4**

In this section we prove Theorems 1.3 and 1.4. Note that the IFS \( \{\phi_i(x) = \frac{1}{N}(x + d_i)\}_{i=1}^{q} \) does not satisfy the OSC in general, which causes some difficulties in studying the dimensions. To overcome these difficulties several methods have been used. Here we follow the approach of [HLR] to obtain a graph-directed system with OSC such that one of the graph-directed sets is \( \partial T \). A different method was used in [SW]. Since \( T(N, D) \subseteq [0, d_q/(N-1)] \), let \( b \) be the minimal integer which is larger than or equal to \( d_q/(N-1) \). We construct an auxiliary tile \( \Gamma := T(N, C) = [0, b] \supseteq T[N, D] \) where \( C = \{0, b, 2b, \ldots, (N-1)b\} \), and define \( \psi_J(x) = \frac{1}{N}(x + (j - 1)b) \), \( j = 1, 2, \ldots, N \). Let \( \Gamma_J = \psi_J(\Gamma) \) for all \( J \in \Sigma_N \). The sequence \( \{\Gamma_J : J \in \Sigma_N^k\}_{k=1}^{\infty} \) forms a nested family of partitions of \([0, b]\). We can select a graph-directed system from these partitions: for \( J \in \Sigma_N^k \), we give a label to \( \Gamma_J \) as

\[
\Delta(J) = \{d_J - c_J : I \in \Sigma_q^k, \phi_I(\Gamma) \cap \psi_J(\Gamma) \neq \emptyset\},
\]

where \( c_J = c_{j_k} + Nc_{j_{k-1}} + \cdots + N^{k-1}c_{j_1} \), if \( J = (j_1, j_2, \ldots, j_k) \in \Sigma_N^k \) and the definition of \( d_I \) is similar. Let \( S_k = \{J \in \Sigma_N^k : \Delta(J) \neq \emptyset\} \), \( S_k^* = \{J \in \Sigma_N^k : \Delta(JJ') \neq \emptyset, \forall J' \in \Sigma_N^k\} \) and \( S_k' = S_k - S_k^* \). It is easy to see that the interval \( \psi_J(\Gamma) \subseteq T \) if and only if \( J \in S_k^* \) and by the construction

\[
T = \bigcap_{k=1}^{\infty} \left( \bigcup_{J \in S_k} \psi_J(\Gamma) \right) \quad \text{and} \quad \partial T = \bigcap_{k=1}^{\infty} \left( \bigcup_{J \in S_k'} \psi_J(\Gamma) \right).
\]

(We remark that it may be necessary to add one or two end points \([0, b]\) to the right side of the second identity above according to the simplicities of 0 and \( b \), which causes trivial changes in the following proofs and no influence at all on the results.)
The crux of this construction is that \( \{\Delta(J) : J \in \Sigma_N^*\} \) is a finite set. This allows us to construct a graph-directed system to reproduce \( T \) and \( \partial T \) in view of (3.1). Here we consider \( \partial T \) only. Let \( \{\Delta(J_i)\}_{i=1}^m \) be all different words in \( S' = \bigcup_{k=1}^{\infty} S'_k \). Then we define the vertices \( V \) as:

\[
V = \{\Delta(0), \Delta(J_1), \ldots, \Delta(J_m)\},
\]

where \( \Delta(0) = \{0\} \) is the “root” (define \( \Delta(J J) = \Delta(J) \)). The corresponding directed edges \( E = \{E_{ij}\}_{i,j=0}^m \) on \( V \) are

\[
E_{ij} = \{c_s \in C : \Delta(J_i s) = \Delta(J_j), 1 \leq s \leq N\},
\]

which come from the partition relationship

\[
\Delta(J_i) \rightarrow \Delta(J_i 1) \Delta(J_i 2) \cdots \Delta(J_i N), \quad i = 0, 1, \ldots, m.
\]

It is clear that, for any vertex \( \Delta(J_j) \), there is a path from the root \( \Delta(0) \) to it. If we let

\[
\phi_{ij}^e = N^{-1}(x + e), \quad e \in E_{i,j}, \quad i, j = 0, 1, \ldots, m,
\]

then according to [HLR, Proposition 3.3], there are nonempty compact subsets \( \{F_0 = \partial T, F_1, \ldots, F_m\} \) satisfying the following graph-directed relationship for \( \partial T \):

\[
(3.2) \quad F_i = \bigcup_{j=0}^m \bigcup_{e \in E_{i,j}} \phi_{ij}^e(F_j) = \bigcup_{j=0}^m N^{-1}(F_j + E_{i,j}), \quad i = 0, 1, \ldots, m.
\]

From (3.2) we can define an \((m+1) \times (m+1)\) matrix \( B \) with the \((i,j)\)th entry given by

\[
b_{ij} = \#E_{i,j}, \quad i, j = 0, 1, \ldots, m
\]

[F] p. 48, where \( B \) is called the adjacency matrix of \( \partial T \). The adjacency matrix is used to count the number of paths of the graph-directed sets in the iteration. Let \( e \) be the \((m+1)\)-vector with all entries equal to 1 and let \( e_i \) be an \((m+1)\)-vector with the \( i \)th entry 1 and zero otherwise. It is not difficult to prove that \( \#S'_n = e_0^t B^n e \) where \( S'_n \) is used in (3.1) [HLR] Proposition 4.1, which satisfies

\[
\lim_{n \to \infty} \left(\frac{\#S'_n}{n}\right)^{1/n} = \lambda_B,
\]

where \( \lambda_B \) is the spectral radius of \( B \). The Hausdorff dimension of \( \partial T \) can be calculated by the following theorem [HLR] Theorem 4.3).

**Theorem 3.1.** Suppose the self-similar set \( T(N, D) \) has nonempty interior and let \( B \) be the adjacency matrix of \( \partial T \). Then

\[
\dim_H(\partial T) = \frac{\log \lambda_B}{\log N},
\]

where \( \lambda_B \) is the spectral radius of \( B \).

To prove Theorem 1.3 we need the following lemmas.

**Lemma 3.2.** Suppose that the adjacency matrix \( B \) of \( \partial T \) is irreducible with spectral radius \( \lambda_B = 1 \). Then the cardinalities of all graph-directed sets are equal to one.

**Proof.** By the Perron-Frobenius Theorem there exists a positive eigenvector \( v \) such that \( v = Bv \). Since \( B \) is irreducible, for each \( j \) there exists an integer \( k \) such that
the \((j,j)\)-entry \(b_{jj}^{(k)}\) of \(B^k\) is positive. Moreover, using \(v = B^k v\), it is easy to get \(b_{ji}^{(k)} = 0\) for \(i \neq j\) and \(b_{jj}^{(k)} = 1\). Hence we have, by iterating (3.2) \(k\) times,

\[N^k F_j = F_j + c_J\]

for some \(J \in \Sigma_N^k\). Consequently \(F_j = \{\sum_{n=1}^{\infty} N^{-kn} c_J = c_J/(N^k-1)\}\) is a singleton. □

Lemma 3.3. Let \(D\) be not a strict product form. Then we can modify all the graph-directed sets in (3.2) such that each of them has no isolated points (may be empty) and keep \(F_0 = \partial T\) invariant.

Proof. From the relation

\[(3.3) \quad F_0 = \bigcup_{j=0}^{m} N^{-1}(F_j + E_{0,j}),\]

we claim that, if \(E_{0,j} \subseteq C\) is nonempty for \(j \neq 0\), then \(F_j\) can be selected so that no isolated points in it can be cancelled without influence on (3.3). Suppose there is an isolated point \(x_0 \in F_j\). If \(x_0 + E_{0,j} \subseteq \bigcup_{k=0, k \neq j} (F_k + E_{0,k})\), then we can omit \(x_0\) from \(F_j\). If the above inclusion is not true, then \(x_0 + c\), for some \(c \in E_{0,j}\), is an isolated point of \(NF_0 = N\partial T\), which is impossible by Theorem 1.2. If \(#F_j\) is finite, that is, all points of it are isolated, then (3.3) holds for each point in \(F_j\), in the case \(F_j\) can be cancelled from the graph-directed relation (3.3) without loss of anything. So the claim follows obviously. Note that in the graph-directed system for the boundary of \(T\) the “root” is \(\partial T = F_0\) and for each \(j\) there is a path from the root \(F_0\) to \(F_j\). Those relations imply that all graph-directed sets can be modified by finite steps with the same method. □

Proof of Theorem 1.3. It is well known that the Hausdorff dimension of the boundary of a self-similar tile is less than one (see e.g. [SW]). Now we make use of Theorem 3.1 to prove that the dimension is positive. Since the adjacency matrix \(B\) can be decomposed as

\[
\begin{bmatrix}
  A_{11} & A_{12} & \cdots & A_{1k} \\
  0 & A_{22} & \cdots & A_{2k} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & A_{kk}
\end{bmatrix},
\]

where all \(A_{ii}, i \geq 2\), are irreducible and \(A_{11}\) is either a zero or an irreducible matrix [S], by Lemma 3.3 we can assume that all cardinalities of graph-directed sets in (3.2) are not finite, and then by Lemma 3.2 the spectral radius of \(A_{kk}\) is larger than one, and so is \(\lambda_B\). Hence the result follows by Theorem 3.1. □

Before proving Theorem 1.4, we recall the definition of box dimension for a nonempty bounded subset \(E\) of \(\mathbb{R}^1\). Let \(N_r(E)\) be the smallest number of sets of diameter \(r\) that can cover \(E\). The lower and upper box dimension of \(E\) are defined as

\[
\dim_B E = \liminf_{r \to 0} \frac{\log N_r(E)}{-\log r} \quad \text{and} \quad \overline{\dim}_B E = \limsup_{r \to 0} \frac{\log N_r(E)}{-\log r}
\]
respectively. If they are equal we refer to the common value as the box dimension of $E$.

**Proof of Theorem 1.4.** Denote $s = \dim_H \partial T$; then $\lambda_B = N^s$ by Theorem 3.1. For any $t > s$ we have
\[
\left( \sum_{\sigma \in \mathcal{S}_n} |\psi_\sigma(\Gamma)|^t \right)^{1/n} = \left( \#\mathcal{S}_n' N^{-nt} b^t \right)^{1/n}.
\]
Thus
\[
\lim_{n \to \infty} \left( \sum_{\sigma \in \mathcal{S}_n} |\psi_\sigma(\Gamma)|^t \right)^{1/n} = \lambda_B N^{-t} = N^{s-t} < 1.
\]
This limit implies that there exists an integer $m$ such that for $n \geq m$ we have
\[
\sum_{\sigma \in \mathcal{S}_n} |\psi_\sigma(\Gamma)|^t < b^t,
\]
which is equivalent to
\[
\#\mathcal{S}_n' < N^{nt}.
\]
For any $r$, $0 < r < bN^{-m}$, there is an integer $n$ larger than $m$ satisfying
\[
bN^{-n} \leq r < bN^{-n+1}.
\]
Note that $\{\psi_\sigma(\Gamma)\}_{\sigma \in \mathcal{S}_n'}$ is a $bN^{-n}$-cover of $\partial T$. Hence
\[
N_r(\partial T) \leq \#\mathcal{S}_n'.
\]
By the definition of box dimension we have
\[
\dim_B(\partial T) = \limsup_{r \to 0} \frac{N_r(\partial T)}{\log r} \leq \limsup_{n \to \infty} \frac{\log (\#\mathcal{S}_n')}{\log N^{nt}} \leq t.
\]
The result follows by letting $t$ tend to $s$. \hfill \Box

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