ANDERSON’S THEOREM FOR COMPACT OPERATORS

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Abstract. It is shown that if $A$ is a compact operator on a Hilbert space with its numerical range $W(A)$ contained in the closed unit disc $\overline{D}$ and with $W(A)$ intersecting the unit circle at infinitely many points, then $W(A)$ is equal to $\overline{D}$. This is an infinite-dimensional analogue of a result of Anderson for finite matrices.

The numerical range $W(A)$ of a bounded linear operator $A$ on a complex Hilbert space $H$ is the subset $\{(Ax, x) : x \in H, \|x\| = 1\}$ of the complex plane, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the inner product and norm in $H$, respectively. Basic properties of the numerical range can be found in [5, Chapter 22] or [4].

In the early 1970s, Joel Anderson proved an interesting result on the numerical ranges of finite matrices. Namely, if $A$ is an $n$-by-$n$ complex matrix, considered as an operator on $\mathbb{C}^n$ equipped with the standard inner product and norm, with its numerical range $W(A)$ contained in the closed unit disc $\overline{D}$ ($\overline{D} \equiv \{ z \in \mathbb{C} : |z| < 1\}$) and intersecting the unit circle $\partial \overline{D}$ at more than $n$ points, then $W(A) = \overline{D}$ (cf. [9, p. 507]). The purpose of this paper is to prove an infinite-dimensional analogue of Anderson’s result for compact operators.

Theorem 1. If $A$ is a compact operator on a Hilbert space with $W(A)$ contained in $\overline{D}$ and $W(A)$ intersecting $\partial \overline{D}$ at infinitely many points, then $W(A) = \overline{D}$.

Anderson never published his proof of the above-mentioned result. As related by him many years later via an e-mail to the second author, his proof was based on the application of Bézout’s theorem to the Kippenhahn curve of the matrix $A$. Generalizations of this result along this line can be found in [3]. In recent years, there appeared three more proofs. One is by Dritschel and Woerdeman [2, Theorem 5.8], based on the canonical decomposition and radial tuples for numerical contractions developed by them. (A numerical contraction is an operator $A$ with $W(A) \subseteq \overline{D}$.)

The second one is due to the second author (cf. [12, Lemma 6]); it depends on the classical Riesz-Fejér theorem on nonnegative trigonometric polynomials. More recently, Hung gave another proof in his Ph.D. dissertation [6, Theorem 4.2.1] by making use of Ando’s characterization of numerical contractions.

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We will prove Theorem 1 using the support function $d_A$ of the compact convex set $W(A)$ of an operator $A$:

$$d_A(\theta) = \max \{ \Re (e^{-i\theta} A) \}$$

$$= \max \{ \Re (\cos \theta \Re A + \sin \theta \Im A) \}$$

for $\theta$ in $\mathbb{R}$, where $\Re A = (A + A^*)/2$ and $\Im A = (A - A^*)/(2i)$ are the real and imaginary parts of $A$. Note that $d_A(\theta)$ is simply the signed distance from the origin to the supporting line $L_\theta$ of $W(A)$ which is perpendicular to the ray $R_\theta$ from the origin that forms angle $\theta$ from the positive $x$-axis (cf. Figure 2).

Our main tool is the next theorem, due to Rellich [10, p. 57], on the analytic perturbation for multiple eigenvalues of Hermitian operators; an elegant and elementary proof can be found in [11, p. 376]. The present form is from [8, Theorem 3.3].

**Theorem 3.** Let $\theta \mapsto A_\theta$ be a real analytic function from an open interval $I$ of $\mathbb{R}$ to Hermitian operators on a fixed Hilbert space, and let $d(\theta) = \max W(A_\theta)$ for $\theta$ in $I$. Assume that for some $\theta_0$ in $I$, $d(\theta_0)$ is an isolated eigenvalue of $A_{\theta_0}$ with finite multiplicity $n$. Then there is an open subinterval $J$ of $I$ which contains $\theta_0$ and there are $m$, $1 \leq m \leq n$, real analytic functions $d_1, \ldots, d_m : J \to \mathbb{R}$ such that

(a) $d_1(\theta_0) = \cdots = d_m(\theta_0) = d(\theta_0)$,

(b) for every $\theta$ in $J \setminus \{ \theta_0 \}$, the $d_j(\theta)$’s are distinct isolated eigenvalues of $A_\theta$ with respective multiplicity $n_j$ independent of $\theta$ which satisfies $\sum_{j=1}^m n_j = n$,

(c) there is some $d_{j_1}$ (resp., $d_{j_2}$) such that $d(\theta) = d_{j_1}(\theta)$ (resp., $d(\theta) = d_{j_2}(\theta)$) for all $\theta$, $\theta < \theta_0$ (resp., $\theta > \theta_0$) in $J$, and

(d) $d(\theta) = \max \{ d_1(\theta), \ldots, d_m(\theta) \}$ for all $\theta$ in $J$.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** We first express our assumptions in terms of $d_A$. The condition $W(A) \subseteq \mathbb{B}$ is obviously equivalent to $d_A(\theta) \leq 1$ for all $\theta$. Under this, we
then have, for a fixed $\theta$, the equivalence of $e^{i\theta} \in \overline{W(A)}$ and $d_A(\theta) = 1$. Indeed, $e^{i\theta}$ belonging to $\overline{W(A)}$ is equivalent to 1 belonging to $\overline{W(e^{-i\theta}A)}$, which is the same as 1 belonging to $\text{Re} W(e^{-i\theta}A) = \overline{\text{Re} (e^{-i\theta}A)}$ (because $W(e^{-i\theta}A) \subseteq \overline{D}$) or $d_A(\theta) = 1$.

Now let $e^{i\theta_n}, n \geq 1, \theta_n \in [0, 2\pi)$, be a sequence of distinct points in $\overline{W(A)} \cap \partial D$. Passing to a subsequence, we may assume that $\theta_n$ converges to $\theta_0$ in $[0, 2\pi]$. Since $d_A(\theta_n) = 1$ for all $n$ and the function $\theta \mapsto \overline{W(\text{Re} (e^{-i\theta}A))}$ is continuous (cf. [5, Solution 220]), we obtain $d_A(\theta_0) = 1$. Moreover, since $\overline{W(\text{Re} (e^{-i\theta_0}A))}$ equals the convex hull of the spectrum of the compact operator $\text{Re} (e^{-i\theta_0}A)$, we infer that $d_A(\theta_0)$ is an isolated eigenvalue of $\text{Re} (e^{-i\theta_0}A)$ with finite multiplicity. Thus Theorem 3 may be applied to obtain two real analytic functions $d_1$ and $d_2$ on some neighborhood $J = (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ of $\theta_0$ such that $d_A = d_1$ on $(\theta_0 - \varepsilon, \theta_0]$ and $d_A = d_2$ on $[\theta_0, \theta_0 + \varepsilon)$. Without loss of generality, we may assume that $(\theta_0 - \varepsilon, \theta_0]$ contains infinitely many $\theta_n$'s. Hence $d_1(\theta_n) = d_A(\theta_n) = 1$ for such $\theta_n$'s. Since $\theta_n$ converges to $\theta_0$ and $d_1$ is analytic on $J$, we obtain $d_1 = 1$ on $J$. Therefore, $d_1 \leq d_A \leq 1$ implies that $d_A = 1$ on $J$. Let $\alpha = \{\theta \in \mathbb{R} : d_A(\theta) = 1\}$. The above arguments also show that if $\theta'$ is a limit point of $\alpha$, then there is some neighborhood $(\theta' - \varepsilon', \theta' + \varepsilon')$ contained in $\alpha$. Now let $a = \sup \{\theta \in \mathbb{R} : [\theta_0, \theta] \subseteq \alpha\}$ and $b = \inf \{\theta \in D : (\theta, \theta_0) \subseteq \alpha\}$. We infer from the above that $a = \infty$ and $b = -\infty$, that is, $\alpha = D$. This shows that $d_A = 1$ is a limit point of $\partial D \subseteq \overline{W(A)}$. As we have seen in the first paragraph of this proof, $d_A(\theta) = 1$ is equivalent to $1 \in W(\text{Re} (e^{-i\theta}A))$. Since this latter set equals the convex hull of the spectrum of the compact operator $\text{Re} (e^{-i\theta}A)$, we infer that 1 is an eigenvalue of $\text{Re} (e^{-i\theta}A)$. Hence 1 is in $W(\text{Re} (e^{-i\theta}A))$ or in $W(e^{-i\theta}A)$ (since $W(e^{-i\theta}A) \subseteq \overline{D}$), which is the same as $e^{i\theta}$ in $W(A)$. We conclude that $\partial D \subseteq W(A)$. The convexity of $W(A)$ then implies that $W(A) = \overline{D}$, completing the proof.

An alternative proof for the last part of the preceding proof is, after obtaining $\overline{W(A)} = \overline{D}$ from $\partial D \subseteq \overline{W(A)}$ and the convexity of $\overline{W(A)}$, to invoke [5, Solution 213] that any compact operator $A$ with $0 \in W(A)$ has $W(A)$ closed, concluding that $W(A) = \overline{D}$.

We end this paper with some further remarks. First, any compact operator $A$ with $W(A) = \overline{D}$ must have norm bigger than one. This is because if $\|A\| \leq 1$, then from the equality case of the Cauchy-Schwarz inequality, we easily derive that $W(A) \cap \partial D = \partial D$ consists of eigenvalues of $A$, which is impossible for the compact $A$. Second, we note that in Theorem 1 the condition that $\overline{W(A)}$ intersects $\partial D$ at infinitely many points cannot be weakened. For example, for each $n \geq 1$, if $A_n$ is the finite-rank operator $\text{diag}(1, \omega_n, \ldots, \omega_n^{n-1}, 0, 0, \ldots)$, where $\omega_n$ is the $n$th primitive root of 1, then $W(A_n) \subseteq \overline{D}$ and $W(A_n)$ intersects $\partial D$ at the $n$ points $1, \omega_n, \ldots, \omega_n^{n-1}$. Finally, Theorem 1 can be generalized from the unit disc to any elliptic disc centered at the origin: if $A$ is a compact operator with $W(A)$ contained in the closed elliptic disc

$$E = \{ x + iy \in \mathbb{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \}, \quad a, b > 0,$$
and with $W(A)$ intersecting $\partial E$ at infinitely many points, then $W(A) = E$. This can be reduced to Theorem 1 by considering the affine transform

$$B = \frac{1}{a} \text{Re} A + \frac{i}{b} \text{Im} A$$

of $A$ since the numerical range of $B$ equals $\mathbb{D}$.

[7] and [1] are the other papers which contain information on the numerical ranges of compact operators.

References


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