PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 12, December 2006, Pages 3691–3696
S 0002-9939(06)08373-0
Article electronically published on May 31, 2006

LENGTH SPECTRUM IN RANK ONE SYMMETRIC SPACE IS NOT ARITHMETIC

INKANG KIM

(Communicated by Jon G. Wolfson)

Abstract. In this paper we show that a nonelementary nonparabolic group in a real semisimple Lie group of rank one has the property that the set of translation lengths of hyperbolic elements is not contained in any discrete subgroup of $\mathbb{R}$.

1. Introduction

In this paper we deal with the problem of discreteness of the set of translation lengths of elements in a nonelementary isometry group of a pinched Hadamard manifold. This problem is related to the mixing property of the geodesic flow [1]. Let Γ be a discrete subgroup of an isometry group of a Hadamard manifold whose sectional curvature is bounded above by -1. One defines that the length spectrum of Γ is arithmetic iff the set of translation lengths of elements of Γ is contained in a discrete subgroup of $\mathbb{R}$. Dal'Bo proved that the length spectrum of Γ being nonarithmetic is equivalent to the condition that the geodesic flow in a nonwandering set is topologically transitive [4].

In particular, we prove that the set of translation lengths cannot be contained in any discrete subgroup of $\mathbb{R}$ for a nonelementary nonparabolic isometry subgroup in a semisimple Lie group of rank one of noncompact type. This result implies that a geodesic flow in a nonwandering set of a negatively curved locally symmetric manifold is topologically transitive.

For the 2-dimensional pinched Hadamard manifold case, Dal'Bo proved nonarithmeticity [5]. A group is called **nonelementary** if its limit set is infinite and is called **nonparabolic** if not all elements of the group fixes some point in the ideal boundary.

Theorem 1. *Let $\Gamma \subset G$ be a nonelementary nonparabolic group in a semisimple Lie group G of noncompact type of rank one. Then the set of translation lengths of hyperbolic isometries in Γ is not contained in any discrete subgroup of $\mathbb{R}$.*

As a corollary, we obtain

Corollary 1. *Let N be a negatively curved nonelementary locally symmetric manifold. Then the geodesic flow in a nonwandering set of N is topologically transitive.*

Received by the editors October 13, 2004 and, in revised form, June 16, 2005.
2000 *Mathematics Subject Classification.* Primary 58D19, 53C23.
This work was partially supported by KOSEF Grant (R01-2005-000-10625-0(2005)).

The marked length rigidity itself is an important problem in Riemannian geometry and especially in symmetric spaces [7, 9, 8, 6]. For unmarked length spectrum, see [10].

2. Boundary of rank one symmetric spaces and the action of the isometry group

Let X be a symmetric space of noncompact type of rank one, $H^n_{\mathbb{R}}, H^n_{\mathbb{C}}, H^n_{\mathbb{H}}, H^2_{\mathbb{O}}$, real, complex, quaternionic, Cayley hyperbolic spaces respectively. The ideal boundary of X is the equivalence classes of geodesic rays under the equivalence relation that two geodesic rays are equivalent if they have bounded Hausdorff distance. But geometrically, it can be identified with the one-point compactification of the nilpotent group N in the Iwasawa decomposition of $\mathrm{Iso}(X) = KAN$. This ideal boundary has a limit metric of the metrics defined on horospheres. There is a nice description of this limit metric in terms of the coordinates in N. N can be identified with $\mathrm{Im}\,\mathbb{F} \rtimes \mathbb{F}^n$, and the multiplication is defined by

$$[t, z][s, w] = [t + s + 2\,\mathrm{Im}\langle z, w\rangle, z + w]$$

where $\langle z, w\rangle = \sum z_i \overline{w_i}$. Define $|[t, k]| = (|t|^2 + |k|^4)^{1/4}$ and $d(g, h) = |h^{-1}g|$. Then this is a left invariant metric. Indeed, N is a Carnot group which is a two-step nilpotent group.

The action of an isometry in X extends continuously to ∂X. In terms of N, this action can be listed as follows:

(1) Dilation: $[t, z] \to [r^2 t, rz]$ where $r > 0$.

If we call $\{(t, 0)\}$ *a vertical set* and $\{(0, z)\}$ a *horizontal set* of N, the dilation acts differently on the vertical set and on the horizontal set. This is a very important fact for future calculations. Indeed this reflects the fact that the ideal boundaries of complex, quaternionic and Cayley hyperbolic spaces have subriemannian geometry.

(2) The nilpotent group N acts on itself by multiplication which is the isometry group.

(3) $(\nu, M) \in O_{\mathbb{F}}(1) \times O_{\mathbb{F}}(m-1)$ acts as $[t, z] \to [\nu t \nu^{-1}, \nu z M]$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

For the Cayley hyperbolic case, the stabilizer of the $\mathbb{O}$-line $L_1 = 0 \times \mathbb{O}$ through $(0, 0)$ and $(0, 1)$ is Spin(8), and the stabilizer of the geodesic through $(0, 0)$ and $(0, 1)$ is Spin(7). Let $L_2 = \mathbb{O} \times 0$. Then Spin(8) acts on L_1 as $SO(8)$ via the even $\frac{1}{2}$-spin representation, and on L_2 by the odd $\frac{1}{2}$-spin representation.

In Heisenberg coordinates, the ideal boundary of L_1 is the vertical set, and the ideal boundary of L_2 is the unit sphere in the horizontal set. So $\nu \in \mathrm{Spin}(7)$ acts by

$$[t, z] \to [\rho(\nu)t, \phi(\nu)z],$$

where ρ and ϕ are the spin representations. See [11] (page 146). For further references, see [2, 3].

In summary, the action of an isometry on the ideal boundary of rank one symmetric space, which fixes 0 and ∞, is of the form $[t, z] \to [\alpha(t), \beta(z)]$, where $\alpha(t) = l^2 \nu t \nu^{-1}, \beta(z) = l\nu z M$ and $l > 0, (\nu, M) \in O_{\mathbb{F}}(1) \times O_{\mathbb{F}}(m-1)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. For the Cayley hyperbolic case, the action belongs to $A \times \mathrm{Spin}(7)$, and α, β are spin representations.

Lemma 1. *Let $[t, z] \to [\alpha(t), \beta(z)]$ be the action of an isometry on the ideal boundary of rank one symmetric space, which fixes 0 and ∞. Then $|\alpha(t)| = |t|, |\beta(z)| = z$.*

If $\sum a_i\beta_i^p(z_i)$ tends to zero as p tends to ∞, then $\sum a_i\beta_i^p(z_i)$ is identically zero. The same is true for $\sum b_i\alpha_i^p(w_i)$.

Proof. Note that the group fixing 0 and ∞ is MAN, where $M \subset K$ and KAN is the Iwasawa decomposition. M is $O_{\mathbb{F}}(1) \times O_{\mathbb{F}}(n)$ of the form $[t, z] \to [\nu t\nu^{-1}, \nu z M]$ for the $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ case and Spin(7) of the form $[t, z] \to [\alpha(t), \beta(z)]$ for the Cayley hyperbolic case.

We first show that M acts isometrically on both the horizontal and the vertical sets. The real, complex, quaternionic case is obvious. But in general this can be done by just understanding how the Carnot metric on the ideal boundary is obtained. See [13] for details. Consider the family of horospheres based at ∞ parametrized by t. Each horosphere has two tangent planes V_2, V_1 invariant under the action of M. V_2 is tangent to the $\mathbb{F}$-plane, and V_1 is an orthogonal complement of the $\mathbb{F}$-plane in the tangent space of the horosphere. V_2 is called vertical and V_1 horizontal. If g_t is the induced metric on the horosphere at t, the metric on X can be written as $g_t \times dt^2$. Then the subriemannian metric on the ideal boundary N is the limit of $e^{-2t}g_t$. The vertical direction becomes the t coordinate and the horizontal direction becomes the z coordinate in $[t, z] \in N$. Since V_1, V_2 is invariant under M, the induced action on N also preserves this dichotomy. This shows that $|\alpha(t)| = |t|, |\beta(z)| = |z|$.

Pick any left invariant metric on $O_{\mathbb{F}}(1) \times O_{\mathbb{F}}(n)$ and Spin(7). We claim that, given $\epsilon > 0$ and z, there exists p such that

$$|\beta^{p+n}z - \beta^n z| < \epsilon, \ \forall n \in \mathbb{Z}.$$

Since $O_{\mathbb{F}}(1) \times O_{\mathbb{F}}(n)$ and Spin(7) are compact, $\{\beta^n\}$ will have an accumulation point. So, for any $\epsilon' > 0$, there exist n, m such that $d(\beta^n, \beta^m) < \epsilon'$. Set $p = n - m$; then since β acts isometrically with respect to the Euclidean norm, by choosing ϵ' small enough,

$$|\beta^{p+k}z - \beta^k z| = |\beta^p z - z| = |\beta^n z - \beta^m z| < \epsilon.$$

Suppose β_1, β_2 are such actions. Since $\{\beta_1^n\}$ accumulate to some β_1', $d(\beta_1', \beta_1^{n_k}) < \epsilon$ for some infinite subset $\{n_k\}$. Then $\{\beta_2^{n_k}\}$ will accumulate to some β_2', so there exist n, m such that

$$d(\beta_1^n, \beta_1^m) < \epsilon, \ d(\beta_2^n, \beta_2^m) < \epsilon.$$

Applying this argument, for given ϵ, a_i and z_i, there exists p such that

$$|\sum a_i\beta_i^{p+n}z_i - \sum a_i\beta_i^n z_i| < \epsilon, \ \forall n \in \mathbb{Z}.$$

Set $f(p) = \sum a_i\beta_i^p z_i$. If f is not identically zero, then $f(p_0) \neq 0$ for some p_0. Choose $N > 0$ large enough so that the ball of radius $\sum_{i=1}^{\infty} N^{-i}$ centered at $f(p_0)$ has positive distance from the origin. For N^{-1}, there is an integer $q_1 \geq 1$ such that $|f(p_0 + q_1) - f(p_0)| < N^{-1}$. Let $p_0 + q_1 = p_1$. Then there is an integer $q_2 \geq 1$ such that $|f(p_1 + q_2) - f(p_1)| < N^{-2}$. Set $p_1 + q_2 = p_2$. In this way find integers $\{p_i\}$ inductively so that $p_i \geq p_{i-1} + 1$ and

$$|f(p_i) - f(p_{i-1})| < N^{-i}.$$

Then $\{f(p_i)\}$ cannot tend to 0 as i tends to ∞ since $|f(p_i) - f(p_0)| < \sum_{i=1}^{\infty} N^{-i}$. This contradicts the fact that $f(p)$ converges to zero as p tends to ∞. The same argument applies to $\sum b_i\alpha_i^p(w_i)$. $\square$

3. Cross-ratio on the ideal boundary of the rank one symmetric space

The **cross-ratio** of four points in the ideal boundary of a Hadamard manifold X is defined as (see [12])

$$[a,b,c,d] = \lim_{(x_i,y_i,z_i,w_i)\to(a,b,c,d)} exp\{d(x_i,z_i)+d(y_i,w_i)-d(x_i,w_i)-d(y_i,z_i)\}.$$

In [7], it is shown that the cross ratio on the ideal boundary is given in terms of the generalized Heisenberg group as follows.

Lemma 2. *If g_1, g_2, g_3, g_4 are four points in the ideal boundary $N \cup \infty$ of rank one symmetric space, the cross ratio is given by*

$$[g_1,g_2,g_3,g_4] = \frac{|g_3^{-1}g_1|^2|g_4^{-1}g_2|^2}{|g_4^{-1}g_1|^2|g_3^{-1}g_2|^2}.$$

4. Arithmetic group

In this section we prove

Theorem 2. *Let $\Gamma \subset G$ be a nonelementary nonparabolic group in a semisimple Lie group G of noncompact type of rank one. Then the set of translation lengths of hyperbolic isometries in Γ is not contained in any discrete subgroup of $\mathbb{R}$.*

Proof. If the length spectrum of Γ is arithmetic, the set of translation lengths of hyperbolic isometries is contained in $a\mathbb{Z}$ for some $a \in \mathbb{R}$.

For a hyperbolic element α, let $l(\alpha)$ denote the translation length of α and let $\alpha^{\pm}$ denote the attracting and the repelling fixed points of α respectively. Then by the formula (see [7], [5])

$$\lim_{n\to\infty} e^{l(\alpha^n)+l(\beta^n)-l(\beta^n\alpha^n)} = [\alpha^-,\beta^-,\alpha^+,\beta^+]$$

the set of cross-ratios on the limit set is contained in $e^{a\mathbb{Z}}$. Here we use the fact that endpoints of periodic geodesics are dense in $L_\Gamma \times L_\Gamma$, where L_Γ is the limit set of Γ; see [7]. This means that 0 is the only accumulation point of cross-ratios on the limit set. Let g be a hyperbolic isometry in Γ with the attracting fixed point ∞ and the repelling fixed point 0 in the Heisenberg coordinates (by conjugation if necessary). We may assume that the translation length l of g is strictly larger than 1 by taking the powers of g. Take two limit points $\xi_1 = [t,z], \xi_2 = [s,w]$ different from 0 and ∞ and $|\xi_1| \neq |\xi_2|$. It is easy to find such elements. For example, if $|\xi_1| = |\xi_2|$, by taking $g(\xi_2)$ we can increase the norm of ξ_2. Let $f(x) = [\xi_1,\xi_2,\infty,x]$ be a continuous function on ∂X, where X is an associated symmetric space of G. Since $f(\infty) = 1$, $\lim_{n\to\infty} f(g^n(x)) = 1$ for $x \neq 0$. The length spectrum of Γ being arithmetic implies that $f(g^n(x)) = [\xi_1,\xi_2,\infty,g^n(x)] = \frac{d(\xi_2,g^n(x))^2}{d(\xi_1,g^n(x))^2} = 1$ for large enough n if $x \neq 0$ (since 0 is the only accumulation point of cross-ratios on the limit set). Set $g([t,z]) = [l^2\alpha(t), l\beta(z)]$, where α and β belong to the appropriate compact groups depending on G.

Then using $x = \xi_1$

$$d([t,z],[l^{2p}\alpha^p(t),l^p\beta^p(z)])^4 = d([s,w],[l^{2p}\alpha^p(t),l^p\beta^p(z)])^4,$$

which is equivalent to

$$\begin{aligned}
d([t,z],g^p([t,z]))^4 &= (|t|^2+|z|^4)\\
+l^{4p}(|\alpha^p(t)|^2+|\beta^p(z)|^4) &+ 2l^{2p}|z|^2|\beta^p(z)|^2\\
+4l^{2p}(\operatorname{Re}\langle\beta^p(z),z\rangle)^2 &+ 4l^{2p}|\operatorname{Im}\langle\beta^p(z),z\rangle|^2\\
-2l^{2p}(\operatorname{Re}\langle\alpha^p(t),t\rangle) &+ 4l^{3p}\operatorname{Re}\langle\alpha^p(t),\operatorname{Im}\langle\beta^p(z),z\rangle\rangle\\
-4l^p\operatorname{Re}\langle t,\operatorname{Im}\langle\beta^p(z),z\rangle\rangle &- 4l^p|z|^2\operatorname{Re}\langle\beta^p(z),z\rangle\\
-4l^{3p}|\beta^p(z)|^2\operatorname{Re}\langle z,\beta^p(z)\rangle &= (|s|^2+|w|^4)\\
+l^{4p}(|\alpha^p(t)|^2+|\beta^p(z)|^4) &+ 2l^{2p}|w|^2|\beta^p(z)|^2\\
+4l^{2p}(\operatorname{Re}\langle\beta^p(z),w\rangle)^2 &+ 4l^{2p}|\operatorname{Im}\langle\beta^p(z),w\rangle|^2\\
-2l^{2p}(\operatorname{Re}\langle\alpha^p(t),s\rangle) &+ 4l^{3p}\operatorname{Re}\langle\alpha^p(t),\operatorname{Im}\langle\beta^p(z),w\rangle\rangle\\
-4l^p\operatorname{Re}\langle s,\operatorname{Im}\langle\beta^p(z),w\rangle\rangle &- 4l^p|w|^2\operatorname{Re}\langle\beta^p(z),w\rangle\\
-4l^{3p}|\beta^p(z)|^2\operatorname{Re}\langle\beta^p(z),w\rangle &= d([s,w],g^p[t,z])^4. \qquad (1)
\end{aligned}$$

Now dividing the equation by $4l^{3p}$ and letting $p\to\infty$, we get

$$\begin{aligned}
|\operatorname{Re}\langle\alpha^p(t),\operatorname{Im}\langle\beta^p(z),z\rangle\rangle &- |\beta^p(z)|^2\operatorname{Re}\langle\beta^p(z),z\rangle\\
-\operatorname{Re}\langle\alpha^p(t),\operatorname{Im}\langle\beta^p(z),w\rangle\rangle &+ |\beta^p(z)|^2\operatorname{Re}\langle\beta^p(z),w\rangle| \le Cl^{-p}. \qquad (2)
\end{aligned}$$

By Lemma 1, this implies that

$$\begin{aligned}
\operatorname{Re}\langle\alpha^p(t),\operatorname{Im}\langle\beta^p(z),z\rangle\rangle &- |\beta^p(z)|^2\operatorname{Re}\langle\beta^p(z),z\rangle\\
=\operatorname{Re}\langle\alpha^p(t),\operatorname{Im}\langle\beta^p(z),w\rangle\rangle &- |\beta^p(z)|^2\operatorname{Re}\langle\beta^p(z),w\rangle. \qquad (3)
\end{aligned}$$

Next dividing by l^{2p} and letting $p\to\infty$ we get the terms after l^{2p} equal. After that, we use l^p. Finally we get $|t|^2+|z|^4=|\xi_1|^4=|\xi_2|^4=|s|^2+|w|^4$, which is a contradiction to the choice of ξ_1 and ξ_2. □

References

[1] M. Babillot, Ergodic theory, Geometric rigidity and number theory conference at Isaac Newton Institute, July 2000.

[2] M. Cowling, A. Dooley, A. Korányi, F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. Math. **87** (1991), no. 1, 1-41. MR1102963 (92e:22017)

[3] M. Cowling, A. Dooley, A. Korányi, F. Ricci, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal. **8** (1998), no. 2, 199-237. MR1705176 (2000m:53071)

[4] F. Dal'Bo, *Topologie du feuilletage fortement stable*, Ann. Inst. Fourier **50** (2000), no. 3, 981-993. MR1779902 (2001i:37045)

[5] F. Dal'Bo, *Remarques sur le spectre des longueurs d'une surface et comptages*, Bol. Soc. Brasil. Mat. **30** (1999), no. 2, 199-221. MR1703039 (2000i:53117)

[6] F. Dal'Bo and I. Kim, *Marked length rigidity for symmetric spaces*, Comment. Math. Helv. **77** (2002), no. 2, 399-407. MR1915048 (2003g:53058)

[7] Inkang Kim, *Marked length rigidity of rank one symmetric spaces and their product*, Topology **40** (2001), no. 6, 1295-1323. MR1867246 (2002m:53064)

[8] Inkang Kim, *Rigidity on symmetric spaces*, Topology **43** (2004), no. 2, 393-405. MR2052969 (2005d:53067)

[9] Inkang Kim, *Ergodic theory and rigidity on the symmetric space of noncompact type*, Ergodic Theory Dynam. Systems **21** (2001), no. 1, 93-114. MR1826662 (2002c:37034)

[10] Inkang Kim, *Isospectral finiteness on hyperbolic 3-manifolds*, Comm. Pure and Appl. Math. **59** (2006), no. 5, 617-625.

[11] G. D. Mostow, Strong rigidity of locally symmetric spaces, Annals of Mathematics Studies, no. 78, Princeton Univ. Press, Princeton, NJ, 1973. MR0385004 (52:5874)

[12] J-P. Otal, *Sur la géometrie symplectique de l'espace des géodésiques d'une variété à courbure négative*, Rev. Mat. Iberoamericana **8** (1992), no. 3, 441-456. MR1202417 (94a:58077)

[13] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. **129** (1989), 1-60. MR0979599 (90e:53058)

Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: inkang@math.snu.ac.kr