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LINKS IN AN OPEN BOOK DECOMPOSITION AND IN THE STANDARD CONTACT STRUCTURE

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Dedicated to Professor Yukio Matsumoto on his sixtieth birthday

ABSTRACT. We study a relationship between arc presentations of links in $\mathbb{R}^3$ and Legendrian links in $\mathbb{R}^3$ with the standard tight contact structure. We determine the arc indices of torus knots.

1. INTRODUCTION

A relationship between closed braid representatives of links in $\mathbb{R}^3$ and transverse and Legendrian links in $\mathbb{R}^3$ with the standard contact structure was first discovered by Bennequin in his pioneering work [1] in "contact topology." He studied this relationship and proved in [1] that the standard contact structure on $\mathbb{R}^3$ is tight, that is, there is no overtwisted disc in $(\mathbb{R}^3, \xi_{\mathrm{std}})$. Since then, some other relationships have been studied; cf. [3], [4]. In this paper, we study a relationship between arc presentations of links in $\mathbb{R}^3$ and Legendrian links in $(\mathbb{R}^3, \xi_{\mathrm{std}})$. The definitions given below suggest that a closed braid representative of a link in $\mathbb{R}^3$ is a transverse link in $\mathbb{R}^3$ with the open book decomposition $\{A(\theta)\}_{0\leq\theta\leq 2\pi}$, and that an arc presentation of a link in $\mathbb{R}^3$ is a (piecewise) Legendrian link in $\mathbb{R}^3$ with $\{A(\theta)\}$. In fact, we show in §2 that one arc presentation of a link in $\mathbb{R}^3$, with its topological link type $\mathcal{L}$, corresponds to one Legendrian link in $(\mathbb{R}^3, \xi_{\mathrm{std}})$, with its topological link type $\overline{\mathcal{L}}$, and vice versa, where $\overline{\mathcal{L}}$ denotes the mirror image of $\mathcal{L}$. Studying this correspondence, we determine the arc index of torus knots, Theorem 1.1.

Let A be the trivial knot in S^3. The complement $S^3 \setminus A$ is then a product $(int\, D^2) \times S^1$. There exists a fibre projection $p_A \colon S^3 \setminus A \to S^1$ whose fibres $A(\theta) = (p_A)^{-1}(e^{i\theta})$ are discs for $0 \leq \theta \leq 2\pi$. An orientation of A induces an orientation of $A(\theta)$, and a positive direction of the fibration $\{A(\theta)\}_{0\leq\theta\leq 2\pi}$. We may think of S^3 as the union $\mathbb{R}^3 \cup \{\infty\}$ and think of A as the union (z-axis) $\cup \{\infty\}$. If we use the polar coordinate, then the fibres $A(\theta)$ are half-planes at angle θ.

Let L be an oriented link in S^3 with $L \cap A = \emptyset$. If we can choose a projection p_A so that L intersects each fibre $A(\theta)$ of the fibration $\{A(\theta)\}$ transversely and

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positively in exactly n points, then L is said to be represented as a *closed n-braid relative to an axis A*.

A link L in S^3 can be embedded in finitely many fibres $A(\theta)$ so that L meets each fibre $A(\theta)$ in a single simple arc. Such an embedding is called an *arc presentation* of L, and the number of fibres used to represent L is the *arc number* of L, denoted by $\alpha(L)$. The minimum number of fibres required to represent a given link L in this fashion is a knot invariant, called the *arc index* of L, and denoted by $\overline{\alpha}(L)$. Note that $\overline{\alpha}(L) = \overline{\alpha}(\overline{L})$, where $\overline{L}$ denotes the mirror image of L. This definition of an arc presentation of a link was suggested by Birman and Menasco [2], and was given by Cromwell [6]. Lyon [10] had given a definition of a presentation of links in *square-bridge position*. Lyon's presentation of links is essentially the same as an arc presentation of links. Brunn [5] showed that every link in $\mathbb{R}^3$ admits a projection onto $\mathbb{R}^2$ with exactly one multiple point. It follows from Brunn's proof that every link in $\mathbb{R}^3$ admits an arc presentation.

A *contact structure* on a 3-manifold M is a completely non-integrable plane field ξ in the tangent bundle of M. Since ξ is a plane field, ξ_x is a two-dimensional subspace of T_xM at each point x in M. There is a locally defined 1-form α such that $\xi_x = \ker(\alpha_x)$. The plane field ξ is completely non-integrable if any 1-form α defining ξ satisfies $\alpha \wedge d\alpha \neq 0$. The *standard contact structure* ξ_{std} on $\mathbb{R}^3$ is given by $\text{span}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\}$, which is also given by the kernel of the 1-form $\alpha = dz - ydx$ defined on $\mathbb{R}^3$.

A *transverse link* T in $(\mathbb{R}^3, \xi_{\text{std}})$ is a union of embedded circles that are always transverse to ξ, that is, $T_xT \oplus \xi_x = T_xM$ for every point x on T. A *Legendrian link* L in $(\mathbb{R}^3, \xi_{\text{std}})$ is a union of embedded circles that are always tangent to ξ, that is, $T_xL \subset \xi_x$ for every point x on L.

Let T^2 denote a standardly embedded torus in S^3, that is, T^2 gives a genus one Heegaard splitting $S^3 = V_0 \cup_{T^2} V_1$ of S^3, where each of V_0 and V_1 is a solid torus. Let μ be the unique closed curve on T^2 that bounds a disc in V_0, and λ the unique closed curve on T^2 that bounds a disc in V_1. Give an orientation to μ arbitrarily, and give an orientation to λ so that μ and λ form a positive basis for $H_1(T^2)$, where T^2 is oriented as the boundary of V_0. Up to homotopy, every closed curve on T^2 can be described as $p\mu + q\lambda$, where p and q are integers. If this closed curve is embedded on T^2, then the closed curve is denoted by $T_{(p,q)}$. If p and q are mutually prime, then $T_{(p,q)}$ is a *torus knot of type* (p, q). Placing $\{\infty\} \in S^3 = \mathbb{R}^3 \cup \{\infty\}$ in the interior of V_1, we may assume that $T_{(p,q)}$ is embedded in $\mathbb{R}^3$.

We prove the following theorem in §3.

Theorem 1.1. *The arc index of $T_{(p,q)}$, $\overline{\alpha}(T_{(p,q)})$, is equal to $|p| + |q|$.*

After the completion of this work, Peter Cromwell informed the author that Elisabetta Beltrami independently had obtained Theorem 1.1 by similar methods as ours.

2. Front projection and arc presentation

Let L_ℓ be a Legendrian link with its topological link type $\mathcal{L}$. Let $\Pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a projection given by $(x, y, z) \mapsto (x, z)$. The image of L_ℓ under the projection Π, $\Pi(L_\ell)$, is called the *front projection* of L_ℓ, denoted by F_ℓ. From a front projection F_ℓ, the Legendrian link L_ℓ can be reconstructed uniquely by using the differential equation $y = \frac{dz}{dx}$. Letting the over arc at each double point of F_ℓ be the one with

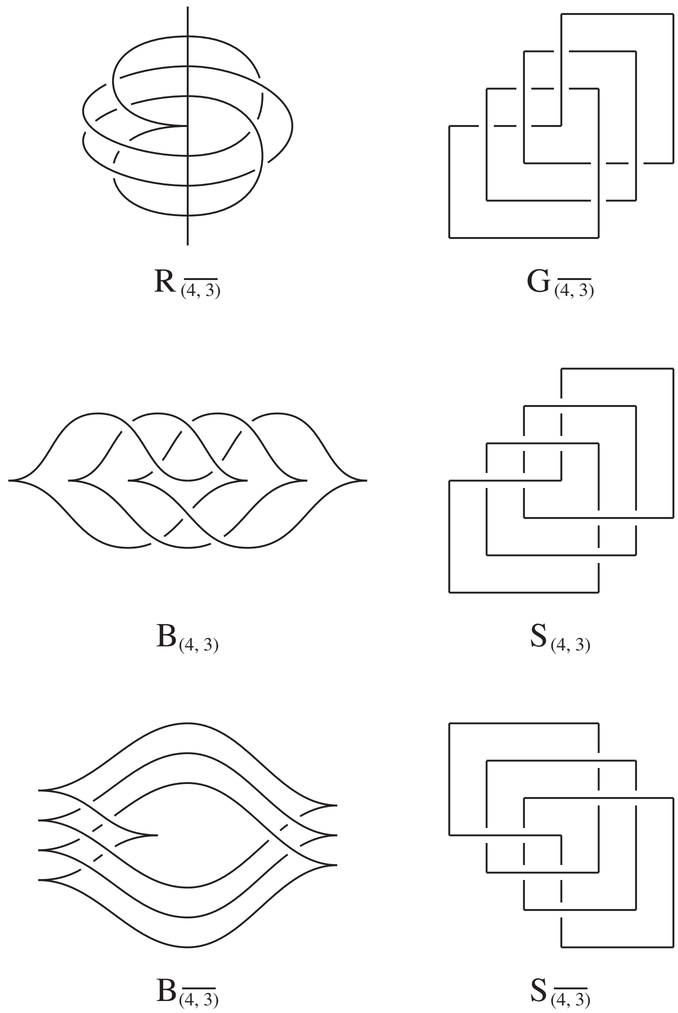

FIGURE 1.

the smaller slope $\frac{dz}{dx} \in \mathbb{R} \setminus \{0\}$, we obtain a *barber-pole diagram* B_ℓ, which is a link diagram of the Legendrian link L_ℓ; cf. [9], [11]. In order to have the standard orientation on $\mathbb{R}^3$, we note that the positive y-axis goes into the page. Let $w(F_\ell)$ be the writhe of the diagram B_ℓ, and let $rc(F_\ell)$ (respectively $\ell c(F_\ell)$) be the number of right (resp. left) cusps of F_ℓ. Note that $rc(F_\ell) = \ell c(F_\ell)$, and that the Thurston-Bennequin number of L_ℓ, $tb(L_\ell)$, is equal to $w(F_\ell) - rc(F_\ell)$.

Rotating B_ℓ 45 degrees counterclockwise, we may construct a diagram S_ℓ in *square-bridge position*, defined by Lyon [10]. See the pair of diagrams in the middle of Figure 1. Each of these two diagrams represents a torus knot of type $(4, 3)$. The horizontal arc passes over the vertical arc at every crossing of the diagram S_ℓ. It is observed by Rudolph [11] that the writhe of S_ℓ, $w(S_\ell)$, is equal to $w(F_\ell)$, and that one right (resp. left) cusp of F_ℓ corresponds to one northeast (resp. southwest) corner of S_ℓ, and vice versa.

Rotating S_ℓ 90 degrees counterclockwise, and switching the over/under information at every crossing, we obtain a diagram $S_{\overline{\ell}}$ in square-bridge position. The diagram $S_{\overline{\ell}}$ represents a link $\overline{L}$ with its topological link type $\overline{\mathcal{L}}$. Rotating $S_{\overline{\ell}}$ 45 degrees clockwise, we may obtain a barber-pole diagram $B_{\overline{\ell}}$ which represents a Legendrian link $L_{\overline{\ell}}$ with its topological link type $\overline{\mathcal{L}}$. See the pair of diagrams at the bottom of Figure 1. Each of these two diagrams represents a torus knot of type $(-4,3)$, the mirror image of a torus knot of type $(4,3)$

Let $G_{\overline{\ell}}$ (resp. G_ℓ) be the diagram which is obtained from S_ℓ (resp. $S_{\overline{\ell}}$) by switching the over/under information at every crossing. So the vertical arc passes over the horizontal arc at every crossing of the diagrams $G_{\overline{\ell}}$ and G_ℓ. Note that $G_{\overline{\ell}}$ (resp. G_ℓ) is also obtained from $S_{\overline{\ell}}$ (resp. S_ℓ) by rotating it 90 degrees clockwise (resp. counterclockwise). We may regard $G_{\overline{\ell}}$ (resp. G_ℓ) as a *rectangular diagram* of a link $\overline{L}$ (resp. L) with its topological link type $\overline{\mathcal{L}}$ (resp. $\mathcal{L}$); cf. [6], [7]. From $G_{\overline{\ell}}$ (resp. G_ℓ), we may construct an arc presentation $R_{\overline{\ell}}$ (resp. R_ℓ) of $\overline{L}$ (resp. L). See the pair of diagrams at the top of Figure 1. Each of these two diagrams represents a torus knot of type $(-4,3)$. This construction shows that the number of vertical arcs of $G_{\overline{\ell}}$ (resp. G_ℓ) is equal to the arc number $\alpha(R_{\overline{\ell}})$ (resp. $\alpha(R_\ell)$) of $R_{\overline{\ell}}$ (resp. R_ℓ). Since $G_{\overline{\ell}}$ (resp. G_ℓ) is a diagram of embedded circles in $\mathbb{R}^3$, the number of vertical arcs of $G_{\overline{\ell}}$ (resp. G_ℓ) is equal to that of horizontal arcs of $G_{\overline{\ell}}$ (resp. G_ℓ), and the number of corners of $G_{\overline{\ell}}$ (resp. G_ℓ) is equal to that of the union of vertical arcs and horizontal arcs of $G_{\overline{\ell}}$ (resp. G_ℓ). The above constructions of $G_{\overline{\ell}}$ and G_ℓ show that the number of vertical (resp. horizontal) arcs of $G_{\overline{\ell}}$ is equal to that of horizontal (resp. vertical) arcs of G_ℓ. It follows that $\alpha(R_{\overline{\ell}}) = \alpha(R_\ell)$, and that the number of corners of each of $G_{\overline{\ell}}$ and G_ℓ is equal to $\alpha(R_{\overline{\ell}}) + \alpha(R_\ell) = 2\alpha(R_{\overline{\ell}}) = 2\alpha(R_\ell)$.

Let $\mathrm{ne}(S_\ell)$ (resp. $\mathrm{nw}(S_\ell)$, $\mathrm{se}(S_\ell)$, $\mathrm{sw}(S_\ell)$) be the number of northeast (resp. northwest, southeast, southwest) corners of the diagram S_ℓ. Since there is a one-to-one correspondence between right (resp. left) cusps of F_ℓ and northeast (resp. southwest) corners of S_ℓ, we have $rc(F_\ell) = \mathrm{ne}(S_\ell)$ and $\ell c(F_\ell) = \mathrm{sw}(S_\ell)$. The equality $\mathrm{ne}(S_\ell) = \mathrm{sw}(S_\ell)$ follows from the equality $rc(F_\ell) = \ell c(F_\ell)$. Since S_ℓ is a diagram of embedded circles in $\mathbb{R}^3$, we obtain the equality $\mathrm{nw}(S_\ell) = \mathrm{se}(S_\ell)$. The equality $\mathrm{ne}(S_\ell) + \mathrm{nw}(S_\ell) + \mathrm{sw}(S_\ell) + \mathrm{se}(S_\ell) = 2\alpha(R_\ell)$ follows from the observation that the number of corners of S_ℓ is equal to that of G_ℓ, which is equal to $2\alpha(R_\ell)$. Then the Thurston-Bennequin number $tb(L_\ell)$ of L_ℓ is equal to $w(F_\ell) - rc(F_\ell) = w(S_\ell) - \mathrm{ne}(S_\ell) = w(S_\ell) - \frac{\mathrm{ne}(S_\ell)+\mathrm{sw}(S_\ell)}{2}$.

The diagram $S_{\overline{\ell}}$ in square-bridge position corresponds to a Legendrian link $L_{\overline{\ell}}$ with its topological link type $\overline{\mathcal{L}}$. Let $F_{\overline{\ell}}$ be a front projection of $L_{\overline{\ell}}$ obtained from $S_{\overline{\ell}}$ by rotating it 45 degrees clockwise. The construction of $S_{\overline{\ell}}$ shows that $\mathrm{ne}(S_\ell) = \mathrm{nw}(S_{\overline{\ell}})$, $\mathrm{nw}(S_\ell) = \mathrm{sw}(S_{\overline{\ell}})$, $\mathrm{sw}(S_\ell) = \mathrm{se}(S_{\overline{\ell}})$ and $\mathrm{se}(S_\ell) = \mathrm{ne}(S_{\overline{\ell}})$, and that $w(S_\ell) = -w(S_{\overline{\ell}})$. Combining the above equalities, we obtain the following lemma.

Lemma 2.1. *The Thurston-Bennequin number $tb(L_{\overline{\ell}})$ of $L_{\overline{\ell}}$ is equal to $-tb(L_\ell) - \alpha(R_\ell)$.*

3. Proof of Theorem 1.1

Changing the roles of $T_{(-p,-q)}$ and $T_{(p,q)}$, and V_0 and V_1 defined in §1, if necessary, and noticing $\overline{\alpha}(L) = \overline{\alpha}(\overline{L})$, it suffices to prove the theorem for torus knots of type (p,q) with $p > q > 0$.

Let $R_{(\overline{p,q})}$ be an arc presentation of a torus knot of type $(-p,q)$ realizing its arc index, that is, $\alpha(R_{(\overline{p,q})}) = \overline{\alpha}(T_{(-p,q)}) = \overline{\alpha}(T_{(p,q)})$. Let $G_{(\overline{p,q})}$ be a rectangular diagram of $T_{(-p,q)}$ corresponding to $R_{(\overline{p,q})}$. Switching the over/under information of $G_{(\overline{p,q})}$ at every crossing, we obtain a diagram $S_{(p,q)}$ of $T_{(p,q)}$ in square-bridge position.

Let $L_{(p,q)}$ be a Legendrian knot corresponding to $S_{(p,q)}$. Theorem 4.1 and Lemma 4.8 in [8] show that $L_{(p,q)}$ is obtained from the unique Legendrian torus knot of type (p,q) with its maximal Thurston-Bennequin number $\overline{tb}(T_{(p,q)})$ by performing stabilizations n times for $n \geq 0$. Then the Thurston-Bennequin number $tb(L_{(p,q)})$ of $L_{(p,q)}$ is equal to $\overline{tb}(T_{(p,q)}) - n = pq - p - q - n$. We define $S_{(\overline{p,q})}$, $L_{(\overline{p,q})}$ and $R_{(p,q)}$ in the same way as in §2. Note that the topological knot type of $L_{(\overline{p,q})}$ is $T_{(-p,q)}$. Lemma 2.1 shows that the Thurston-Bennequin number $tb(L_{(\overline{p,q})})$ of $L_{(\overline{p,q})}$ is equal to $-tb(L_{(p,q)}) - \alpha(R_{(p,q)}) = -tb(L_{(p,q)}) - \alpha(R_{(\overline{p,q})}) = -(pq - p - q - n) - \overline{\alpha}(T_{(p,q)})$. A theorem of Etnyre and Honda (Theorem 4.1 in [8]) shows that the maximal Thurston-Bennequin number $\overline{tb}(T_{(-p,q)})$ of a torus knot of type $(-p,q)$ is equal to $-pq$, where $p > q > 0$. Therefore, we have the inequality $tb(L_{(\overline{p,q})}) = -pq + p + q + n - \overline{\alpha}(T_{(p,q)}) \leq -pq$, so we have $\overline{\alpha}(T_{(p,q)}) \geq p + q + n$.

As observed by Cromwell on p. 39 in [6], we have an arc presentation $R'_{(p,q)}$ of $T_{(p,q)}$ with $\alpha(R'_{(p,q)}) = p + q$. So we have $\overline{\alpha}(T_{(p,q)}) \leq p + q$. Thus, the above two inequalities show that $n = 0$ and $\overline{\alpha}(T_{(p,q)}) = p + q$.

4. Some remarks

Let L_ℓ be a Legendrian link in $(\mathbb{R}^3, \xi_{\mathrm{std}})$. Each point of the intersection of L_ℓ with the plane $\{y = 0\}$ corresponds to a point on F_ℓ where the tangent line to F_ℓ is parallel to the x-axis, and each of these points of F_ℓ corresponds to a corner of S_ℓ. The *contact bridge number* $b_\xi(L_\ell)$, of a Legendrian link L_ℓ is defined to be half of the number of intersections of L_ℓ with the plane $\{y = 0\}$. The *contact bridge index* $\overline{b_\xi}(\mathcal{L})$ is the minimum of $b_\xi(L_\ell)$ among all Legendrian links L_ℓ representing the topological link type $\mathcal{L}$. Note that the contact bridge index is a topological link invariant. The definitions of the arc index and the contact bridge index show the following.

Proposition 4.1. *For every topological link type $\mathcal{L}$, we have $\overline{b_\xi}(\mathcal{L}) = \overline{\alpha}(\mathcal{L})$.*

We make another remark on a relationship between knots in an open book decomposition and knots in the standard contact structure. Let K be an oriented knot. As described by Cromwell in Proposition on p. 34 of [6], two closed braid representatives K^+ and $-K^-$ of K are constructed from one arc presentation K^r of K, where $-K^-$ denotes K^- with the opposite orientation. A closed braid representative K^+ (resp. $-K^-$) is obtained from K^r (resp. $-K^r$) by tilting each arc so that the tilted arcs intersect the fibration $\{A(\theta)\}$ positively, and by connecting these arcs with short arcs on $\partial N(A)$, each of which intersects $\{A(\theta)\}$ positively. From one Legendrian knot K^ℓ, a positive transverse knot $T_+(K^\ell)$ and a negative transverse knot $T_-(K^\ell)$ are constructed as positive and negative transverse push-offs of K^ℓ. It is clear from these constructions that K^+ corresponds to $T_+(K^\ell)$, and K^- to $T_-(K^\ell)$.

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