A SIMPLE PROOF OF ZAGIER DUALITY
FOR HILBERT MODULAR FORMS

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Abstract. In this paper, we give a simple proof of an identity between the Fourier coefficients of the weakly holomorphic modular forms of weight 0 arising from Borcherds products of Hilbert modular forms and those of the weakly holomorphic modular forms of weight 2 satisfying a certain property.

1. Introduction and results

A certain sequence of modular forms of weight 1/2 arises in the theory of Borcherds products for modular forms with Heegner divisors. Zagier proved that there exists a duality, called Zagier duality, between the Fourier coefficients of these modular forms and those of some modular forms of weight 3/2. In [2] Rouse obtained an analog of Zagier duality between the Fourier coefficients of the weakly holomorphic modular forms of weight 0 arising from Borcherds products of Hilbert modular forms and those of the weakly holomorphic modular forms of weight 2 satisfying a certain property. In this paper, we give a simple proof of the identity proved in [2], by constructing a linear relation among the Fourier coefficients of weakly holomorphic modular forms.

Let \( \chi_p \) be the Dirichlet character \( \left( \frac{\cdot}{p} \right) \). For an integer \( k \geq 0 \) and \( \epsilon = \pm 1 \) let \( A_k^\epsilon(\Gamma_0(p), \chi_p) \) denote the subspace of weakly holomorphic modular forms \( F(z) \) of weight \( k \) and character \( \chi_p \) such that if \( F(z) = \sum_{n \in \mathbb{Z}} c(n)q^n \), then \( c(n) = 0 \) for \( \chi_p(n) = -\epsilon \).

Proposition 1.1. Suppose that \( f = \sum_{n \in \mathbb{Z}} a_f(n)q^n \in A_k(f,\Gamma_0(p), \chi_p) \) and \( g = \sum_{n \in \mathbb{Z}} a_g(n)q^n \in A_k(g,\Gamma_0(p), \chi_p) \). For \( t \in \{0, \infty\} \) let

\[
f|z\gamma_t = \sum_{n \in \mathbb{Z}} a_f^t(n)q^n \quad \text{and} \quad g|0\gamma_t = \sum_{n \in \mathbb{Z}} a_g^t(n)q^n,
\]

where \( \gamma_0 = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) and \( \gamma_\infty = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \). If \( k(f) + k(g) = 2 \), then

\[
\sum_{i+j=0} (a_f^0(i)a_g^0(j) + a_f^\infty(i)a_g^\infty(j)) = 0.
\]
Let $p$ be a prime and
\[ s(m) = \begin{cases} 
2 & \text{if } m \equiv 0 \pmod{p}, \\
1 & \text{if } m \not\equiv 0 \pmod{p}. 
\end{cases} \]

From Proposition 1.1 we obtain the following theorem that gives a connection between a weakly holomorphic modular form of weight 0 arising in Borcherds products for Hilbert modular forms (see Theorem 6 in [1]) and a certain weakly holomorphic modular form of weight 2.

**Theorem 1.2.** Suppose that, for a nonnegative integer $m$ and a positive integer $d$ with $\chi(m) \neq -\epsilon$ and $\chi(d) \neq -\epsilon$, there is a function $F_{d,p}^\epsilon(z) \in A_0^\epsilon(\Gamma_0(p), \chi_p)$ such that
\[ F_{d,p}^\epsilon(z) = \frac{1}{s(d)} q^{-d} + O(1) = \sum_{n \in \mathbb{Z}} A_{d,p}(n) q^n \]
and $G_{m,p}^\epsilon(z) \in A_2^\epsilon(\Gamma_0(p), \chi_p)$ such that
\[ G_{m,p}^\epsilon(z) = \frac{1}{s(m)} q^{-m} + O(q) = \sum_{n \in \mathbb{Z}} B_{m,p}(n) q^n. \]

Then
\[ A_{d,p}(m) = -B_{m,p}(d). \]

**Remark 1.3.** Let $S_k^+(\Gamma_0(p), \chi_p)$ denote the subspace of holomorphic cusp forms in $A_k(\Gamma_0(p), \chi_p)$. When $\epsilon = 1$ and $p \in \{5, 13, 17\}$, Theorem 1.2 was proved in [2] by the arithmetic of operators for modular forms.

**Remark 1.4.** If $p = 5, 13$ or $17$, then for each $m \geq 1$ with $\chi_p(m) \neq -1$, there exists a unique $F_{m,p}^+(z) \in A_1^+(\Gamma_0(p), \chi_p)$ and a unique $G_{m,p}^+(z) \in A_2^+(\Gamma_0(p), \chi_p)$.

**Proof of Proposition 1.1.** Suppose $G$ is a meromorphic modular form of weight 2 on $\Gamma_0(p)$ for a prime $p$. We denote the set of distinct cusps of $\Gamma_0(p)$ as $S_p = \{1, \infty\}$. For $\tau \in \mathbb{H} \cup S_p$, let $D_{\tau}$ be the image of $\tau$ under the canonical map from $\mathbb{H} \cup S_p$ to $X_0(p)$, where $\mathbb{H}$ denotes the complex upper half-plane. The residue of $G$ at $D_{\tau}$ on $X_0(p)$, denoted by $\text{Res}_{D_{\tau}} Gdz$, is well defined since we have a canonical correspondence between meromorphic modular forms of weight 2 on $\Gamma_0(p)$ and meromorphic 1-forms of $X_0(p)$. If $\text{Res}_\tau G$ denotes the residue of $G$ at $\tau$ on $\mathbb{H}$, then we obtain
\[ \text{Res}_{D_{\tau}} Gdz = \frac{1}{l_{\tau}} \text{Res}_\tau G. \]

Here, $\lambda_{\tau}$ is the order of the isotropy group at $\tau$. The residue of $G$ at each cusp $t$ in $S_p$ is
\[ \text{Res}_{D_{\tau}} Gdz = \frac{a_t(0)}{2\pi i}, \]
where $G(z) \mid 2 \gamma_t = \sum_{n=m}^{\infty} a_t(n) q^n$ at $\infty$.

To prove Proposition 1.1 we take $G = fg$. Since $\chi_p$ is a quadratic character, $G$ is a meromorphic modular form of weight 2 on $\Gamma_0(p)$. We have
\[ fg|2 \gamma_t = \left( \sum_{n \in \mathbb{Z}} a_f^t(n) q^n \right) \left( \sum_{n \in \mathbb{Z}} a_g^t(n) q^n \right). \]

Therefore, by the residue theorem we give a proof of Proposition 1.1. \qed
Proof of Theorem 1.2. We begin by stating the following lemma.

**Lemma 1.5** (Lemma 3 of [1]). Let \( F(z) = \sum_{n \in \mathbb{Z}} A(n)q^n \in A_k(\Gamma_0(p), \chi_p) \) and \( \epsilon \in \{1, -1\} \). Then \( F(z) \in A_k^\epsilon(\Gamma_0(p), \chi_p) \) if and only if
\[
p^{1-k/2}(F|U_p) = \epsilon \sqrt{p}(F|\gamma_0),
\]
where \( F|U_p = \sum_{n \in \mathbb{Z}} A(pn)q^n \).

From Lemma 1.5 we have
\[
\epsilon \cdot (F_{d,p}|0\gamma_0) = p^{\frac{s(d)}{2}} A_{d,p}(pm)q^n + p^{\frac{s(d)}{2}} \sum_{n \geq 0} A_{d,p}(pn)q^n
\]
and
\[
\epsilon \cdot (G_{m,p}|\gamma_0) = p^{\frac{s(m)}{2}} B_{m,p}(pm)q^n + p^{\frac{s(m)}{2}} \sum_{n \geq 1} B_{m,p}(pn)q^n.
\]
Using Proposition 1.1, we have
\[
\left(\frac{1}{s(m)} + \frac{s(m) - 1}{s(m)}\right) A_{d,p}(m) + \left(\frac{1}{s(d)} + \frac{s(d) - 1}{s(d)}\right) B_{m,p}(d) = 0.
\]
This proves Theorem 1.2. \(\square\)

**Remark 1.6.** Following the method used in the proof of Proposition 1.1, one can also obtain the similar identity between the Fourier coefficients of \( f_{k_1} \) in \( A_k^\epsilon(\Gamma_0(p), \chi_p) \) and those of \( f_{k_2} \) in \( A_k^\epsilon(\Gamma_0(p), \chi_p) \) where \( k_1 + k_2 = 2 \).

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**References**


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