A VARIATIONAL METHOD IN FIXED POINT RESULTS WITH INWARDNESS CONDITIONS

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(Communicated by Jonathan M. Borwein)

Abstract. We generalize, in a metric space setting, the result due to Lim (2000), that a weakly inward multivalued contraction, defined on a nonempty closed subset of a Banach space, has a fixed point. The simple proof of this generalization, avoiding the use of a transfinite induction as in Lim’s paper, is based on Ekeland’s variational principle (1974), along the lines of Hamel (1994) and Takahashi (1991). Moreover, we give a sharp estimate for the distance from any point to the fixed point set.

1. Introduction and notation

In this note, we consider a metric space \((X,d)\), a nonempty subset \(D\) of \(X\), and a multifunction \(T : D \to 2^X\) with nonempty values, and we let \(F_T := \{x \in D : x \in T(x)\}\) denote the (possibly empty) set of fixed points of \(T\). We shall assume that \(T\) has a closed graph \(G_T := \{(x,y) \in D \times X : y \in T(x)\}\). Given \(0 \leq \kappa < 1\), for \(0 < \varepsilon < (1 - \kappa)(1 + \kappa)^{-1}\), we let

\[
\sigma_\varepsilon := \frac{1 - \varepsilon}{1 + \varepsilon} - \kappa > 0,
\]

and we consider the following property, hereafter denoted by \((P)_{\sigma_\varepsilon}\): for every \((x,y) \in G_T\) with \(y \neq x\), there exist \(u \in X\) and \(z \in D\) such that

\[
(1) \quad d(x,y) = d(x,u) + d(u,y), \quad d(u,z) < \varepsilon d(u,x),
\]

and

\[
(2) \quad d(y,T(z)) \leq \kappa d(x,z).
\]

(Note that since \(\varepsilon \leq 1\), both \(u\) and \(z\) differ from \(x\).) We say that \(T\) satisfies property \((P)_{1-\kappa}\) if \(T\) satisfies property \((P)_{\sigma_\varepsilon}\) for all \(0 < \varepsilon < (1 - \kappa)(1 + \kappa)^{-1}\).

Let us first compare these properties with more standard notions. Recall that \(T\) is a contraction of modulus \(\kappa\) \((0 \leq \kappa < 1)\) if

\[
(3) \quad e(T(x),T(z)) := \sup_{y \in T(x)} d(y,T(z)) \leq \kappa d(x,z) \quad \text{for all } x,z \in D
\]

(where \(d(y,T(z)) := \inf_{w \in T(z)} d(y,w)\)). Of course, this is equivalent to

\[
\mathcal{H}(T(x),T(z)) \leq \kappa d(x,z) \quad \text{for all } x,z \in D,
\]

Received by the editors June 24, 2005.
2000 Mathematics Subject Classification. Primary 47H10; Secondary 49J53.

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where \( \mathcal{H}(T(x), T(z)) := \max\{e(T(x), T(z)), e(T(z), T(x))\} \) is the classical Hausdorff distance between \( T(x) \) and \( T(z) \). If \( T \) is a contraction of modulus \( \kappa \), and if for every \((x, y) \in \mathcal{G}_T \) with \( y \neq x \) there exists \( u \in X \) such that

\[
d(x, y) = d(x, u) + d(u, x) \quad \text{and} \quad d(u, D) < \varepsilon d(u, x),
\]

then \((1)\) and \((2)\) are satisfied, for some \( z \in D \). Note further that if \( D \) is closed and if \( T \) is a contraction with (nonempty) closed values, then \( \mathcal{G}_T \) is closed.

On the other hand, if \((X, \|\cdot\|)\) is a normed space (and \( d(x, y) := \|y - x\| \)), and if for some \( \varepsilon > 0 \) we have

\[
\inf_{t \geq 1, x \in D} \|y - x - t(z - x)\| < \varepsilon \|y - x\|
\]

for all \((x, y) \in \mathcal{G}_T, y \neq x\),

then \( T \) satisfies \((1)\) (indeed, if \((x, y) \in \mathcal{G}_T \) with \( y \neq x \) and if \( z \in D \) and \( t \geq 1 \) are such that \( \|y - x - t(z - x)\| < \varepsilon \|y - x\| \), then \((3)\) holds with \( u := x + t^{-1}(y - x) \) since \( \|u - z\| < \varepsilon \|u - x\| \) — of course, \((1)\) and \((5)\) are not equivalent, for the set \( \{u \in X : \|x - y\| = \|x - u\| + \|u - y\|\} \) may in general differ from the geometrical segment joining \( x \) and \( y \). Now, \((5)\) clearly holds for every \( \varepsilon > 0 \) if and only if

\[
T(x) - x \subset [1, +\infty([D - x]) \quad \text{for all } x \in D.
\]

Summing up this discussion, we see that if \( D \) is a (nonempty) closed subset of a normed space \( X \), and if \( T : D \to 2^X \setminus \{\emptyset\} \) is a closed-valued contraction of modulus \( \kappa \) verifying \((6)\), then \( T \) satisfies property \((P)_{1-\kappa}\).

Assume now that \( D = X \). Given \( 0 \leq \kappa < 1 \), we say that the multifunction \( T : X \to 2^X \setminus \{\emptyset\} \) defined on the metric space \( (X, d) \) is a directional contraction of modulus \( \kappa \) if \( \mathcal{G}_T \) is closed and if for every \((x, y) \in \mathcal{G}_T \) with \( y \neq x \), there exists \( z \in X \setminus \{x\} \) such that

\[
d(x, y) = d(x, z) + d(z, y) \quad \text{and} \quad d(y, T(z)) \leq \kappa d(x, z).
\]

Observe that a closed-valued contraction is a directional contraction: \( \mathcal{G}_T \) is closed, while if \((x, y) \in \mathcal{G}_T \) with \( y \neq x \) and if \( T \) satisfies \((3)\), then \((7)\) is satisfied with \( z := y \). Moreover, if \( T \) satisfies \((7)\), then \( T \) satisfies property \((P)_\sigma \) (with \( u := z \)) for all \( 0 < \varepsilon < (1 - \kappa)(1 + \kappa)^{-1} \). Thus, if \( T : X \to 2^X \setminus \{\emptyset\} \) is a directional contraction of modulus \( \kappa \), then \( T \) satisfies property \((P)_{1-\kappa}\).

In \([7]\), Lim proved the following nice result: if \( D \) is a closed subset of a Banach space \( X \) and \( T : D \to 2^X \setminus \{\emptyset\} \) is a closed-valued contraction satisfying \((6)\) — that is to say, \( T \) is a weakly inward contraction — then \( T \) has a fixed point. Here, we give an elementary proof of a generalization of this result, using property \((P)_\sigma \) and a variational method directly derived from the Ekeland variational principle, thus avoiding the use of transfinite induction as in the quoted reference. Moreover, we also provide an estimate for the distance \( d(x, F_T) \). The use of variational methods in fixed point theory dates back to the celebrated paper of Caristi \([3]\). However, our point of view is quite different, more in line with the method of Hamel \([6]\) and Takahashi \([12]\), and is based upon an immediate application of the basic Ekeland’s principle in the spirit of Penot \([10]\) \([11]\).

Our main result also provides an extension of a result in Clarke \([4]\), dealing with continuous single-valued directional contractions, as well as containing the classical result of Nadler \([9]\).

In the sequel, we shall use the following notation: given a function \( f : X \to \mathbb{R} \cup \{+\infty\} \), if \(-\infty < \alpha < \beta \leq +\infty \), we set

\[
[f \leq \beta] := \{x \in X : f(x) \leq \beta\}
\]
(so that \([f \leq +\infty] = X\), and we define in a similar way the sets \([f < \beta]\), \([f > \alpha]\), and \([\alpha < f < \beta] := [f < \beta] \setminus [f \leq \alpha]\). We say that \(f\) is proper if \(\text{dom} f := [f < +\infty] \neq \emptyset\).

2. Fixed point results

Let \(X\) be a metric space endowed with the metric \(d\), and let \(f : X \to \mathbb{R} \cup \{+\infty\}\) be a function. A point \(x \in X\) is said to be a \(d\)-point of \(f\) if
\[
f(x) < f(z) + d(z, x) \quad \text{for all } z \in X, z \neq x.
\]
(Observe that \(d\)-points are in \(\text{dom} f\), and that global minima are \(d\)-points.) With our notation, the basic form of Ekeland’s variational principle is the following (see [11] Theorem B; see also [2] Theorem 2.1):

**Theorem 2.1.** If \((X, d)\) is complete and \(f : X \to \mathbb{R} \cup \{+\infty\}\) is proper, lower semicontinuous, and bounded from below, then \(f\) has a \(d\)-point.

This result is proved through a simple iterative procedure involving a sequence of sets of the type \(M_x := \{z \in X : f(z) + d(z, x) \leq f(x)\} \subset [f \leq f(x)], x \in X\). It is readily seen that \(x\) is a \(d\)-point of \(f\) if and only if \(M_x = \{x\}\), and that, using the triangle inequality we have
\[
\bar{x} \in M_x \text{ is a } d\text{-point of the restriction of } f \text{ to } M_x \implies \bar{x} \text{ is a } d\text{-point of } f.
\]
Thus, applying Theorem 2.1 to the restriction of \(f\) to \(M_x\), we immediately obtain the following.

**Corollary 2.1.** Let \((X, d)\) be a metric space and let \(f : X \to \mathbb{R} \cup \{+\infty\}\) be proper, lower semicontinuous, and bounded from below. Assume that for every \(x \in \text{dom} f\), the set \([f \leq f(x)]\) is complete. Then, for every \(x \in X\), \(f\) has a \(d\)-point in \(M_x\).

Note that Corollary 2.1 applies to \(([f < \beta], d), \beta \in \mathbb{R} \cup \{+\infty\}\), whenever \((X, d)\) is complete. The result with \((X, d)\) complete is given in [11] Theorem B. We finally state the following immediate consequence of Corollary 2.1 which we shall use for the proof of our main result.

**Corollary 2.2.** Let \((X, d)\) be a metric space and let \(f : X \to \mathbb{R} \cup \{+\infty\}\) be a lower semicontinuous function. Assume that for every \(x \in \text{dom} f\) the set \([f \leq f(x)]\) is complete, and that \(\mu := \inf_X f \in \mathbb{R}\). Set \(S := [f \leq \mu]\). Then, the following are equivalent:

(a) \(f\) has no \(d\)-point in \(X \setminus S\);
(b) for every \(x \in X\), there exists \(\bar{x} \in M_x \cap S\) (i.e., \(f(x) - \mu \geq d(x, \bar{x})\)).

(Indeed, a \(d\)-point of \(f\) in \(M_x\), given by Corollary 2.1 must be in \(S\) if (a) holds, and conversely, \(M_x \neq \{x\}\) for \(x \in X \setminus S\) if (b) holds.) Corollary 2.2 slightly extends [6] Theorem 2], which was itself an extension of the results of [12].

**Remark 2.1.** Let \((X, d)\) be complete, let \(f : X \to \mathbb{R} \cup \{+\infty\}\) be lower semicontinuous, and let \(-\infty < \alpha < \beta \leq +\infty\). Then, applying Corollary 2.2 to the metric space \(([f < \beta], d)\) and to the function \(g : [f < \beta] \to \mathbb{R} \cup \{+\infty\}\) defined by \(g(x) := (f(x) - \alpha)^+\), yields that the following are equivalent:

(a) for every \(x \in [\alpha < f < \beta]\) there exists \(y \in [f < \beta] \setminus \{x\}\) such that \(f(x) - \alpha \geq (f(y) - \alpha)^+ + d(y, x)\);
(b) for every \(x \in [\alpha < f < \beta]\) there exists \(\bar{x} \in [f \leq \alpha]\) such that \(f(x) - \alpha \geq d(x, \bar{x})\).

See [2] and the references therein for related results.
The following is the main result of this note.

**Theorem 2.2.** Let $D$ be a nonempty subset of a complete metric space $(X,d)$, and let $T : D \to 2^X$ be a multifunction with nonempty values and closed graph. Let $0 \leq \kappa < 1$, let $\varepsilon > 0$ be such that

$$\sigma_\varepsilon := \frac{1-\varepsilon}{1+\varepsilon} - \kappa > 0,$$

and assume that $T$ satisfies property $(P)_{\sigma_\varepsilon}$ (recall (1). Then, $\mathcal{F}_T \neq \emptyset$ and

$$d(x,T(x)) \geq \sigma_\varepsilon d(x,\mathcal{F}_T) \quad \text{for all } x \in D.$$

**Proof.** Assume first that $\kappa > 0$. Let $(x,y) \in \mathcal{G}_T$, $y \neq x$, be fixed. According to property $(P)_{\sigma_\varepsilon}$, we find $u \in X$ and $z \in D$ such that $d(x,y) = d(x,u) + d(u,y)$, $d(u,z) < \varepsilon d(u,x)$, and $d(y,T(z)) \leq \kappa d(x,z)$. Then

$$d(x,z) \leq d(x,u) + d(u,z) < (1+\varepsilon)d(x,u)$$

and

$$d(y,T(z)) \leq \kappa d(x,z) < \kappa(1+\varepsilon)d(x,u),$$

so that we find $w \in T(z)$ with

$$d(y,w) \leq \kappa(1+\varepsilon)d(x,u).$$

On the other hand, we also have

$$d(z,w) \leq d(z,u) + d(u,y) + d(y,w) = d(z,u) + d(x,y) - d(x,u) + d(y,w) < \varepsilon d(u,x) + d(x,y) - d(x,u) + \kappa(1+\varepsilon)d(x,u),$$

so that

$$d(x,y) > d(z,w) + (1-\varepsilon - \kappa(1+\varepsilon))d(x,u) = d(z,w) + \sigma_\varepsilon(1+\varepsilon)d(x,u).$$

Now, consider $X \times X$ as endowed with the metric

$$\tilde{d}((x',y'),(x'',y'')) := \sigma_\varepsilon \max \left\{d(x',x''), \frac{1}{\kappa}d(y',y'')\right\}.$$

Since $\mathcal{G}_T$ is closed in $(X \times X, \tilde{d})$, $(\mathcal{G}_T, \tilde{d})$ is complete. Consider the continuous, nonnegative function $f := d_{\mathcal{G}_T}$ (the restriction of $d$ to $\mathcal{G}_T$). Then, combining (8), (9), and (10), we see that for every $(x,y) \in \mathcal{G}_T$ with $y \neq x$, we find $(z,w) \in \mathcal{G}_T$ such that

$$f(x,y) > f(z,w) + \tilde{d}((x,y),(z,w)).$$

Thus, $f$ has no $\tilde{d}$-point in $[f>0]$, and we deduce from Corollary 2.2 that $\inf f = 0$, that $[f\leq 0] = \{(x,x) : x \in \mathcal{F}_T\} \neq \emptyset$, and that for every $(x,y) \in \mathcal{G}_T$ there exists $(z,w) \in \mathcal{G}_T$ such that

$$d(x,y) \geq \tilde{d}((x,y),(z,w)) \geq \sigma_\varepsilon d(x,\tilde{x}) \geq \sigma_\varepsilon d(x,\mathcal{F}_T),$$

so that $d(x,T(x)) \geq \sigma d(x,\mathcal{F}_T)$, since $y$ is arbitrary in $T(x)$.

If $\kappa = 0$, considering $\tilde{\kappa} > 0$ such that $\tilde{\sigma}_\varepsilon := (1-\varepsilon)(1+\varepsilon)^{-1} - \tilde{\kappa} > 0$, we see that $T$ satisfies property $(P)_{\tilde{\sigma}_\varepsilon}$, so that, arguing as above and then letting $\tilde{\kappa} \to 0$ yields the conclusion. \qed
Corollary 2.3. Let $D$ be a nonempty subset of a complete metric space $(X,d)$, let $T : D \to 2^X$ be a multifunction with nonempty values and closed graph, and let $0 \leq \kappa < 1$. Assume that $T$ satisfies property $(P)_{1-\kappa}$. Then, $F_T \neq \emptyset$ and $d(x, T(x)) \geq (1 - \kappa)d(x, F_T)$ for all $x \in D$.

Proof. Recall that $T$ satisfying property $(P)_{1-\kappa}$ means that it satisfies property $(P)_{\sigma}$ for any $0 < \varepsilon < (1 - \kappa)(1 + \kappa)^{-1}$. Thus, the conclusion follows from Theorem 2.2.

Remark 2.2. Taking into account what we already said in the introduction, Corollary 2.3 is an extension of Lim’s result [27, Theorem 1], where $D$ is a nonempty closed subset of a Banach space $X$, and $T$ is a contraction with nonempty, closed values satisfying the inwardness condition (9). This result was previously established for single-valued maps by Martinez-Yanez [8], and by Xu [13, Theorem 3.3] under the additional assumption that each $x \in D$ has a nearest point in $T(x)$. Corollary 2.3 also extends Clarke’s Theorem 7.6.2 in [4], where $T$ is a continuous single-valued, directional contraction from the complete metric space $X$ into itself, as well as it contains Nadler’s theorem: every multi-valued contraction with nonempty, closed values, from a complete metric space into itself, has a fixed point (indeed, Nadler considers multifunctions with bounded values — see Remark 2.3 below). A simple example of a (single-valued) directional contraction which is not a contraction is given in [4, Remark 7.6.3].

We note that a variational proof of Nadler’s theorem was already given by Takahashi [12], while Clarke’s proof of his result also relies on Ekeland’s principle, applied to the function $f(x) := d(x, T(x))$, which is continuous if so is (the single-valued map) $T$. Our proof of Theorem 2.2 suggests that it is more appropriate to work in $G_T$ — that is, in the product space $X \times X$ — when possible (in the same spirit, see [11, Section 5]). Still, we now give a variant of Theorem 2.2 where we apply Corollary 2.2 to the function $x \mapsto d(x, T(x))$; see Remark 2.3 below for comments.

Theorem 2.3. Let $D$ be a nonempty closed subset of a complete metric space $(X,d)$, and let $T : D \to 2^X$ be a multifunction with nonempty, closed values. Assume that every $x \in D$ has a nearest point in $T(x)$, and that the function $x \mapsto d(x, T(x))$ is lower semicontinuous on $D$. Let further $0 \leq \kappa < 1$ and $\varepsilon > 0$ be such that

$$\sigma := \frac{1 - \varepsilon}{1 + \varepsilon} - \kappa > 0,$$

and assume that for every $(x,y) \in G_T$ with $y \neq x$, there exist $u \in X$ and $z \in D$ such that

$$d(x, y) = d(x, u) + d(u, y), \quad d(u, z) < \varepsilon d(u, x),$$

and

$$d(z, T(z)) \leq d(z, y) + \kappa d(z, x).$$

Then, $F_T \neq \emptyset$ and

$$d(x, T(x)) \geq \sigma d(x, F_T) \quad \text{for all } x \in D.$$

Proof. We consider $D$ as endowed with the metric $\hat{d}(x, y) := \sigma d(x, y)$ and the lower semicontinuous function $f : D \to \mathbb{R}_+$ defined by $f(x) := d(x, T(x))$. Then, $(D, \hat{d})$ is complete and $[f \leq 0] = F_T$. Let $x \in D$ be such that $f(x) > 0$, and let $y \in T(x)$
be such that \( f(x) = d(x, y) \) — so that \( y \neq x \). Then let \( u \in X \) and \( z \in D \) be as in (11) and (12). We obtain from (11) that \( d(z, x) < (1 + \varepsilon)d(u, x) \) and

\[
\begin{align*}
d(z, y) + \frac{1-\varepsilon}{1+\varepsilon}d(z, x) &< d(z, y) + (1 - \varepsilon)d(u, x) \\
&= d(z, y) + d(x, y) - d(u, y) - \varepsilon d(u, x) \\
&< d(z, y) - (d(u, y) + d(u, z)) + d(x, y) \leq d(x, y).
\end{align*}
\]

Then, using also (12), we have

\[
d(z, T(z)) + \sigma d(z, x) \leq d(z, y) + \frac{1 - \varepsilon}{1 + \varepsilon} d(z, x),
\]

so that, combining the previous inequalities, we have \( f(z) + \bar{d}(z, x) < f(x) \). Thus, \( f \) has no \( \bar{d} \)-point in \([f > 0]\), and we deduce from Corollary 2.2 that \( [f \leq 0] = {\mathcal F}_T \neq \emptyset \), and that for every \( x \in D \) there exists \( \bar{x} \in {\mathcal F}_T \) such that \( d(x, T(x)) \geq \sigma d(x, \bar{x}) \geq \sigma d(x, {\mathcal F}_T) \).

**Remark 2.3.** The assumption that \( x \mapsto d(x, T(x)) \) is lower semicontinuous, in Theorem 2.3, is satisfied, for example, if \( T \) is \( {\mathcal H} \)-upper semicontinuous (“\( {\mathcal H} \)” is for Hausdorff), that is, if \( \lim_{x \to x}\epsilon(T(z), T(x)) = 0 \) for all \( x \in D \).

Note that (2) implies (12) since \( d(z, T(z)) \leq d(z, y) + d(y, T(z)) \). Similarly, if \( T \) is a contraction of modulus \( \kappa \), then \( T \) satisfies \( d(z, T(z)) \leq d(z, T(x)) + \kappa d(z, x) \) for all \( z, x \in D \).

When the multifunction \( T \) has unbounded values, the requirement that it be a contraction is difficult to satisfy, in general. We show in the following simple example that condition (13) is strictly weaker than the contractiveness of \( T \).

**Example 2.1.** Let \( X := \mathbb{R}, D := [0, 1] \), and \( T : D \to 2^X \) be defined by

\[
T(x) := \begin{cases} 
(-\infty, \frac{x}{2} + \frac{1}{4} ] & \text{if } x > 0, \\
[0, \frac{1}{4}] & \text{if } x = 0.
\end{cases}
\]

Then, for any \( x, z \in [0, 1] \) we have \( d(z, T(x)) = (z - \frac{x}{2} - \frac{1}{4})^+ \), whence an easy computation yields \( d(z, T(z)) - d(z, T(x)) \leq \frac{1}{2}|z - x| \), so that (13) is verified with \( \kappa := \frac{1}{4} \). But \( T \) is not a contraction since \( \mathcal{H}(T(0), T(x)) = +\infty \) for any \( x > 0 \).

Observe that \( T \) indeed satisfies property (\( P_1 \))\( -\kappa \), so that Theorem 2.3 applies, as well as Theorem 2.2. Since \( \mathcal{F}_T = [0, \frac{1}{4}] \), we see in this example that the estimate \( d(x, T(x)) \geq (1 - \kappa)d(x, \mathcal{F}_T) \) is exact.

**Acknowledgement**

We thank the referee for drawing our attention to Clarke’s result, which led us to greatly improve the presentation of the paper.

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