CONTINUOUSLY EXTENDING PARTIAL FUNCTIONS

PHILLIP ZENOR

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Abstract. We characterize those Hausdorff spaces in which continuous functions defined on compact subsets can be continuously extended to continuous functions defined on the space.

1. Introduction

$C(X)$ will denote the set of continuous real valued functions defined on the space $X$, $T_{\text{norm}}$ will denote the norm topology on $C(X)$, while $T_{co}$ will denote the compact-open topology on $C(X)$. Let $K(X)$ denote the space of compact subsets of $X$ endowed with the Vietoris topology. Each function in $C_K(X) = \{ f \in C(H) : H \in K(X) \}$ is identified with its graph so that $C_K(X)$ is a subspace of $K(X) \times \mathbb{R}$. The space $C_K(X)$ was first studied by Kuratowski in [5], [6]. In [8], the author shows that if $X$ is a compact metric space, then there is a continuous function $e : C_K(X) \rightarrow (C(X), T_{\text{norm}})$ such that $ef(x) = f(x)$ for all $x \in \text{Domain}(f)$. (We will call such a function $e$ an extender.) In [7], Stepanova defines a separating function to be a function $\varphi : X^2 \setminus \Delta \rightarrow C(X)$ such that $\varphi(x,y)(x) \neq \varphi(x,y)(y)$ and proves that the following conditions are equivalent for a paracompact p-space $X$:

1. $X$ admits a continuous separating function $\varphi : X^2 \setminus \Delta \rightarrow (C(X), T_{\text{norm}})$,
2. there is a continuous extender $e : C_K(X) \rightarrow (C(X), T_{\text{norm}})$, and
3. $X$ is metrizable.

(Paracompact p-spaces can be characterized as those spaces that admit perfect maps onto metric spaces [4].)

In [4], it is shown that if $X$ is metrizable, then there is a continuous extender $e : C_K(X) \rightarrow (C(X), T_{\text{norm}})$ that is linear on functions with common domains.

Definition 1. A function $f : X \rightarrow C(Y)$ can be thought of as a function $\varphi_f : X \times Y \rightarrow \mathcal{R}$, defined by $\varphi_f(x,y) = [f(x)](y)$. We will say that $f : X \rightarrow C(Y)$ is naturally continuous if $\varphi_f : X \times Y \rightarrow \mathcal{R}$ is continuous. Unless otherwise stated, we will think of a separating function for $X$ as a function $\varphi : (X^2 \setminus \Delta) \times X \rightarrow \mathcal{R}$ such that $\varphi(x,y,x) \neq \varphi(x,y,y)$.
Proposition 2. If $y$ containing $T$ basic member of rating function $f$ the following conditions are equivalent on the Hausdorff space Theorem 1.

Proof. Suppose that $f : X 	o (C(Y), T_{co})$ is continuous, then $\varphi_f : X \times Y \to \mathcal{R}$ is continuous. We will see in Proposition 1 that if $\varphi_f : X \times Y \to \mathcal{R}$ is continuous, then $f : X \to (C(Y), T_{co})$ is continuous.

In [3], $X$ is defined to be continuously Urysohn if $X$ admits a continuous separating function $f : X^2 \setminus \Delta \to (C(X), T_{norm})$.

Definition 2. We will say that $X$ is weakly continuously Urysohn if $X$ admits a continuous separating function $\varphi : (X^2 \setminus \Delta) \times X \to \mathcal{R}$.

The main result of this paper is the following.

Theorem 1. The following conditions are equivalent on the Hausdorff space $X$:

1. $X$ is weakly continuously Urysohn,
2. there is a naturally continuous extender $e : C_K(X) \to C(X)$, and
3. there is a naturally continuous extender $e : C_K(X) \to C(X)$ such that if $\text{Domain}(f) = \text{Domain}(g)$ and $f(x) \leq g(x)$ for all $x \in \text{Domain}(f)$, then $ef(x) \leq eg(x)$ for all $x \in X$.

2. Background, constructions and proofs

If $U \subset Y$ and $V \subset \mathcal{R}$, then we will let $[U, V] = \{f \in C(Y) : f(U) \subset V\}$.

Proposition 1. A function $f : X \to (C(Y), T_{co})$ is continuous provided that the function $\varphi_f : X \times Y \to \mathcal{R}$ is continuous.

Proof. Suppose that $\varphi_f : X \times Y \to \mathcal{R}$ is continuous, $x \in X$ and $[K, W]$ is a sub-basic member of $T_{co}$ containing $f(x)$. For each $k \in K$, there are open sets $U_k$ in $X$ and $V_k$ in $Y$ such that $x \in U_k$, $k \in V_k$, and $\varphi_f(U_k \times V_k) \subset W$. There is a finite subset $\{k_1, \ldots, k_n\}$ of $K$ such that $\{V_{k_1}, \ldots, V_{k_n}\}$ covers $K$. It follows that if $x' \in \cap \{U_{k_1}, \ldots, U_{k_n}\}$, then $f(x') \in [K, W]$.

Proposition 2. If $Y$ is locally compact, then $f : X \to (C(Y), T_{co})$ is continuous if and only if $\varphi_f : X \times Y \to \mathcal{R}$ is continuous.

Proof. Suppose that $f : X \to (C(Y), T_{co})$ is continuous. Let $W$ be an open set in $\mathcal{R}$ containing $\varphi_f(x, y) = f(x)(y)$. Since $f(x) \in C(Y)$, we may let $V$ be an open set containing $y$ with compact closure such that $f(x) \in [V, W]$. Since $f$ is continuous, there is an open set $U$ in $X$ containing $x$ such that if $x' \in U$, then $f(x') \in [V, W]$. Thus, if $(x', y') \in U \times V$, then $\varphi_f(x', y') = f(x')(y') \in W$. It follows that $\varphi_f$ is continuous.

Recall that if $X$ is compact, then the compact-open topology and the norm topology on $C(X)$ are the same topologies. Thus we have

Theorem 2 (Stepanova [4]). The compact space $X$ is metrizable if and only if $X$ is weakly continuously Urysohn.

In [4], Stepanova points out that a submetrizable space admits a continuous separating function. Indeed, we have the following observation.

Proposition 3. If $X$ has a zero-set diagonal, then $X$ is weakly continuously Urysohn.
Proof. Let $\rho : X^2 \to [0, 1]$ be a continuous function such that $\Delta = \rho^{-1}(0)$. Since $\rho$ is continuous,
\[
\varphi(x, y, t) = \frac{1}{2} \left[ -\rho(y, t) + \rho(x, t)(1 - \rho(y, t)) \right] + 1
\]
is a continuous function from $(X^2 \setminus \Delta) \times X$ into $R$ such that $\varphi(x, y, x) = 0$ and $\varphi(x, y, y) = 1$.

In [2], Bennett and Lutzer give us an example of a linearly ordered space that is continuously Urysohn but that does not have a $G_\delta$-diagonal.

We let $\mathcal{M}(X) = \{(H, K) \in K(X) \times K(X) : H \cap K = \emptyset\}$ and $\mathcal{A}(X) = \{(x, K) \in X \times K(X) : x \notin K\}$.

Lemma 1 (Stepanova [2]). If $X$ admits a continuous separating function $\varphi : (X^2 \setminus \Delta) \times X \to R$, then there is a continuous function $\varphi' : (X^2 \setminus \Delta) \times X \to [0, 1]$ such that $\varphi'(x, y, x) = 0$ and $\varphi'(x, y, y) = 1$.

Lemma 2. If $X$ admits a separating continuous function $\varphi : (X^2 \setminus \Delta) \times X \to R$, then there is a continuous function $\tilde{\varphi} : \mathcal{M}(X) \times X \to [0, 1]$ such that

1. if $x \in H$, then $\tilde{\varphi}(H, K, x) = 0$, and if $x \in K$, then $\tilde{\varphi}(H, K, x) = 1$, and
2. if $(H, K)$ and $(H', K') \in \mathcal{M}(X)$ with $H \subset H'$ and $K' \subset K$, then $\tilde{\varphi}(H', K', x) \leq \tilde{\varphi}(H, K, x)$.

Proof. By Lemma 1 we may assume that $0 \leq \varphi(x, y, t) \leq 1$, $\varphi(x, y, x) = 0$, and $\varphi(x, y, y) = 1$.

Define $\varphi_1 : \mathcal{A}(X) \times X \to R$ by $\varphi(x, H, t) = \max\{\varphi(x, y, t) : y \in H\}$.

Clearly, $\varphi_1(x, H, x) = 0$. If $t \in H$, then $\varphi_1(x, H, t) = 1$, and if $H \subset K$, then $\varphi_1(x, H, t) \leq \varphi_1(x, K, t)$. We will show that $\varphi_1$ is continuous.

Let $\varepsilon > 0$ and let $W = (\varphi_1(x_0, K_0, t_0) - \varepsilon, \varphi_1(x_0, K_0, t_0) + \varepsilon)$. For each $y \in K_0$ there are open sets $O_y$, $\Lambda_y$, and $\Omega_y$ containing $x_0$, $y$, and $t_0$, respectively, such that if $x' \in O_y$, $y' \in \Lambda_y$, and $t' \in \Omega_y$, then $\varphi(x', y', t') \in (\varphi(y, x, t, y) - \varepsilon, \varphi(y, x, t, y) + \varepsilon)$. Since $K_0$ is compact, there is a finite subset $\{y_1, \ldots, y_n\}$ of $K_0$ such that $\{\Lambda_{y_1}, \ldots, \Lambda_{y_n}\}$ covers $K$. We may assume that $\varphi_1(x_0, K_0, t_0) = \varphi_1(x_0, y_i, t_0)$. Let

$O = \bigcap_{i=1}^n O_{y_i}$, $\Lambda = \bigcap_{i=1}^n \Lambda_{y_i}$, and $\Omega = \bigcap_{i=1}^n \Omega_{y_i}$. Let $(x', K', t') \in O \times \Lambda \times \Omega$. Then $\varphi(x', K', t') \geq \varphi(x', K', t') \geq \varphi(x_0, k_1, t_0) - \varepsilon = \varphi_1(x_0, y_i, t_0) - \varepsilon$. Now, let $k \in K'$ be such that $\varphi(x', k, t') = \varphi_1(x', K', t')$. Then $k \in \Lambda_{y_i}$ for some $1 \leq i \leq n$. We have $\varphi(x', k, t') \leq \varphi(x_0, y_i, t_0) + \varepsilon \leq \varphi(x_0, y_i, t_0) + \varepsilon = \varphi_1(x_0, K_0, t_0) + \varepsilon$.

We now define $\tilde{\varphi}(H, K, x) = \min\{\varphi_1(x, K, t) : x \in H\}$. We need only show that $\tilde{\varphi}$ is continuous. To this end, let $\varepsilon > 0$. For each $x \in H$, there are open sets $U_x$ in $X$ containing $x$, $V_x$ in $K(X)$ containing $K$, and $\Lambda_x$ containing $t$ such that if $(x', K', t') \in U_x \times V_x \times \Lambda_x$, $\varphi_1(x, K, t) - \varepsilon < \varphi_1(x', K', t') < \varphi_1(x, K, t) + \varepsilon$.

There is a finite set $\{x_1, \ldots, x_n\} \subset H$ such that $\{U_{x_1}, \ldots, U_{x_n}\}$ covers $H$. We may assume that $\tilde{\varphi}(H, K, x) = \varphi_1(x_1, K, t)$. Let $U = \bigcup_{i=1}^n U_{x_i}$, $V = \bigcup_{i=1}^n V_{x_i}$, and $\Lambda = \bigcup_{i=1}^n \Lambda_{x_i}$. Let $(H', K', t') \in U \times V \times \Lambda$. There is an $x' \in H' \cap U_{x_1}$, and so, $\tilde{\varphi}(H', K', t') \leq \varphi_1(x', K', t') < \varphi_1(x_1, K, t) + \varepsilon = \varphi_1(H, K, t) + \varepsilon$. Now, let $x' \in H'$ be such that $\tilde{\varphi}(H', K', t') = \varphi_1(x', H', t')$. Then $x' \in U_{x_j}$ for some $j$. Thus, $\tilde{\varphi}(H', K', t') = \varphi_1(x', K', t') > \varphi_1(x_j, K, t) - \varepsilon \geq \tilde{\varphi}(H, K, t) - \varepsilon$. 
Observation 1. It follows from Part 2 of Lemma 2 that if \((H, K, t) \in \mathcal{M}(X) \times X\), and \(\varepsilon > 0\), then

1. there are open sets \(U \times V\) in \(\mathcal{M}(X)\) containing \((H, K)\) and \(W\) containing \(t\) such that if \(\{(A, B), (A', B')\} \subset \mathcal{M}(X)\) with \(A' \subset A\), \(A' \subset U\), \(B' \subset B\), \(A' \subset V\), and \(x \in W\), then \(\varphi(A, B, x) < \varphi(H, K, t) + \varepsilon\)

2. there are open sets \(U' \times V'\) in \(\mathcal{M}(X)\) containing \((H, K)\) and \(W'\) containing \(t\) such that if \(\{(A, B), (A', B')\} \subset \mathcal{M}(X)\) with \(A' \subset U', B' \subset B, B' \subset V'\), and \(x \in W'\), then \(\varphi(A, B, x) > \varphi(H, K, t) - \varepsilon\).

Let \(C_\varepsilon(X)\) denote the set \(\{f : H \to (0, 1) \mid H \in K(X), f\) is continuous\} endowed with the Vietoris topology.

Lemma 3. Let \(\varphi : \mathcal{M}(X) \times X \to [0, 1]\) be the function \(\bar{\varphi}\) given in Lemma 2. We extend \(\varphi\) so that if \(H \neq \emptyset\), then \(\varphi(\emptyset, H, x) = 1\) and \(\varphi(H, \emptyset, x) = 0\) for all \(x \in X\). Suppose that \(f \in C_\varepsilon(X)\), and \(0 < a < b < 1\).

1. If \(f^{-1}[0, a] \neq \emptyset\), \(\varphi(f^{-1}[0, a], f^{-1}[b, 1], x) < \zeta\), and \(\varepsilon > 0\), then there are open sets \(\Gamma\) in \(C_\varepsilon(X)\) containing \(f\) and \(O\) in \(X\) containing \(x\) such that if \(a' < b', a' < a + \varepsilon, b' > b + \varepsilon\), \(y \in \Gamma\), and \(x' \in O\), then

\[\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') < \zeta.\]

2. If \(f^{-1}[b, 1] \neq \emptyset\), \(\varphi(f^{-1}[0, a], f^{-1}[b, 1], x) > \zeta\), and \(\varepsilon > 0\), then there are open sets \(\Gamma\) in \(C_\varepsilon(X)\) containing \(f\) and \(O\) in \(X\) containing \(x\) such that if \(a' < b', a' < a - \varepsilon, b' < b - \varepsilon\), \(y \in \Gamma\), and \(x' \in O\), then

\[\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') > \zeta.\]

Proof of Lemma 3(1). Case I: \(f^{-1}[b, 1] = \emptyset\). Let \(c = \sup(f)\) and \(\delta < \frac{1}{2}\min\{\varepsilon, b - c\}\). Then \(\varphi(f^{-1}[0, c], f^{-1}[b, 1], x) = 0\) for all \(x \in X\). Let \(\langle \gamma_1, \ldots, \gamma_k \rangle\) be an open set in \(K(X)\) containing the domain of \(f\) such that \(f^{-1}[0, a] \cap \gamma_1 \neq \emptyset\) and such that if \(\langle y, z \rangle \subset \gamma_1 \cap \text{Domain}(f)\), then \(|f(y) - f(z)| < \delta\). For each \(i \leq k\), choose \(x_i \in \gamma_i \cap \text{Domain}(f)\) such that \(x_i \in f^{-1}[0, a]\) and let \(\gamma_i^* = \gamma_i \times (f(x_i) - \delta, f(x_i) + \delta)\). Then \(\Gamma = \langle \gamma_1^*, \ldots, \gamma_k^* \rangle\) is an open set in \(C_\varepsilon(X)\) containing \(f\).

Let \(g \in \Gamma\). We will show that \(\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') = 0\) for all \(x' \in X\). First, \(g^{-1}[b', 1] = \emptyset\). To see that this is true, let \(\langle y, g(y) \rangle \in \gamma_i^*\). Then \(g(y) < f(x_i) + \delta \leq c + \delta < b + \varepsilon < b'\). Now, we show that \(g^{-1}[0, a'] \neq \emptyset\). To this end, let \(\langle y, g(y) \rangle \in \gamma_i^*\). Then \(g(y) < f(x_i) + \delta \leq a + \delta < a'\) and \(g^{-1}[0, a'] \neq \emptyset\). We have that \(\varphi(g^{-1}[0, a'], [b', 1], x') = 0\) for all \(x \in X\). Case I follows.

Case II: \(f^{-1}[b, 1] \neq \emptyset\). According to Observation 1, since \(f^{-1}[0, a] \cap f^{-1}[b, 1] = \emptyset\), there are mutually exclusive open sets in \(K(X)\), \(A = \langle \alpha_1, \ldots, \alpha_n \rangle\) containing \(f^{-1}[0, a]\) and \(B = \langle \beta_1, \ldots, \beta_n \rangle\) containing \(f^{-1}[b, 1]\), and \(O\), open in \(X\) containing \(x\), such that if \((A', B', x') \in A \times B \times O\), \((A, B) \in \mathcal{M}(X)\), and \(A' \subset A\), and \(B \subset B'\), then \(\varphi(A, B, x') < \zeta\).

Let \(\delta > 0\) be such that \(2\delta < \varepsilon\), \(f^{-1}[0, a + 2\delta] \in A\), and \(f^{-1}[b - 2\delta, 1] \in B\). Choose open sets in \(X\), \(\langle \gamma_1, \ldots, \gamma_k \rangle\), such that (1) \(\text{Domain}(f) \in \langle \gamma_1, \ldots, \gamma_k \rangle\), (2) \(f^{-1}[0, a] \subset \langle \gamma_1, \ldots, \gamma_k \rangle\), (3) \(f^{-1}[b, 1] \subset \langle \gamma_1, \ldots, \gamma_k \rangle\), and (4) if \(t_1\) and \(t_2\) are in \(\text{Domain}(f) \cap \gamma_i\), then \(|f(t_1) - f(t_2)| < \delta/2\).

Let \(g \in \Gamma\). For each \(i \leq k\), choose \(x_i\) in \(\text{Domain}(f) \cap \gamma_i\) such that if \(i \leq k_1\), then \(f(x_i) \leq a\) and if \(i \geq k_2\), then \(f(x_i) \geq b\), and let \(\gamma_i^* = \gamma_i \times (f(x_i) - \delta, f(x_i) + \delta)\). Then \(\Gamma = \langle \gamma_1^*, \ldots, \gamma_k^* \rangle\) is an open set in \(C_\varepsilon(X)\) containing \(f\). For each \(i \leq k_1\),
let \( y_i \in \text{Domain}(g) \) such that \((y_i, g(y_i)) \in \gamma_i^*\) and let \( A' = \{ y_1, \ldots, y_k \} \). Then \( g(y_i) < f(x_i) + \delta < a' \). Thus, \( A' \subset g^{-1}[0, a'] \) and \( A' \in \mathcal{A} \). If \( g^{-1}[b', 1] = \emptyset \), then 
\[
\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') = 0
\]
for all \( x' \in X \), and we are done. So let \( y \in g^{-1}[b', 1] \).

There is an \( i \) such that \((y, g(y)) \in \gamma_i^*\). Since \( f(x_i) > g(y) - \delta \geq b' - \delta > b \), \( i \geq k_2 \),
and there is an integer \( j \) such that \( y \in \beta_j \). Therefore, if we let \( B' = \{ x_{k_2}, \ldots, x_k \} \cup g^{-1}[b', 1] \), then \( g^{-1}[b', 1] \subset B' \in \mathcal{B} \). We have that if \( g \in \Gamma \) and \( x' \in O \), then
\[
\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') < \zeta.
\]

The proof of Lemma 3(2) follows in the same way.

A construction: Let \( \varphi : \mathcal{M}(X) \times X \to [0, 1] \) be as given in Lemma 2. For convience, if \( H \neq \emptyset \), then we define \( \varphi(\emptyset, H, x) = 1 \), and \( \varphi(H, \emptyset, x) = 0 \). For \( f \in \mathcal{C}_K(X) \), \( z \in [0, 1] \), and \( n \in \mathcal{N} \), define

- \( f_{n,z}(x) = \begin{cases} \varphi(f^{-1}[0, z], f^{-1}[z + 1/n, 1], x) & \text{if } f^{-1}[0, z] \neq \emptyset, \\ 1 & \text{otherwise}, \end{cases} \)
- \( \overline{f}_{n,z}(x) = \begin{cases} \varphi(f^{-1}[0, z - 1/n], f^{-1}[z, 1], x) & \text{if } f^{-1}[z, 1] \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases} \)
- \( U_{f,n,z} = \{ x \mid f_{n,z}(x) < z/n \} \), and 
- \( V_{f,n,z} = \{ x \mid \overline{f}_{n,z}(x) > 1 - 1/n + z/n \} \).

We define \( W_{f,z} = \bigcup_{n \in \mathcal{N}} (U_{f,n,z} \setminus \overline{f}_{f,z}) \). Then \( W_{f,z} \) is an open set containing \( f^{-1}[0, z] \) and \( \overline{f}_{f,z} \cap f^{-1}(z, 1] = \emptyset \).

Note that (1) if \( z_1 < z_2 \), then \( f_{n,z_1}(x) \leq f_{n,z_2}(x) \) and \( \overline{f}_{n,z_2}(x) \leq \overline{f}_{n,z_1}(x) \). Thus, \( \overline{U}_{f,n,z_2} \subset U_{f,n,z_2} \) and \( \overline{V}_{f,n,z_2} \subset V_{f,n,z_2} \). (2) If \( z \in [0, 1] \), then \( \overline{U}_{f,n+1,z} \subset U_{f,n,z} \) and \( \overline{V}_{f,n+1,z} \subset V_{f,n,z} \). (3) If \( z > \sup(f) \), then \( W_{f,z} = X \). (4) If \( z < \inf(f) \), then \( W_{f,z} = \emptyset \).

Lemma 4. Suppose that \( 0 < z_1 < z_2 < 1 \) and \( 1/N < (z_2 - z_1)/3 \). Then \( V_{f,z} \cap U_{f,N,z_1} = \emptyset \).

Proof. We are given that \( z_1 + 1/N < z_2 < 1/N \). If \( f^{-1}[0, z_1] = \emptyset \), then \( U_{f,N,z_2} = \emptyset \). If \( f^{-1}[z_2, 1] = \emptyset \), then \( V_{f,N,z_2} = \emptyset \). If \( f^{-1}[0, z_1] \neq \emptyset \) and \( f^{-1}[z_2, 1] \neq \emptyset \), then \( \{ \eta \mid \varphi(f^{-1}[0, z_1], f^{-1}[z_1 + 1/N, 1], \eta < 1 \} \cup \{ \eta \mid \varphi(f^{-1}[0, z_2 - 1/N], f^{-1}[z_2, 1], \eta) = 1 \} \subset \{ \eta \mid \varphi(f^{-1}[0, z_2 - 1/N], f^{-1}[z_2, 1], \eta) > 1 - 1/N + z_1/N \} \).

Thus, \( \{ \eta \mid \varphi(f^{-1}[0, z_1], f^{-1}[z_1 + 1/N, 1], \eta < z_1/N \} \cap \{ \eta \mid \varphi(f^{-1}[0, z_2 - 1/N], f^{-1}[z_2, 1], \eta) > 1 - 1/N + z_1/N \} = \emptyset \).

Lemma 5. If \( z_1 < z_2 \), then \( \overline{W}_{f,z_1} \subset W_{f,z_2} \).

Proof. Suppose that \( x \in \overline{W}_{f,z_1} \setminus W_{f,z_2} \). By Lemma 3 we may let \( N \) denote the first integer such that \( x \notin U_{f,N,z_1} \cap V_{f,N,z_2} \). Since \( x \in \overline{U}_{f,z_1} \), \( x \in \overline{U}_{f,z_1} \), and \( f^{-1}[0, z_1] \neq \emptyset \). But, \( f^{-1}[z_1 + 1, 1] = \emptyset \); so, \( U_{f,z_1} = X \). Since \( x \in \overline{U}_{f,z_1} \subset U_{f,z_1} \) and \( x \notin W_{f,z_2} \), it must be that \( x \in V_{f,z_2} \) and \( f^{-1}[z_2, 1] \neq \emptyset \). Since \( f^{-1}[0, z_2 - 1] = \emptyset \), \( V_{f,z_1} = X \). Thus, \( N > 1 \).

If \( x \in \overline{V}_{f,N,z_1} \), then \( x \notin U_{f,N,z_1} \), \( x \in V_{f,N,z_1} \), and \( x \notin \overline{U}_{f,N+1,z_1} \). Then \( x \in (V_{f,N,z_1} \setminus \overline{U}_{f,N+1,z_1}) \) and \((V_{f,N,z_1} \setminus \overline{U}_{f,N+1,z_1}) \cap W_{f,z_1} = \emptyset \), which would be a contradiction. So, \( x \notin \overline{V}_{f,N,z_1} \). Since \( x \notin W_{f,z_2} \), \( x \notin U_{f,N,z_2} \), and so, \( x \notin \overline{U}_{f,N,z_2} \).

But then \( x \in (V_{f,N-1,z_1} \setminus \overline{U}_{f,N,z_1}) \cap W_{f,z_1} = \emptyset \), which is a contradiction from which the lemma follows.
Lemma 6. If $X$ is weakly continuously Urysohn, then there is a continuous function $e : C^*_K(X) \times X \to [0, 1]$ such that

1. $ef(x) = f(x)$ for all $x \in \text{Domain}(f)$, and
2. if $f$ and $g$ have a common domain and if $f(x) \leq g(x)$ for all $x \in \text{Domain}(f)$, then $ef(x) \leq eg(x)$ for all $x \in X$.

Proof. For $f \in C^*_K(X)$ and $x \in X$, define $ef(x) = \text{glb}\{z \in [0, 1] \mid x \in W_{f,z}\}$. It follows from the construction that $ef : X \to [0, 1]$ is an extension of $f$ such that if $f$ and $g$ share a common domain, then $f \leq g$ implies that $ef \leq eg$. Furthermore, the image of $ef$ is contained in the convex hull of the image of $f$. It is a standard argument that $ef$ is continuous. It remains to show that $e : (C^*_K(X)) \times X \to [0, 1]$ is continuous. To this end, let $f \in C^*_K(X)$, $x \in X$, and $\varepsilon > 0$. Let $z = ef(x)$. We will obtain an open set $\Gamma \times O$ in $C^*_K(X) \times X$ containing $(f, x)$ such that if $(g, x') \in \Gamma \times O$, then $|ef(x) - eg(x')| < \varepsilon$.

**Part I:** First, we obtain an open set $\Gamma_1 \times O_1$ in $C^*_K(X) \times X$ containing $(f, x)$ such that if $(g, x') \in \Gamma_1 \times O_1$ and $z_0 > z'' + \varepsilon = z + \varepsilon$, then $x' \in W_{g,z_0}$. To this end, choose $z''$ and $z'''$ in $[0, 1]$ such that $z < z' < z'' < z + \varepsilon$.

Since $x \in W_{f,z'}$, there is an $N > 0$ such that $x \in U_{f,N,z'} \setminus W_{f,N,z''}$. Let $\varepsilon' = \varepsilon + z - z''$. Since $V_{f,N,z'} \neq X$, $f^{-1}[0, z' - 1/N] \neq \emptyset$ and $f^{-1}[0, z'' - 1/N] \neq \emptyset$. Let

$$\varphi(f^{-1}[0, z'' - 1/N], f^{-1}[z''', 1], x) \leq \varphi(f^{-1}[0, z'' - 1/N], f^{-1}[z', 1], x) \leq 1 - 1/N + z'/N < 1 - 1/N + z''/N.$$ 

By Lemma 3, there is an open set $\Lambda_1 \times W_1$ in $C^*_K(X) \times X$ containing $(f, x)$ such that if $(g, x') \in \Lambda_1 \times W_1$ and $z_0 > z'' + \varepsilon = z + \varepsilon$, then $\varphi(g^{-1}[0, z_0 - 1/N], g^{-1}[z_0, 1], x') < 1 - 1/N + z'/N < 1 - 1/N + z''/N$.

**Step A:** Let $\varepsilon' = \varepsilon + z - z''$. Since $V_{f,N,z'} \neq X$, $f^{-1}[0, z'' - 1/N] \neq \emptyset$. Since $\varphi(f^{-1}[0, z'], f^{-1}[z'' + 1/N, 1], x) < 1/N$, we may employ Lemma 2 to obtain an open set $\Lambda_2 \times U_2$ in $C^*_K(X) \times X$ containing $(f, x)$ such that if $z_0 > z'' + \varepsilon = z + \varepsilon$ and $(g, x') \in \Lambda_2 \times U_2$, then $\varphi(g^{-1}[0, z_0], g^{-1}[z_0 + 1/N, 1], x') < 1/N$. Thus, we can choose $x' \in U_{g,N,z_0}$ to complete the argument for Part I, i.e., $\Lambda_1 \cap \Lambda_2$ and $O_1 = U_1 \cap U_2$.

**Part II:** Second, we find an open set $\Gamma_2 \times O_2$ in $C^*_K(X) \times X$ containing $(f, x)$ such that if $(g, x') \in \Gamma_2 \times O_2$, then $eg(x') \geq z - \varepsilon$; that is, if $z_0 < z - \varepsilon$, then $x' \notin W_{g,z_0}$. We consider three cases.

**Case A:** $f^{-1}[0, z - \varepsilon] = \emptyset$. Choose $z'$ so that $z - \varepsilon < z' < z$ and $f^{-1}[0, z'] = \emptyset$. Set $2\varepsilon' = z' - (z - \varepsilon)$. Let $\{U_1 \ldots U_n\}$ be an irreducible open cover of $f^{-1}[z', 1] = \text{Domain}(f)$ such that if $1 \leq i \leq n$ and $\{a_i, b_i\} \subset \text{Domain}(f) \cap U_i$, then $|f(a) - f(b)| < \varepsilon'$. For each $1 \leq i \leq n$, choose $x_i \in U_i \cap \text{Domain}(f)$ and let $U_i^* = U_i \times (x_i - \varepsilon', x_i + \varepsilon')$. Then $\Gamma_2 = \langle U_1^*, \ldots, U_n^* \rangle$ is an open set containing $f$ such that if $g \in \Gamma_2$, $z_0 < z' - \varepsilon' = z - \varepsilon$, and $x' \in X$, then $g^{-1}[0, z_0] = \emptyset$. Letting $O_2 = X$, we have an open set $\Gamma_2 \times O_2$ in $C^*_K(X) \times X$ containing $(f, x)$ such that if $(g, x') \in \Gamma_2 \times O_2$, then $eg(x') \geq z - \varepsilon$.

In both Cases B and C, we will choose $z''$ and $z'''$ in $(0, 1)$ such that $z > z' > z'' > z''' > z - \varepsilon$. Lemma 4 gives us a first integer $N$ such that $x \notin U_{f,N,z'''} \cap V_{f,N,z'}$.

**Case B:** $f^{-1}[0, z - \varepsilon] \neq \emptyset$ and $x \in V_{f,N,z'''}$. Then $x \notin U_{f,N,z'''}$ so

$$\varphi(f^{-1}[0, z'''], f^{-1}[z'''' + 1/N, 1], x) \geq z''''/N.$$
and $f^{-1}[z'' + 1/N, 1] \neq \emptyset$. It follows that $f^{-1}[z'' + 1/N, 1] \neq \emptyset$ and

$$\varphi(f^{-1}[0, z''], f^{-1}[z'' + 1/N, 1], x) > z''/N > z''/N.$$  

We let $\epsilon' = z'' - (z - \varepsilon)$ and apply Lemma \[\text{K}\] to obtain an open set $\Omega_1 \times V_1$ in $C^*_K(X) \times X$ containing $(f, x)$ such that if $(g, x') \in \Omega_1 \times V_1$ and $z_0 < z'' - \epsilon' = z - \varepsilon$, then

$$\varphi(g^{-1}[0, z_0], g^{-1}[z_0 + 1/N, 1], x') > z''/N > z_0/N.$$  

Now, since $x \in \overline{V}_{f,N,z''}, x \in V_{f,N,z''}$ and $\varphi(f^{-1}[0, z'', 1/N], f^{-1}[z'' + 1/N, 1], x) > 1 - 1/N + z''/N$. Letting $\epsilon' = z'' - (z - \varepsilon)$, we apply Lemma \[\text{K}\] to obtain an open set $\Omega_2 \times V_2 \in C^*_K(X) \times X$ containing $(f, x)$ such that if $(g, x') \in C^*_K(X) \times X$ and $z_0 < z'' = z'' = z - \varepsilon$, then $x' \notin U_{g_1, z_0}$ for all $i \geq N$.

To complete Case B, we let $\Gamma_2 = \Omega_1 \cap \Omega_2$ and $O_2 = V_1 \cap V_2$.

**Case C:** $f^{-1}[0, z - \varepsilon] \neq \emptyset$ and $x \notin \overline{V}_{f,N,z''}$. Then since $x \notin W_{f, z''}$, $x \notin U_{f,N,z''}$. Thus, $\varphi(f^{-1}[0, z''], f^{-1}[z'' + 1/N, 1], x) > z''/N$. It follows that $f^{-1}[z'' + 1/N, 1] \neq \emptyset$ and $\varphi(f^{-1}[0, z''], f^{-1}[z'' + 1/N, x]) > z''/N > z''/N$. Again, Lemma \[\text{K}\] gives us an open set $\Omega_1 \times V_1 \in C^*_K(X)$ such that if $(g, x') \in \Omega_1 \times V_1$, then $x' \notin U_{g_1, z_0}$.

Since $f^{-1}[0, z - \varepsilon] \neq \emptyset$, $f^{-1}[0, z''] \neq \emptyset$, and $U_{f_1, z''} = X$. Similarly, since $f^{-1}[z'' + 1/N, 1] \neq \emptyset$, $V_{f_1, z''} = X$. Therefore, $N > 1$. By hypothesis, $x' \subseteq V_{f,N^{-1},z''} \subseteq V_{f,N-1,z''}$. Letting $\epsilon' = z'' - (z - \varepsilon)$, we employ Lemma \[\text{K}\] to obtain an open set $\Omega_2 \times V_2$ in $C^*_K(X) \times X$ such that if $(g, x') \in \Omega_2 \times V_2$ and $z_0 < z'' - \epsilon' = z - \varepsilon$, then $x' \notin W_{g_2, z_0}$.

To complete Case C, we let $\Gamma_2 = \Omega_1 \cap \Omega_2$ and $O_2 = V_1 \cap V_2$. To complete the proof of the lemma, we let $\Gamma = \Gamma_1 \cap \Gamma_2$ and $O = O_1 \cap O_2$.

We are now in a position to obtain our main result.

**Theorem 1.** If $X$ is a Hausdorff space, then the following conditions are equivalent:

1. $X$ is weakly continuously Urysohn.
2. There is a continuous extender $e : C_K(X) \times X \to R$.
3. There is a continuous extender $e : C_K(X) \times X \to R$ such that if $f$ and $g$ have a common domain and if $f(x) \leq g(x)$ for all $x \in \text{Domain}(f)$, then $e f(x) \leq e g(x)$ for all $x \in X$.

**Proof.** Clearly, (3) $\Rightarrow$ (2). We will use a standard argument to show that (1) $\Rightarrow$ (3). Let $e' : C^*_K(X) \times X \to [0, 1]$ be the extender given in Lemma \[\text{K}\]. Let $h : R \to (0, 1)$ be an order preserving homeomorphism taking $R$ onto $(0, 1)$. For $f \in C_K(X)$, define $e f = h^{-1}e h \circ h \circ f$. To obtain (2) $\Rightarrow$ (1), for each $(x, y) \in X^2 \setminus \Delta$, let $\psi_{(x,y)} : \{x, y\} \to [0, 1]$ be defined by $\psi_{(x,y)}(x) = 0$ and $\psi_{(x,y)}(y) = 1$. Let $\varphi(x, y, t) = e \psi_{(x,y)}(t)$.

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**References**


