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CONTINUOUSLY EXTENDING PARTIAL FUNCTIONS

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Abstract. We characterize those Hausdorff spaces in which continuous functions defined on compact subsets can be continuously extended to continuous functions defined on the space.

1. Introduction

$C(X)$ will denote the set of continuous real valued functions defined on the space X. $\mathcal{T}_{norm}$ will denote the norm topology on $C(X)$, while $\mathcal{T}_{co}$ will denote the compact-open topology on $C(X)$. Let $K(X)$ denote the space of compact subsets of X endowed with the Vietoris topology.[1] Each function in $\mathcal{C}_K(X) = \{f \in C(H) : H \in K(X)\}$ is identified with its graph so that $\mathcal{C}_K(X)$ is a subspace of $K(X \times \mathcal{R})$. The space $\mathcal{C}_K(X)$ was first studied by Kuratowski in [5], [6]. In [8], the author shows that if X is a compact metric space, then there is a continuous function $e : \mathcal{C}_K(X) \to (C(X), \mathcal{T}_{norm})$ such that $ef(x) = f(x)$ for all $x \in \text{Domain}(f)$. (We will call such a function e an extender.) In [7], Stepanova defines a separating function to be a function $\varphi : X^2 \setminus \Delta \to C(X)$ such that $\varphi(x, y)(x) \neq \varphi(x, y)(y)$ and proves that the following conditions are equivalent for a paracompact p-space X:

(1) X admits a continuous separating function $\varphi : X^2 \setminus \Delta \to (C(X), \mathcal{T}_{norm})$,
(2) there is a continuous extender $e : \mathcal{C}_K(X) \to (C(X), \mathcal{T}_{norm})$, and
(3) X is metrizable.

(Paracompact p-spaces can be characterized as those spaces that admit perfect maps onto metric spaces [1].)

In [4], it is shown that if X is metrizable, then there is a continuous extender $e : \mathcal{C}_K(X) \to (C(X), \mathcal{T}_{norm})$ that is linear on functions with common domains.

Definition 1. A function $f : X \to C(Y)$ can be thought of as a function $\varphi_f : X \times Y \to \mathcal{R}$, defined by $\varphi_f(x, y) = [f(x)](y)$. We will say that $f : X \to C(Y)$ is naturally continuous if $\varphi_f : X \times Y \to \mathcal{R}$ is continuous. Unless otherwise stated, we will think of a separating function for X as a function $\varphi : (X^2 \setminus \Delta) \times X \to \mathcal{R}$ such that $\varphi(x, y, x) \neq \varphi(x, y, y)$.

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[1] If $\{U_1, \ldots, U_n\}$ is a finite collection of open subsets of X, then the set $\langle U_1, \ldots, U_n\rangle = \{H \in K(X) : H \subset \bigcup_{i=1}^n U_i$, and if $1 \leq i \leq n$, then $H \cap U_i \neq \emptyset\}$ is a basic open set for the Vietoris topology on $K(X)$.

Clearly, if $f : X \to (C(Y), \mathcal{T}_{norm})$ is continuous, then $\varphi_f : X \times Y \to \mathcal{R}$ is continuous. We will see in Proposition 1 that if $\varphi_f : X \times Y \to \mathcal{R}$ is continuous, then $f : X \to (C(Y), \mathcal{T}_{co})$ is continuous.

In [3], X is defined to be continuously Urysohn if X admits a continuous separating function $f : X^2 \setminus \Delta \to (C(X), \mathcal{T}_{norm})$.

Definition 2. We will say that X is weakly continuously Uryshon if X admits a continuous separating function $\varphi : (X^2 \setminus \Delta) \times X \to \mathcal{R}$.

The main result of this paper is the following.

Theorem 1. *The following conditions are equivalent on the Hausdorf space X:*

(1) *X is weakly continuously Urysohn,*
(2) *there is a naturally continuous extender $e : \mathcal{C}_K(X) \to C(X)$, and*
(3) *there is a naturally continuous extender $e : \mathcal{C}_K(X) \to C(X)$ such that if Domain(f) = Domain(g) and $f(x) \leq g(x)$ for all $x \in$ Domain(f), then $ef(x) \leq eg(x)$ for all $x \in X$.*

2. Background, constructions and proofs

If $U \subset Y$ and $V \subset \mathcal{R}$, then we will let $[U, V] = \{f \in C(Y) : f(U) \subset V\}$.

Proposition 1. *A function $f : X \to (C(Y), \mathcal{T}_{co})$ is continuous provided that the function $\varphi_f : X \times Y \to \mathcal{R}$ is continuous.*

Proof. Suppose that $\varphi_f : X \times Y \to \mathcal{R}$ is continuous, $x \in X$ and $[K, W]$ is a subbasic member of $\mathcal{T}_{co}$ containing $f(x)$. For each $k \in K$, there are open sets U_k in X and V_k in Y such that $x \in U_k$, $k \in V_k$, and $\varphi_f(U_k \times V_k) \subset W$. There is a finite subset $\{k_1, \ldots, k_n\}$ of K such that $\{V_{k_1}, \ldots, V_{k_n}\}$ covers K. It follows that if $x' \in \cap\{U_{k_1}, \ldots, U_{k_n}\}$, then $f(x') \in [K, W]$.

Proposition 2. *If Y is locally compact, then $f : X \to (C(Y), \mathcal{T}_{co})$ is continuous if and only if $\varphi_f : X \times Y \to \mathcal{R}$ is continuous.*

Proof. Suppose that $f : X \to (C(Y), \mathcal{T}_{co})$ is continuous. Let W be an open set in $\mathcal{R}$ containing $\varphi_f(x, y) = f(x)(y)$. Since $f(x) \in C(Y)$, we may let V be an open set containing y with compact closure such that $f(x) \in [\overline{V}, W]$. Since f is continuous, there is an open set U in X containing x such that if $x' \in U$, then $f(x') \in [\overline{V}, W]$. Thus, if $(x', y') \in U \times V$, then $\varphi_f(x', y') = f(x')(y') \in W$. It follows that φ_f is continuous.

Recall that if X is compact, then the compact-open topology and the norm topology on $C(X)$ are the same topologies. Thus we have

Theorem 2 (Stepanova [7])**.** *The compact space X is metrizable if and only if X is weakly continuously Urysohn.*

In [7], Stepanova points out that a submetrizable space admits a continuous separating function. Indeed, we have the following observation.

Proposition 3. *If X has a zero-set diagonal, then X is weakly continuously Urysohn.*

Proof. Let $\rho : X^2 \to [0,1]$ be a continuous function such that $\Delta = \rho^{-1}(0)$. Since ρ is continuous,

$$\varphi(x,y,t) = \frac{1}{2}\left[\frac{-\rho(y,t) + \rho(x,t)(1-\rho(y,t))}{\rho(x,t)+\rho(y,t)} + 1\right]$$

is a continuous function from $(X^2 \setminus \Delta) \times X$ into $\mathcal{R}$ such that $\varphi(x,y,x) = 0$ and $\varphi(x,y,y) = 1$.

In [2], Bennett and Lutzer give us an example of a linearly ordered space that is continuously Uryshon but that does not have a G_δ-diagonal.

We let $\mathcal{M}(X) = \{(H,K) \in K(X) \times K(X) : H \cap K = \emptyset\}$ and $\mathcal{A}(X) = \{(x,K) \in X \times K(X) : x \notin K\}$.

Lemma 1 (Stepanova [7])**.** *If X admits a continuous separating function $\varphi : (X^2 \setminus \Delta) \times X \to \mathcal{R}$, then there is a continuous function $\varphi' : (X^2 \setminus \Delta) \times X \to [0,1]$ such that $\varphi'(x,y,x) = 0$ and $\varphi'(x,y,y) = 1$.*

Lemma 2. *If X admits a continuous separating function $\varphi : (X^2 \setminus \Delta) \times X \to \mathcal{R}$, then there is a continuous function $\tilde{\varphi} : \mathcal{M}(X) \times X \to [0,1]$ such that*

(1) *if $x \in H$, then $\tilde{\varphi}(H,K,x) = 0$, and if $x \in K$, then $\tilde{\varphi}(H,K,x) = 1$, and*
(2) *if (H,K) and $(H',K') \in \mathcal{M}(X)$ with $H \subset H'$ and $K' \subset K$, then $\tilde{\varphi}(H',K',x) \le \tilde{\varphi}(H,K,x)$.*

Proof. By Lemma 1, we may assume that $0 \le \varphi(x,y,t) \le 1$, $\varphi(x,y,x) = 0$, and $\varphi(x,y,y) = 1$.

Define $\varphi_1 : \mathcal{A}(X) \times X \to \mathcal{R}$ by $\varphi_1(x,H,t) = \max\{\varphi(x,y,t) : y \in H\}$.

Clearly, $\varphi_1(x,H,x) = 0$. If $t \in H$, then $\varphi_1(x,H,t) = 1$, and if $H \subset K$, then $\varphi_1(x,H,t) \le \varphi_1(x,K,t)$. We will show that φ_1 is continuous.

Let $\varepsilon > 0$ and let $W = (\varphi_1(x_0,K_0,t_0) - \varepsilon, \varphi_1(x_0,K_0,t_0) + \varepsilon)$. For each $y \in K_0$ there are open sets O_y, Λ_y, and Ω_y containing x_0, y, and t_0, respectively, such that if $x' \in O_y$, $y' \in \Lambda_y$, and $t' \in \Omega_y$, then $\varphi(x',y',t') \in (\varphi(x_0,y,t_0) - \varepsilon, \varphi(x_0,y,t_0) + \varepsilon)$. Since K_0 is compact, there is a finite subset $\{y_1,\dots,y_n\}$ of K_0 such that $\{\Lambda_{y_1},\dots\Lambda_{y_n}\}$ covers K. We may assume that $\varphi_1(x_0,K_0,t_0) = \varphi(x_0,y_1,t_0)$. Let $O = \bigcap_{i=1}^n O_{y_i}$, $\Lambda = \langle \Lambda_{y_1},\dots,\Lambda_{y_n}\rangle$, and $\Omega = \bigcap_{i=1}^n \Omega_{y_i}$. Let $(x',K',t') \in O \times \Lambda \times \Omega$. Let $k \in K' \cap \Lambda_{y_1}$. Then $\varphi_1(x',K',t') \ge \varphi(x',k,t') \ge \varphi(x_0,k_1,t_0) - \varepsilon = \varphi_1(x_0,y_0,t_0) - \varepsilon$. Now, let $k \in K'$ be such that $\varphi(x',k,t') = \varphi_1(x',K',t')$. Then $k \in \Lambda_{y_i}$ for some $1 \le i \le n$. We have $\varphi(x',k,t') \le \varphi(x_0,y_i,t_0) + \varepsilon \le \varphi(x_0,y_1,t_0) + \varepsilon = \varphi_1(x_0,K_0,t_0) + \varepsilon$.

We now define $\tilde{\varphi}(H,K,t) = \min\{\varphi_1(x,K,t) : x \in H\}$. We need only show that $\tilde{\varphi}$ is continuous. To this end, let $\varepsilon > 0$. For each $x \in H$, there are open sets U_x in X containing x, V_x in $K(X)$ containing K, and Λ_x containing t such that if $(x',K',t') \in U_x \times V_x \times \Lambda_x$, $\varphi_1(x,K,t) - \epsilon < \varphi_1(x',K',t') < \varphi_1(x,K,t) + \varepsilon$. There is a finite set $\{x_1,\dots,x_n\} \subset H$ such that $\{U_{x_1},\dots,U_{x_n}\}$ covers H. We may assume that $\tilde{\varphi}(H,K,t) = \varphi_1(x_1,K,t)$. Let $U = \langle U_{x_1},\dots,U_{x_n}\rangle$, $V = \bigcap_{i=1}^n V_{x_i}$, and $\Lambda = \bigcap_{i=1}^n \Lambda_{x_i}$. Let $(H',K',t') \in U \times V \times \Lambda$. There is an $x' \in H' \cap U_{x_1}$, and so, $\tilde{\varphi}(H',K',t') \le \varphi_1(x',K',t') < \varphi_1(x_1,K,t) + \varepsilon = \varphi_1(H,K,t) + \varepsilon$. Now, let $x' \in H'$ be such that $\tilde{\varphi}(H',K',t') = \varphi_1(x',H',t')$. Then $x' \in U_{x_j}$ for some j. Thus, $\tilde{\varphi}(H',K',t') = \varphi_1(x',K',t') > \varphi_1(x_j,K,t) - \varepsilon \ge \tilde{\varphi}(H,K,t) - \varepsilon$.

Observation 1. It follows from Part 2 of Lemma 2 that if $(H, K, t) \in \mathcal{M}(X) \times X$, and $\varepsilon > 0$, then

(1) there are open sets $U \times V$ in $\mathcal{M}(X)$ containing (H, K) and W containing t such that if $\{(A, B), (A', B')\} \subset \mathcal{M}(X)$ with $A' \subset A,\ A' \in U,\ B \subset B' \in V$, and $x \in W$, then $\varphi(A, B, x) < \varphi(H, K, t) + \varepsilon$ and
(2) there are open sets $U' \times V'$ in $\mathcal{M}(X)$ containing (H, K) and W' containing t such that if $\{(A, B), (A', B')\} \subset \mathcal{M}(X)$ with $A \subset A' \in U',\ B' \subset B,\ B' \in V'$ and $x \in W'$, then $\varphi(A, B, x) > \varphi(H, K, t) - \varepsilon$.

Let $\mathcal{C}_K^\star(X)$ denote the set $\{f : H \to (0, 1) \,\big|\, H \in K(X), f \text{ is continuous}\}$ endowed with the Vietoris topology.

Lemma 3. *Let $\varphi : \mathcal{M}(X) \times X \to [0, 1]$ be the function $\tilde{\varphi}$ given in Lemma 2. We extend φ so that if $H \neq \emptyset$, then $\varphi(\emptyset, H, x) = 1$ and $\varphi(H, \emptyset, x) = 0$ for all $x \in X$. Suppose that $f \in \mathcal{C}_K^\star(X)$, and $0 < a < b < 1$.*

(1) *If $f^{-1}[0, a] \neq \emptyset,\ \varphi(f^{-1}[0, a], f^{-1}[b, 1], x) < \zeta$, and $\varepsilon > 0$, then there are open sets Γ in $\mathcal{C}_K^\star(X)$ containing f and O in X containing x such that if $a' < b',\ a' > a + \varepsilon, b' > b + \varepsilon,\ g \in \Gamma$, and $x' \in O$, then*
$$\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') < \zeta.$$

(2) *If $f^{-1}[b, 1] \neq \emptyset,\ \varphi(f^{-1}[0, a], f^{-1}[b, 1], x) > \zeta$, and $\varepsilon > 0$, then there are open sets Γ in $\mathcal{C}_K^\star(X)$ containing f and O in X containing x such that if $a' < b',\ a' < a - \varepsilon, b' < b - \varepsilon,\ g \in \Gamma$, and $x' \in O$, then*
$$\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') > \zeta.$$

Proof of Lemma 3(1). Case I: $f^{-1}[b, 1] = \emptyset$. Let $c = \sup(f)$ and $\delta < \frac{1}{2}\min\{\varepsilon, b - c\}$. Then $\varphi(f^{-1}[0, c], f^{-1}[b, 1], x) = 0$ for all $x \in X$. Let $\langle \gamma_1, \ldots, \gamma_k \rangle$ be an open set in $K(X)$ containing the domain of f such that $f^{-1}[0, a] \cap \gamma_1 \neq \emptyset$ and such that if $\{y, z\} \subset \gamma_i \cap \text{Domain}(f)$, then $|f(y) - f(z)| < \delta$. For each $i \leq k$, choose $x_i \in \gamma_i \cap \text{Domain}(f)$ such that $x_1 \in f^{-1}[0, a]$ and let $\gamma_i^\star = \gamma_i \times (f(x_i) - \delta, f(x_i) + \delta)$. $\Gamma = \langle \gamma_1^\star, \ldots, \gamma_k^\star \rangle$ is an open set in $\mathcal{C}_K(X)$ containing f.

Let $g \in \Gamma$. We will show that $\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') = 0$ for all $x' \in X$. First, $g^{-1}[b', 1] = \emptyset$. To see that this is true, let $(y, g(y)) \in \gamma_i^\star$. Then $g(y) < f(x_i) + \delta \leq c + \delta < b + \varepsilon < b'$. Now, we show that $g^{-1}[0, a'] \neq \emptyset$. To this end, let $(y, g(y)) \in \gamma_1^\star$. Then $g(y) < f(x_1) + \delta \leq a + \delta < a'$ and $g^{-1}[0, a'] \neq \emptyset$. We have that $\varphi(g^{-1}[0, a'], [b', 1], x') = 0$ for all $x \in X$. Case I follows.

Case II: $f^{-1}[b, 1] \neq \emptyset$. According to Observation 1, since $f^{-1}[0, a] \cap f^{-1}[b, 1] = \emptyset$, there are mutually exclusive open sets in $K(X)$, $\mathcal{A} = \langle \alpha_1, \ldots, \alpha_n \rangle$ containing $f^{-1}[0, a]$ and $\mathcal{B} = \langle \beta_1, \ldots, \beta_n \rangle$ containing $f^{-1}[b, 1]$, and O, open in X containing x, such that if $(A', B', x') \in \mathcal{A} \times \mathcal{B} \times O,\ (A, B) \in \mathcal{M}(X)$, and $A' \subset A$, and $B \subset B'$, then $\varphi(A, B, x') < \zeta$.

Let $\delta > 0$ be such that $2\delta < \varepsilon,\ f^{-1}[0, a + 2\delta] \in \mathcal{A}$, and $f^{-1}[b - 2\delta, 1] \in \mathcal{B}$. Choose open sets in X, $\{\gamma_1, \ldots, \gamma_{k_1}, \ldots, \gamma_{k_2}, \ldots, \gamma_k\}$, such that (1) $\text{Domain}(f) \in \langle \gamma_1, \ldots, \gamma_k \rangle$, (2) $f^{-1}[0, a] \in \langle \gamma_1, \ldots, \gamma_{k_1} \rangle \subset \mathcal{A}$, (3) $f^{-1}[b, 1] \in \langle \gamma_{k_2}, \ldots, \gamma_{k_n} \rangle \subset \mathcal{B}$, and (4) if t_1 and t_2 are in $\text{Domain}(f) \cap \gamma_i$, then $|f(t_1) - f(t_2)| < \delta/2$.

Let $g \in \Gamma$. For each $i \leq k$, choose x_i in $\text{Domain}(f) \cap \gamma_i$ such that if $i \leq k_1$, then $f(x_i) \leq a$ and if $i \geq k_2$, then $f(x_i) \geq b$, and let $\gamma_i^\star = \gamma_i \times (f(x_i) - \delta, f(x_i) + \delta)$. Then $\Gamma = \langle \gamma_1^\star, \ldots \gamma_k^\star \rangle$ is an open set in $\mathcal{C}_K(X)$ containing f. For each $i \leq k_1$,

let $y_i \in \text{Domain}(g)$ such that $(y_i, g(y_i)) \in \gamma_i^\star$ and let $A' = \{y_1, \ldots, y_{k_1}\}$. Then $g(y_i) < f(x_i) + \delta < a'$. Thus, $A' \subset g^{-1}[0, a']$ and $A' \in \mathcal{A}$. If $g^{-1}[b', 1] = \emptyset$, then $\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') = 0$ for all $x' \in X$, and we are done. So let $y \in g^{-1}[b', 1]$. There is an i such that $(y, g(y)) \in \gamma_i^\star$. Since $f(x_i) > g(y) - \delta \geq b' - \delta > b,\ i \geq k_2$, and there is an integer j such that $y \in \beta_j$. Therefore, if we let $B' = \{x_{k_2}, \ldots, x_k\} \cup g^{-1}[b', 1]$, then $g^{-1}[b', 1] \subset B' \in \mathcal{B}$. We have that if $g \in \Gamma$ and $x' \in O$, then

$$\varphi(g^{-1}[0, a'], g^{-1}[b', 1], x') < \zeta.$$

The proof of Lemma 3(2) follows in the same way.

A construction: Let $\varphi : \mathcal{M}(X) \times X \to [0, 1]$ be as given in Lemma 2. For convience, if $H \neq \emptyset$, then we define $\varphi(\emptyset, H, x) = 1$, and $\varphi(H, \emptyset, x) = 0$. For $f \in \mathcal{C}_K^\star(X),\ z \in [0, 1]$, and $n \in \mathcal{N}$, define

- $\underline{f}_{n,z}(x) = \begin{cases} \varphi(f^{-1}[0, z], f^{-1}[z + 1/n, 1], x), & \text{if } f^{-1}[0, z] \neq \emptyset, \\ 1 \text{ otherwise}, \end{cases}$
- $\overline{f}_{n,z}(x) = \begin{cases} \varphi(f^{-1}[0, z - 1/n], f^{-1}[z, 1], x), & \text{if } f^{-1}[z, 1] \neq \emptyset, \\ 0 \text{ otherwise}, \end{cases}$
- $U_{f,n,z} = \{x \,|\, \underline{f}_{n,z}(x) < z/n\}$, and
- $V_{f,n,z} = \{x \,|\, \overline{f}_{n,z}(x) > 1 - 1/n + z/n\}$.

We define $W_{f,z} = \bigcup_{i \in \mathcal{N}} \left(U_{f,i,z} \setminus \overline{V}_{f,i,z}\right)$. Then $W_{f,z}$ is an open set containing $f^{-1}[0, z)$ and $\overline{W}_{f,z} \cap f^{-1}(z, 1] = \emptyset$.

Note that (1) if $z_1 < z_2$, then $\underline{f}_{n,z_2}(x) \leq \underline{f}_{n,z_1}(x)$ and $\overline{f}_{n,z_2}(x) \leq \overline{f}_{n,z_1}(x)$. Thus, $\overline{U}_{f,n,z_1} \subset U_{f,n,z_2}$ and $\overline{V}_{f,n,z_2} \subset V_{f,n,z_1}$. (2) If $z \in [0, 1]$, then $\overline{U}_{f,n+1,z} \subset U_{f,n,z}$ and $\overline{V}_{f,n+1,z} \subset V_{f,n,z}$. (3) If $z > \sup(f)$, then $W_{f,z} = X$. (4) If $z < \inf(f)$, then $W_{f,z} = \emptyset$.

Lemma 4. *Suppose that* $0 < z_1 < z_2 < 1$ *and* $1/N < (z_2 - z_1)/3$. *Then* $V_{f,N,z_2} \cap U_{f,N,z_1} = \emptyset$.

Proof. We are given that $z_1 + 1/N < z_2 - 1/N$. If $f^{-1}[0, z_1] = \emptyset$, then $U_{f,N,z_1} = \emptyset$. If $f^{-1}[z_2, 1] = \emptyset$, then $V_{f,N,z_2} = \emptyset$. If $f^{-1}[0, z_1] \neq \emptyset$ and $f^{-1}[z_2, 1] \neq \emptyset$, then $\{\eta \,|\, \varphi(f^{-1}[0, z_1], f^{-1}[z_1 + 1/N, 1], \eta) < 1\} \subset \{\eta \,|\, \varphi(f^{-1}[0, z_2 - 1/N], f^{-1}[z_2, 1], \eta) = 0\}$. Thus, $\{\eta \,|\, \varphi(f^{-1}[0, z_1], f^{-1}[z_1 + 1/N, 1], \eta) < z_1/N\} \cap \{\eta \,|\, \varphi(f^{-1}[0, z_2 - 1/N], f^{-1}[z_2, 1], \eta) > 1 - 1/N + z_2/N\} = \emptyset$.

Lemma 5. *If* $z_1 < z_2$, *then* $\overline{W}_{f,z_1} \subset W_{f,z_2}$.

Proof. Suppose that $x \in \overline{W}_{f,z_1} \setminus W_{f,z_2}$. By Lemma 4, we may let N denote the first integer such that $x \notin U_{f,N,z_1} \cap V_{f,N,z_2}$. Since $x \in \overline{W}_{f,z_1},\ x \in \overline{U}_{f,1,z_1}$, and $f^{-1}[0, z_1] \neq \emptyset$. But, $f^{-1}[z_1 + 1, 1] = \emptyset$; so, $U_{f,1,z_1} = X$. Since $x \in \overline{U}_{f,1,z_1} \subset U_{f,1,z_2}$ and $x \notin W_{f,z_2}$, it must be that $x \in \overline{V}_{f,1,z_2}$ and $f^{-1}[z_2, 1] \neq \emptyset$. Since $f^{-1}[0, z_2 - 1] = \emptyset$, $V_{f,1,z_2} = X$. Thus, $N > 1$.

If $x \in \overline{V}_{f,N,z_2}$, then $x \notin U_{f,N,z_1},\ x \in V_{f,N,z_1}$, and $x \notin \overline{U}_{f,N+1,z_1}$. Then $x \in (V_{f,N,z_1} \setminus \overline{U}_{f,N+1,z_1})$ and $(V_{f,N,z_1} \setminus \overline{U}_{f,N+1,z_1}) \cap W_{f,z_1} = \emptyset$, which would be a contradiction. So, $x \notin \overline{V}_{f,N,z_2}$. Since $x \notin W_{f,z_2},\ x \notin U_{f,N,z_2}$, and so, $x \notin \overline{U}_{f,N,z_1}$. But then $x \in (V_{f,N-1,z_1} \setminus \overline{U}_{f,N,z_1}) \cap W_{f,z_1} = \emptyset$, which is a contradiction from which the lemma follows.

Lemma 6. *If X is weakly continuously Urysohn, then there is a continuous function $e : \mathcal{C}_K^\star(X) \times X \to [0,1]$ such that*

(1) *$ef(x) = f(x)$ for all $x \in$ Domain(f), and*
(2) *if f and g have a common domain and if $f(x) \le g(x)$ for all $x \in$ Domain(f), then $ef(x) \le eg(x)$ for all $x \in X$.*

Proof. For $f \in \mathcal{C}_K^\star(X)$ and $x \in X$, define $ef(x) = \mathrm{glb}\{z \in [0,1] \,\big|\, x \in W_{f,z}\}$. It follows from the construction that $ef : X \to [0,1]$ is an extension of f such that if f and g share a common domain, then $f \le g$ implies that $ef \le eg$. Furthermore, the image of ef is contained in the convex hull of the image of f. It is a standard argument that ef is continuous. It remains to show that $e : (\mathcal{C}_K^\star(X)) \times X \to [0,1]$ is continuous. To this end, let $f \in \mathcal{C}_K^\star(X)$, $x \in X$, and $\varepsilon > 0$. Let $z = ef(x)$. We will obtain an open set $\Gamma \times O$ in $\mathcal{C}_K^\star(X) \times X$ containing (f,x) such that if $(g,x') \in \Gamma \times O$, then $|ef(x) - eg(x')| < \varepsilon$.

Part I: First, we obtain an open set $\Gamma_1 \times O_1$ in $\mathcal{C}_K^\star(X) \times X$ containing (f,x) such that if $(g,x') \in \Gamma_1 \times O_1$, then $eg(x) \le z + \varepsilon$; that is, if $1 > z_0 > z + \varepsilon$, then $x' \in W_{g,z_0}$. To this end, choose z' and z'' in $[0,1]$ such that $z < z' < z'' < z + \varepsilon$. Since $x \in W_{f,z'}$, there is an $N > 0$ such that $x \in U_{f,N,z'} \setminus \overline{V}_{f,N,z'}$.

Step A: Let $\varepsilon' = \varepsilon + z - z''$. Since $V_{f,N,z'} \ne X$, $f^{-1}[0, z' - 1/N] \ne \emptyset$ and $f^{-1}[0, z'' - 1/N] \ne \emptyset$.

$$\begin{aligned}\varphi(f^{-1}[0, z'' - 1/N], f^{-1}[z'', 1], x) &\le \varphi(f^{-1}[0, z' - 1/N], f^{-1}[z', 1], x)\\ &\le 1 - 1/N + z'/N < 1 - 1/N + z''/N.\end{aligned}$$

By Lemma 3, there is an open set $\Lambda_1 \times W_1$ in $\mathcal{C}_K^\star(X) \times X$ containing (f,x) such that if $(g,x') \in \Lambda_1 \times W_1$ and $z_0 > z'' + \varepsilon' = z + \varepsilon$, then $\varphi(g^{-1}[0, z_0 - 1/N], g^{-1}[z_0, 1], x') < 1 - 1/N + z'' < 1 - 1/N + z_0/N$ and $x' \notin \overline{V}_{g,N,z_0}$.

Step B: Let $\varepsilon' = \varepsilon + z - z'$. Since $f^{-1}[0, z' - 1/N] \ne \emptyset$, $f^{-1}[0, z'] \ne \emptyset$. Since $\varphi(f^{-1}[0, z'], f^{-1}[z' + 1/N, 1], x) < z'/N$, we may employ Lemma 3 to obtain an open set $\Lambda_2 \times U_2$ in $\mathcal{C}_K^\star \times X$ containing (f,x) such that if $z_0 > z' + \varepsilon' = z + \varepsilon$ and $(g,x') \in \Lambda_2 \times U_2$, then $\varphi(g^{-1}[0, z_0], g^{-1}[z_0 + 1/N, 1], x') < z'/N < z_0/N$ and $x' \in U_{g,N,z_0}$.

To complete the argument for Part I, let $\Gamma_1 = \Lambda_1 \cap \Lambda_2$ and $O_1 = U_1 \cap U_2$.

Part II: Second, we find an open set $\Gamma_2 \times O_2$ in $\mathcal{C}_K^\star(X) \times X$ containing (f,x) such that if $(g,x') \in \Gamma_2 \times O_2$, then $eg(x') \ge z - \varepsilon$; that is, if $z_0 < z - \varepsilon$, then $x' \notin W_{g,z_0}$. We consider three cases.

Case A: $f^{-1}[0, z - \varepsilon] = \emptyset$.
Choose z' so that $z - \varepsilon < z' < z$ and $f^{-1}[0, z'] = \emptyset$. Set $2\varepsilon' = z' - (z - \varepsilon)$. Let $\{U_1 \dots U_n\}$ be an irreducible open cover of $f^{-1}[z', 1] = \mathrm{Domain}(f)$ such that if $1 \le i \le n$ and $\{a, b\} \subset \mathrm{Domain}(f) \cap U_i$, then $|f(a) - f(b)| < \varepsilon'$. For each $1 \le i \le n$, choose $x_i \in U_i \cap \mathrm{Domain}(f)$ and let $U_i^\star = U_i \times (x_i - \varepsilon', x_i + \varepsilon')$. Then $\Gamma_2 = \langle U_1^\star, \dots, U_n^\star \rangle$ is an open set containing f such that if $g \in \Gamma_2$, $z_0 < z' - \varepsilon' = z - \varepsilon$, and $x' \in X$, then $g^{-1}[0, z_0] = \emptyset$. Letting $O_2 = X$, we have an open set $\Gamma_2 \times O_2$ in $\mathcal{C}_K^\star(X) \times X$ containing (f,x) such that if $(g,x') \in \Gamma_2 \times O_2$, then $eg(x') \ge z - \varepsilon$

In both Cases B and C, we will choose z', z'' and z''' in $(0,1)$ such that $z > z' > z'' > z''' > z - \varepsilon$. Lemma 4 gives us a first integer N such that $x \notin U_{f,N,z''} \cap V_{f,N,z'}$.

Case B: $f^{-1}[0, z - \varepsilon] \ne \emptyset$ and $x \in \overline{V}_{f,N,z'}$. Then $x \notin U_{f,N,z''}$ so

$$\varphi(f^{-1}[0, z''], f^{-1}[z'' + 1/N, 1], x) \ge z''/N$$

and $f^{-1}[z''+1/N,1]\neq\emptyset$. It follows that $f^{-1}[z'''+1/N,1]\neq\emptyset$ and

$$\varphi(f^{-1}[0,z'''],f^{-1}[z'''+1/N,1],x)>z''/N>z'''/N.$$

We let $\varepsilon'=z'''-(z-\varepsilon)$ and apply Lemma 3 to obtain an open set $\Omega_1\times V_1$ in $\mathcal{C}_K^\star(X)\times X$ containg (f,x) such that if $(g,x')\in\Omega_1\times V_1$ and $z_0<z'''-\varepsilon'=z-\varepsilon$, then

$$\varphi(g^{-1}[0,z_0],g^{-1}[z_0+1/N,1],x')>z'''/N>z_0/N.$$

Now, since $x\in\overline{V}_{f,N,z'}$, $x\in V_{f,N,z'''}$ and $\varphi(f^{-1}[0,z'''-1/N],f^{-1}[z''',1],x)>1-1/N+z'''/N$. Letting $\varepsilon'=z'''-(z-\varepsilon)$, we apply Lemma 3 to obtain an open set $\Omega_2\times V_2\in\mathcal{C}_K^\star(X)\times X$ containing (f,x) such that if $(g,x')\in\mathcal{C}_K^\star(X)\times X$ and $z_0<z'''-\varepsilon'=z-\varepsilon$, then $x'\notin U_{g,i,z_0}$ for all $i\geq N$.

To complete Case B, we let $\Gamma_2=\Omega_1\cap\Omega_2$ and $O_2=V_1\cap V_2$.

Case C: $f^{-1}[0,z-\varepsilon]\neq\emptyset$ and $x\notin\overline{V}_{f,N,z'}$. Then since $x\notin W_{f,z'}$, $x\notin U_{f,N,z'}$. Thus, $\varphi(f^{-1}[0,z'],f^{-1}[z'+1/N,1],x)\geq z'/N$. It follows that $f^{-1}[z'+1/N,1]\neq\emptyset$ and $\varphi(f^{-1}[0,z''],f^{-1}[z''+1/N],x)\geq z'/N>z''/N$. Again, Lemma 3 gives us an open set $\Omega_1\times V_1\in\mathcal{C}_K^\star(X)$ such that if $(g,x')\in\Omega_1\times V_1$, then $x'\notin U_{g,N,z_0}$.

Since $f^{-1}[0,z-\varepsilon]\neq\emptyset$, $f^{-1}[0,z'']\neq\emptyset$, and $U_{f,1,z''}=X$. Similarly, since $f^{-1}[z'+1/N,1]\neq\emptyset$, $V_{f,1,z'}=X$. Therefore, $N>1$. By hypothesis, $x'\in V_{f,N-1,z'}\subset V_{f,N-1,z''}$. Letting $\varepsilon'=z''-(z-\epsilon)$, we employ Lemma 3 to obtain an open set $\Omega_2\times V_2$ in $\mathcal{C}_K^\star(X)\times X$ such that if $(g,x')\in\Omega_2\times V_2$ and $z_0<z''-\varepsilon'=z-\varepsilon$, then $z'\notin W_{g,z_0}$.

To complete Case C, we let $\Gamma_2=\Omega_1\cap\Omega_2$ and $O_2=V_1\cap V_2$. To complete the proof of the lemma, we let $\Gamma=\Gamma_1\cap\Gamma_2$ and $O=O_1\cap O_2$.

We are now in a position to obtain our main result.

Theorem 1. *If X is a Hausdorff space, then the following conditions are equivalent:*

1. *X is weakly continuously Urysohn.*
2. *There is a continuous extender $e:\mathcal{C}_K(X)\times X\to\mathcal{R}$.*
3. *There is a continuous extender $e:\mathcal{C}_K(X)\times X\to\mathcal{R}$ such that if f and g have a common domain and if $f(x)\leq g(x)$ for all $x\in$ Domain(f), then $ef(x)\leq eg(x)$ for all $x\in X$.*

Proof. Clearly, (3) $\Rightarrow$ (2). We will use a standard argument to show that (1) $\Rightarrow$ (3). Let $e^\star:\mathcal{C}_K^\star(X)\times X\to[0,1]$ be the extender given in Lemma 6. Let $h:\mathcal{R}\to(0,1)$ be an order preserving homeomorphism taking $\mathcal{R}$ onto $(0,1)$. For $f\in\mathcal{C}_K(X)$, define $ef=h^{-1}\circ e^\star\circ h\circ f$. To obtain (2) $\Rightarrow$ (1), for each $(x,y)\in X^2\setminus\Delta$ let $\psi_{(x,y)}:\{x,y\}\to[0,1]$ be defined by $\psi_{(x,y)}(x)=0$ and $\psi_{(x,y)}(y)=1$. Let $\varphi(x,y,t)=e\psi_{(x,y)}(t)$.

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