CONVOLUTION CONGRUENCES FOR THE PARTITION FUNCTION

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Abstract. Ahlgren and Boylan recently proved the uniqueness of the Ramanujan congruences for the primes 5, 7, and 11 by using the modularity of a certain partition function. Here we use their result to find universal congruences, of a different type, which hold for the partition function modulo all primes \( \ell \geq 5 \).

1. Introduction and statement of results

Let \( p(n) \) denote the number of partitions of the positive integer \( n \), that is, the number of representations of \( n \) as the sum of a sequence of non-increasing positive integers. For more on partitions see [And98] or [AO01]. For convenience, we let \( p(0) = 1 \), and \( p(\alpha) = 0 \) if \( \alpha < 0 \). Euler introduced the generating function

\[
\sum_{n \geq 0} p(n)q^n = \prod_{n \geq 1} \frac{1}{1 - q^n}.
\]

If we denote the pentagonal numbers

\[
\omega(k) := \frac{1}{2}k(3k + 1), \quad k \in \mathbb{Z},
\]

then we have

\[
\sum_{k \in \mathbb{Z}} (-1)^k \omega(k) \cdot \sum_{n \geq 0} p(n)q^n = 1.
\]

This gives rise to the recursive formula

\[
p(n) = \sum_{k \in \mathbb{Z} - \{0\}} (-1)^{k+1} p(n - \omega(k)),
\]

which holds for all positive integers \( n \). While this formula is useful for quickly computing values of \( p(n) \), it has not offered much information about the arithmetic of the partition function.

The famous Ramanujan congruences

\[
\begin{align*}
p(5n + 4) & \equiv 0 \pmod{5}, \\
p(7n + 5) & \equiv 0 \pmod{7}, \\
p(11n + 6) & \equiv 0 \pmod{11},
\end{align*}
\]

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may be reformulated in the following peculiar way. For primes $\ell \geq 5$, define

$$\delta_\ell := \frac{\ell^2 - 1}{24},$$

and let

$$\mathcal{P}_\ell(z) := \sum_{k \in \mathbb{Z}} (-1)^k q^{\omega(k)} \sum_{n \geq 0} p(\ell n - \delta_\ell) q^n. \quad (1.2)$$

The Ramanujan congruences are equivalent to the assertion that $\mathcal{P}_\ell(z) \equiv 0 \pmod{\ell}$ for $\ell \in \{5, 7, 11\}$. For example, we have

$$\mathcal{P}_5(z) = 5q + 30q^2 + 135q^3 + 490q^4 + 1575q^5 + \cdots \equiv 0 \pmod{5},$$

$$\mathcal{P}_7(z) = 7q + 77q^2 + 490q^3 + 2436q^4 + 10143q^5 + \cdots \equiv 0 \pmod{7},$$

$$\mathcal{P}_{11}(z) = 11q + 297q^2 + 3718q^3 + 31185q^4 + 204226q^5 + \cdots \equiv 0 \pmod{11}. \quad (1.3)$$

Ahlgren and Boylan [AB03] have proven Ramanujan’s famous conjecture that congruences of the form $p(\ell n - \beta) \equiv 0 \pmod{\ell}$ exist only for primes $5, 7, 11$. Therefore, it is natural to ask whether some refinement of (1.3) continues to hold for all primes $\ell \geq 5$. Here we show that this is indeed the case.

To make this precise, let $M_k$ denote the space of weight $k$ holomorphic modular forms on $\text{SL}_2(\mathbb{Z})$. For more on modular forms see [Kob93]. We shall denote a modular form $f(z) \in M_k$ by its Fourier expansion

$$f(z) := \sum_{n=0}^{\infty} a_f(n) q^n, \quad q := e^{2\pi i z}. \quad (1.4)$$

For each prime $\ell$ we have the $U(\ell)$-operator defined by

$$f(z)|U(\ell) = \sum_{n=0}^{\infty} a_f(\ell n) q^n. \quad (1.4)$$

Furthermore, for even $k \geq 4$, let $E_k(z) \in M_k$ denote the normalized Eisenstein series of weight $k$,

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad (1.5)$$

where the rational numbers $B_k$ are the Bernoulli numbers defined by

$$\sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2} t + \frac{1}{12} t^2 + \cdots$$

and

$$\sigma_{k-1}(n) = \sum_{1 \leq d \mid n} d^{k-1}. \quad (1.5)$$

Our refinement of (1.3) is the following.

**Theorem 1.1.** Suppose $\ell \geq 5$ is prime and $m \in \{4, 6, 8, 10, 14\}$. If $\mathcal{P}_\ell^{(m)}(z) := E_m(z) \mathcal{P}_\ell(z)$, then

$$\mathcal{P}_\ell^{(m)}(z)|U(\ell) \equiv 0 \pmod{\ell}. \quad (1.6)$$
Remark. Note that $E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \equiv 1 \pmod{5}$. Similarly, $E_6(z) \equiv 1 \pmod{7}$ and $E_{10}(z) \equiv 1 \pmod{11}$. While this does not yield the full Ramanujan congruences, it does show
\[ p(\ell^2 n + \delta) \equiv 0 \pmod{\ell} \quad \text{for } \ell = 5, 7, 11. \]

Example. If $m = 4$ and $\ell = 13$, then
\[ \mathcal{P}^{(4)}_{13}(z) = 11q + 3130q^2 + 149709q^3 + 3225214q^4 + 43626740q^5 + \cdots, \]
\[ \mathcal{P}^{(4)}_{13}(z)|U(13) = 64580467571997q + 166693095587328996218q^2 + \cdots \equiv 0 \pmod{13}. \]
If we let $b(4) := 240, b(6) := -540, b(8) := 480, b(10) := -264, b(14) := -24,$ then Theorem 1.1 implies the following recursive congruence.

**Theorem 1.2.** For each prime $\ell \geq 5$ and each $m \in \{4, 6, 8, 10, 14\}$ we have
\[ p(\ell^2 N - \delta) \equiv \sum_{k \in \mathbb{Z} - \{0\}} (-1)^{k+1} p(\ell^2 (N - \omega(k)) - \delta) \]
\[ + b(m) \sum_{k \geq 1}^{k \in \mathbb{Z}} \sigma_{m-1}(n)(-1)^{k+1} p(\ell^2 (N - \omega(k) - n/\ell) - \delta) \pmod{\ell} \]
for all positive integers $N$.

Theorem 1.2 is a consequence of a generic theorem for cusp forms on $\text{SL}_2(\mathbb{Z})$ with integer coefficients. Let $S_k$ denote the space of cusp forms of weight $k$, that is, the subspace of modular forms $f(z) \in M_k$ such that $a_f(0) = 0$. For example,
\[ \Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + \cdots \in S_{12} \cap \mathbb{Z}[q]. \]

**Theorem 1.3.** Fix a prime $\ell \geq 5$. Suppose $k = a(\ell - 1) + b$ for some $1 \leq a \leq 12$ and $b \in \{4, 6, 8, 10, 14\}$ with $b \geq a + 2$. If $f(z) \in S_k \cap \mathbb{Z}[q]$, then $f(z)|U(\ell) \equiv 0 \pmod{\ell}$. We prove all three of the above theorems in the next section. We then conclude with a discussion of an analogous function to $\mathcal{P}_\ell$ for the small primes $\ell = 2, 3$ in the final section.

2. Proof of the Theorems

We begin with some preliminary information about modular forms. Recall that for even $k$ we know $M_k$ is a finite-dimensional $\mathbb{C}$-vector space of dimension
\begin{equation}
(2.1) \quad d(k) := \dim_{\mathbb{C}}(M_k) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \not\equiv 0 \pmod{12}, \\ \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 0 \pmod{12}. \end{cases}
\end{equation}

For even $k$ define the function $\delta(k) \in \{0, 4, 6, 8, 10, 14\}$ by the congruence
\begin{equation}
(2.2) \quad \delta(k) \equiv k \pmod{12}.
\end{equation}

The above function $\delta(k)$ gives information about the zeroes of $f(z) \in M_k$ by the valence formula below. Let $\omega := e^{2\pi i/3}$. If $f(z) \in M_k$ and $v_\omega(f)$ is the order vanishing of $f$ at $\tau$, then
\begin{equation}
(2.3) \quad \frac{k}{12} = v_\infty(f) + \frac{1}{2}v_1(f) + \frac{1}{3}v_\omega(f) + \sum_{\tau \in \mathbb{H}/U} v_\tau(f),
\end{equation}
where $U$ is any congruence subgroup of $\text{SL}_2(\mathbb{Z})$.
Theorem 2.1

where $\mathcal{H}$ denotes the upper half plane and $\Gamma = \text{SL}_2(\mathbb{Z})$. The valence formula tells us, for example, that $E_{\ell-1}(z)/E_{\delta(\ell-1)}(z)$ and $E_b(z)/E_4(z)$ are holomorphic modular forms. In addition, we have some well-known facts about the Eisenstein series $E_k(z)$. For convenience we define $E_0(z) := 1 \in M_0$.

1. $E_k(z) = 1 + \cdots \in M_k \cap \mathbb{Z}[q]$ for each $k \in \{0, 4, 6, 8, 10, 14\}$.
2. $E_{\ell-1}(z) \equiv 1 \pmod{\ell}$ for $\ell \geq 5$ prime.

We will also need the following theorem of Choie, Kohnen, and Ono [CKO05].

**Theorem 2.1** ([CKO05 Theorem 3.1]). If $f(z) \in M_k$ and $g(z) \in M_{12N}$, then

$$\text{const} \left( \frac{E_{14-\ell(k)}(z)g(z)}{\Delta(z)^{N}} \cdot f(z) \right) = 0,$$

where $\text{const}(h(z))$ denotes the constant term of the power series $h(z)$.

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Fix a prime $\ell \geq 5$. Let $b$ be of the specified form. Fix a cusp form $f(z) \in S_k \cap \mathbb{Z}[q]$. It suffices to show that $a_f(\ell m) \equiv 0 \pmod{\ell}$ for each $n \geq 1$. This will follow by induction on $n$.

Recall, as $f$ is a cusp form, $a_f(0) = 0$. Fix an integer $n$ and suppose $a_f(\ell m) \equiv 0 \pmod{\ell}$ for each $0 \leq m \leq n - 1$. Consider the quantity

$$c(a, b, \ell) := \delta(a(\ell - 1)) - 12(d(b + \delta(a(\ell - 1))) - 1).$$

Define the function

$$g_n(z) := E_{14-b}(z)^{t-1} E_{\ell-1}(z)^{b-a-2} E_4(z)^{3d(n-1)} \tilde{g}(z),$$

where

$$\tilde{g}(z) := \begin{cases} E_{\delta(a(\ell-1))}(z) & \text{if } c(a, b, \ell) > 2, \\ E_{14-b}/E_{12-b}(z) & \text{if } c(a, b, \ell) = 2, \\ E_0(z) & \text{if } c(a, b, \ell) = 0, \\ E_{14-b}/E_{16-b}(z) & \text{if } c(a, b, \ell) = -2, \\ 1/E_{c(a,b,\ell)}(z) & \text{if } c(a, b, \ell) < -2. \end{cases}$$

We first show that the above function $g_n(z)$ is a holomorphic modular form by considering each of the three non-trivial possibilities for $c(a, b, \ell)$ separately.

Case i) $c(a, b, \ell) = 2$.

$$g_n(z) = E_{14-b}(z)^{t-1} E_{\ell-1}(z)^{b-a-2} E_4(z)^{3d(n-1)} E_{14-b}(z)/E_{12-b}(z).$$

If $c(a, b, \ell) = 2$, then $d(b + \delta(a(\ell - 1))) = 2$ and $\delta(a(\ell - 1)) = 14$. Hence, $\delta(\ell - 1) = 10$. Furthermore, $b \in \{4, 6, 8\}$. Since $\delta(\ell - 1) = 10$, it follows by the valence formula that $E_{\ell-1}(z)E_{14-b}(z)/E_{12-b}(z)$ is holomorphic for each $b \in \{4, 6, 8\}$. It suffices to show that if $c(a, b, \ell) = 2$, then $b - a - 2 > 0$. If $b - a - 2 = 0$, then $a$ is even, which contradicts $\delta(a(\ell - 1)) = 14$.

Case ii) $c(a, b, \ell) = -2$.

$$g_n(z) = E_{14-b}(z)^{t-1} E_{\ell-1}(z)^{b-a-2} E_4(z)^{3d(n-1)} E_{14-b}(z)/E_{16-b}(z).$$

If $c(a, b, \ell) = -2$, then $d(b + \delta(a(\ell - 1))) = 2$ and $\delta(a(\ell - 1)) = 10$, so $\delta(\ell - 1) = 10$ and $b \in \{6, 8, 10\}$. Again, the valence formula implies that $E_{\ell-1}(z)E_{14-b}(z)/E_{16-b}(z)$ is holomorphic for $b \in \{6, 10\}$ if $b - a - 2 > 0$ and for $b = 8$ if $b - a - 2 > 1$. If $b - a - 2 = 0$, then $a$ is even, which contradicts the fact that $\delta(a(\ell - 1)) = 10$. If $b = 8$ and $b - a - 2 = 1$, then $a = 5$, but then $a(\ell - 1) \equiv 5(10) \not\equiv 10 \pmod{12}$. 


Case iii) \(c(a, b, \ell) < -2\).

\[ g_n(z) = E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}E_4(z)^{3(n-1)} / E_{c(a, b, \ell)}(z). \]

As \(c(a, b, \ell) < -2\), we have \(d(b + \delta(a(\ell - 1))) > 2\). As \(b > 14\) and \(\delta(a(\ell - 1)) \leq 14\), we have \(d(b + \delta(a(\ell - 1))) \leq 3\). Moreover, if \(\delta(a(\ell - 1)) \in \{10, 14\}\), then \(d(b + \delta(a(\ell - 1))) = 3\), and if \(\delta(a(\ell - 1)) \in \{4, 6, 8\}\), then \(d(b + \delta(a(\ell - 1))) = 2\).

First suppose \(\delta(a(\ell - 1)) \in \{10, 14\}\), so \(\delta(\ell - 1) = 10\). If \(\delta(a(\ell - 1)) = 10\), then \(-c(a, b, \ell) = 14\), and if \(\delta(a(\ell - 1)) = 14\), then \(-c(a, b, \ell) = 10\). By the valence formula, \(E_{\ell-1}(z)^{b-a-2}/E_{c(a, b, \ell)}(z)\) is a holomorphic modular form if \(\delta(a(\ell - 1)) = 14\) and \(b - a - 2 > 0\) or \(\delta(a(\ell - 1)) = 10\) and \(b - a - 2 > 1\). As in the previous cases, \(\delta(\ell - 1) = 10\) implies \(b - a - 2 > 0\). Suppose \(\delta(a(\ell - 1)) = 10\) and \(b - a - 2 = 1\). As \(d(b + \delta(a(\ell - 1))) = 3\), this implies that \(b = 14\). Thus, \(a = 11\), which yields \(11(\ell - 1) \equiv 11(10) \not\equiv 10 \pmod{12}\), a contradiction.

Now suppose \(\delta(a(\ell - 1)) \in \{4, 6, 8\}\), so \(E_{c(a, b, \ell)}(z) \in \{E_8(z), E_6(z), E_4(z)\}\), respectively. Begin with \(\delta(a(\ell - 1)) = 4\); then \(\delta(\ell - 1) = 4\) or \(10\). By the valence formula, \(E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}/E_4(z)\) is a holomorphic modular form if \(b - a - 2 > 1\). As \(d(b + a(\ell - 1)) = 2\), this implies that \(b = 8, 14\), and so we need only check that \(a \neq 5, 6, 11, 12\). Note that in each of these cases \(a(\ell - 1) \neq 4 \pmod{12}\).

Next suppose \(\delta(a(\ell - 1)) = 6\), so \(\delta(\ell - 1) = 6\) or \(10\). It again follows by the valence formula that \(E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}/E_4(z)\) is a holomorphic modular form if \(b - a - 2 > 0\). If \(\delta(a(\ell - 1)) = 6\) and \(b - a - 2 = 0\), then \(a\) is even, which contradicts \(\delta(a(\ell - 1)) = 6\).

Finally, consider \(\delta(a(\ell - 1)) = 8\), so \(\delta(\ell - 1) = 4\) or \(10\). If \(b - a - 2 > 0\) or \(b \neq 8, 14\), then \(E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}/E_4(z)\) is a holomorphic modular form. If \(b = 8, 14\), then \(b - a - 2 = 0\) implies \(a = 6, 12\), for which \(\delta(a(\ell - 1)) \neq 8 \pmod{12}\).

We can apply Theorem 2.1 to show \(a_f(\ell n) \equiv 0 \pmod{\ell}\). Take \(f(z)\) as above and \(g(z) = g_n(z)\). We have \(k = a(\ell - 1) - \delta(a(\ell - 1)) + \delta(a(\ell - 1)) + b\); it follows that

\[ d(k) = \frac{a(\ell - 1) - \delta(a(\ell - 1))}{12} + d(b + \delta(a(\ell - 1))). \]

Similarly,

\[ \delta(k) = \delta(b + \delta(a(\ell - 1))) \]
\[ = b + \delta(a(\ell - 1)) - 12(d(b + \delta(a(\ell - 1))) - 1) \]
\[ = b + c(a, b, \ell). \]

Moreover, \(\tilde{g}(z)\) has weight \(c(a, b, \ell)\); hence, \(g_n(z)\) has weight

\[ (14 - b)(\ell - 1) + (\ell - 1)(b - a - 2) + 12(\ell n - 1) + c(a, b, \ell) = 12N, \]

where

\[ N = \ell n - \frac{a(\ell - 1) - \delta(a(\ell - 1))}{12} - d(b + \delta(a(\ell - 1))) \]
\[ = \ell n - d(k). \]

Thus, by Theorem 2.1

\[ \text{const} \left( \frac{E_{14-b-c(a, b, \ell)}(z)g_n(z)}{\Delta(z)^{\ell n}} \cdot f(z) \right) = 0. \]
Each possibility for $g_n(z)$ allows us to conclude that
\[
\text{const} \left( \frac{E_{14-b}(z)E_{\ell-1}(z)^{b-a-2}E_4(z)^{3\ell(n-1)}}{\Delta(z)^{\ell n}} \cdot f(z) \right) = 0,
\]
where we use the observation that
\[
E_{14-b-c(a,b,\ell)}(z)\tilde{g}(z) = E_{14-b}(z).
\]

Now consider the above function modulo $\ell$, recalling $E_{\ell-1}(z) \equiv 1 \pmod{\ell}$. The constant term of
\[
\frac{E_{14-b}(z)E_{\ell-1}(z)^{b-a-2}E_4(z)^{3\ell(n-1)}}{\Delta(z)^{\ell n}} \cdot f(z)
\]
\[
\equiv \frac{E_{14-b}(\ell z)E_4(\ell z)^{3\ell(n-1)}}{\Delta(\ell z)^n} \cdot f(z) \quad \pmod{\ell}
\]
\[
(2.4) \equiv E_{14-b}(\ell z)E_4(\ell z)^{3\ell(n-1)} (q^{-\ell} + 24 + 324q^3 + \cdots)^n \sum_{n=0}^{\infty} a_f(n)q^n \quad \pmod{\ell}
\]
is congruent to zero modulo $\ell$. Recall that $E_{14-b}(\ell z)$ and $E_4(\ell z)^{3\ell(n-1)}$ have leading coefficient 1, so by multiplying out the above expression, we find for each $0 \leq m \leq n-1$ integers $a_{b,n}(m)$ such that
\[
(2.5) \quad a_f(\ell n) + \sum_{m=0}^{n-1} a_{b,n}(m)a_f(\ell m) \equiv 0 \pmod{\ell}.
\]
In particular, $a_f(\ell m) \equiv 0 \pmod{\ell}$ for each $0 \leq m \leq n-1$ by the inductive hypothesis, and hence $a_f(\ell n) \equiv 0 \pmod{\ell}$ by induction. \hfill \Box

**Remark.** The above proof fails for the small primes $\ell = 2, 3$; for example, $E_{\ell-1}(z)$ is not a holomorphic modular form for these primes. We can, however, modify the proof using the fact that $E_k(z) \equiv 1 \pmod{2, 3}$ for $k \in \{0, 4, 6, 8, 10, 14\}$. It suffices to show that for each $n \geq 1$ we can find a non-negative integer $N$ such that $d(k) + N = \ell n$, as we can then conclude by Theorem 2.1 that
\[
\text{const} \left( \frac{E_{14-b}(k)E_4(z)^{3N}}{\Delta(z)^{d(k)+N}} \cdot f(z) \right) \equiv \text{const} \left( \Delta(\ell z)^{-n} \cdot f(z) \right) \quad \pmod{\ell}.
\]
This yields equations analogous to (2.4) and (2.5). For $\ell = 3$ we have $2a + b \leq 38$, so for each $n \geq 1$ we can take $N = 3n - d(k) \geq 0$, with the exception of $k = 36$, where $d(36) = 4$. Similarly, for $\ell = 2$ we have $a + b \leq 26$, so again we can take $N = 2n - d(k) \geq 0$ with the exception of $k = 24$, where $d(24) = 3$. For these exceptions the conclusion of Theorem 1.3 does not hold as $\Delta(z)^{\ell} = q^{\ell} + \cdots \in S_{12\ell} \cap \mathbb{Z}[q]$.

We are now ready to prove the main theorem of the paper, which follows as a consequence of Theorem 1.3 above and the following theorem of Ahlgren and Boylan [AB03].

**Theorem 2.2 ([AB03 Theorem 3]).** If $\ell \geq 5$ is prime and $j \geq 1 \in \mathbb{Z}$, then let $1 \leq \beta_{\ell,j} \leq \ell^2$ be the unique integer for which $24\beta_{\ell,j} \equiv 1 \pmod{\ell^2}$. Define $k_{\ell,j}$ by
\[
k_{\ell,j} := \begin{cases} \frac{\ell^{-1}(\ell^j - 1)}{2} - \frac{1}{2} \left( \frac{24\beta_{\ell,j}^{-1}}{\ell} - 1 \right) & \text{if } j \text{ is odd}, \\ \frac{\ell^{-1}(\ell^j - 1)}{2} - \frac{1}{2} \left( \frac{24\beta_{\ell,j}^{-1}}{\ell} - 1 \right) & \text{if } j \text{ is even}. \end{cases}
\]
Then there exists a modular form $F_{\ell,j} \in M_{k_{\ell,j}} \cap \mathbb{Z}[q]$ such that
\[
\sum_{n=0}^{\infty} p(\ell n + \beta_{\ell,j}) q^n \equiv \prod_{n=1}^{\infty} (1 - q^n)^{24n/24} \cdot F_{\ell,j}(z) \pmod{\ell}.
\]

**Proof of Theorem 1.1.** We now apply Theorem 2.2 in the case where $\ell$ is any prime at least 5 and $j = 1$. In this case
\[
k_{\ell} := k_{\ell,1} = (\ell - 1) - 12 (\lfloor \ell/24 \rfloor + 1),
\]
\[
\beta_{\ell} := \beta_{\ell,1} = -\ell (\lfloor \ell/24 \rfloor + 1) = -\delta_{\ell} + \ell (\lfloor \ell/24 \rfloor + 1).
\]
Thus, we have
\[
\sum_{n=0}^{\infty} p(\ell n + \beta_{\ell}) q^n \equiv \prod_{n=1}^{\infty} (1 - q^n)^{24n/24^{\ell+1}} F_{\ell,1} \equiv \prod_{n=1}^{\infty} (1 - q^n)^{24(\lfloor \ell/24 \rfloor + 1) - \ell} F_{\ell,1} \pmod{\ell}.
\]

Multiplication by $q^{(\ell/24)+1} \prod_{n=1}^{\infty} (1 - q^n)^{\ell}$ yields
\[
q^{(\ell/24)+1} \prod_{n=1}^{\infty} (1 - q^n)^{\ell} \sum_{n=0}^{\infty} p(\ell n + \beta_{\ell}) q^n \equiv \Delta(z)^{(\ell/24)+1} \cdot F_{\ell,1} \pmod{\ell},
\]
\[
\prod_{n=1}^{\infty} (1 - q^\ell n) \sum_{n=0}^{\infty} p(\ell n - \delta_{\ell}) q^n \equiv \Delta(z)^{(\ell/24)+1} \cdot F_{\ell,1} \pmod{\ell}.
\]
Note that $\Delta(z)^{(\ell/24)+1} \cdot F_{\ell,1}$ (mod $\ell$) $\in S_{\ell-1} \cap \mathbb{Z}[q]$, and so, by Theorem 1.3, we can multiply both sides of the equation above by any Eisenstein series $E_m(z)$ for $m \in \{4, 6, 8, 10, 14\}$ to find
\[
(2.6) \quad \left( E_m(z) \prod_{n=1}^{\infty} (1 - q^\ell n) \sum_{n=0}^{\infty} p(\ell n - \delta_{\ell}) \right) |U(\ell) \equiv 0 \pmod{\ell}.
\]
Applying Euler’s Pentagonal Number Theorem, which states
\[
\prod_{n=1}^{\infty} (1 - q^\ell n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{\ell \omega(k)},
\]
completes the proof. \qed

**Equation (2.6) brings us to Theorem 1.2.**

**Proof of Theorem 1.2.** Suppose $\ell \geq 5$ is prime. For each $m \in \{4, 6, 8, 10, 14\}$ define the numbers $A_m(n)$ by
\[
(2.7) \quad \sum_{n \geq 0} A_m(n) q^n := \left( 1 + b(m) \sum_{n \geq 1} \sigma_{m-1}(n) q^n \right) \left( \sum_{k \in \mathbb{Z}} (-1)^k q^{\ell \omega(k)} \right) \left( \sum_{n \geq 0} p(\ell n - \delta_{\ell}) q^n \right).
\]
In particular,
\[
A_m(N) = p(\ell N - \delta_{\ell}) + \sum_{k \in \mathbb{Z} - \{0\}} (-1)^k p(\ell(N - \ell \omega(k)) - \delta_{\ell}) + b(m) \sum_{n \geq 1 \atop k \in \mathbb{Z}} \sigma_{m-1}(n)(-1)^k p(\ell(N - \ell \omega(k) - n) - \delta_{\ell}).
\]
The result now follows by Theorem 1.1 as
\[ \sum_{n \geq 0} A_m(n)q^n U(\ell) \equiv 0 \pmod{\ell}. \]

3. The primes 2 and 3

It is difficult to prove anything about the distribution of the partition function \( p(n) \) modulo 2 and 3. For these primes \( \ell = 2, 3 \), we cannot define \( P_\ell(z) \) since \( \delta_\ell \) is no longer an integer. We can, however, define the following analog for these small primes. For \( \ell = 2, 3 \) define
\[ P_\ell^*(z) := \sum_{k \in \mathbb{Z}} (-1)^k q^{\omega(k)} \sum_{n \geq 0} p(n)q^n. \]

Note that for these small primes
\[ P_\ell^*(z) \equiv \prod_{n \geq 0} (1 - q^n)^{\ell - 1} \pmod{\ell}. \]

For \( \ell = 2, 3 \) define the numbers \( a_\ell(n) \) by
\[ \sum_{n \geq 0} a_\ell(n)q^n = P_\ell^*(z). \]

It follows that
\[ P_2^*(z) \equiv \sum_{k \in \mathbb{Z}} (-1)^k q^{\omega(k)} \pmod{2}, \]
and so \( a_2(N) \equiv 0 \pmod{2} \) precisely when \( N \) is not a pentagonal number. Similarly,
\[ P_3^*(z) \equiv \sum_{m,n \in \mathbb{Z}} (-1)^{m+n} q^{\omega(m)+\omega(n)} \pmod{3}. \]

In particular, \( a_3(N) \equiv 0 \pmod{3} \) for \( N \) that do not satisfy \( 24N + 2 = (6m + 1)^2 + (6n + 1)^2 \), where \( m, n \in \mathbb{Z} \). Note that while it is difficult to prove anything about \( p(n) \pmod{2, 3} \), in sharp contrast we have that \( a_\ell(n) \equiv 0 \pmod{\ell} \) with arithmetic density one.

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References


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