ON SUBMULTIPLICATIVITY OF SPECTRAL RADIUS 
AND TRANSITIVITY OF SEMIGROUPS

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Abstract. It is shown that a transitive, closed, homogeneous semigroup of 
linear transformations on a finite-dimensional space either has zero divisors or 
is simultaneously similar to a group consisting of scalar multiples of unitary 
transformations. The proof begins with the result that for each closed homo-
geneous semigroup with no zero divisors there is a $k$ such that the spectral 
radius satisfies $r(AB) \leq kr(A)r(B)$ for all $A$ and $B$ in the semigroup.

It is also shown that the spectral radius is not $k$-submultiplicative on any 
transitive semigroup of compact operators.

1. Introduction

We consider semigroups of linear transformations or operators, by which we 
simply mean collections closed under multiplication. A semigroup $S$ is said to be 
transitive if $\{Ax : A \in S\}$ is dense in the space for every non-zero vector $x$ (this 
is sometimes called ‘topological transitivity’, to distinguish it from the case where 
$\{Ax : A \in S\}$ is the entire space for each $x \neq 0$). A semigroup is said to be irreducible 
if it has no non-trivial invariant subspaces; a transitive semigroup is irreducible (but an irreducible semigroup need not be transitive).

The spectral radius $r$ is said to be $k$-submultiplicative on a semigroup if $r(AB) \leq 
kr(A)r(B)$ for all $A$ and $B$ in the semigroup. This paper was partially stimulated 
by [2], which contains results relating $k$-submultiplicativity of spectral radius to 
$\frac{1}{k}$-supermultiplicativity (i.e., $r(AB) \geq \frac{1}{k}r(A)r(B)$).

We begin with several results on finite-dimensional spaces and then establish a 
theorem for semigroups of compact operators.

2. Semigroups of linear transformations on finite-dimensional spaces

In this section we restrict our attention to linear operators on finite-dimensional 
complex vector spaces.

Definition 2.1. A semigroup of linear transformations is said to be homogeneous 
if it is closed under multiplication by complex scalars. By a closed semigroup we 
mean one that is closed in the norm topology.
Theorem 2.2. If a closed, homogeneous semigroup has no (non-zero) zero divisors, then there exists a $k \geq 1$ such that
\[
\frac{1}{k} r(A)r(B) \leq r(AB) \leq kr(A)r(B)
\]
for all $A$ and $B$ in the semigroup.

Proof. Suppose there is no $k$ satisfying $r(AB) \leq kr(A)r(B)$ for all $A$ and $B$. For each $n$, choose a pair $\{A_n, B_n\}$ in the semigroup such that
\[
r(A_nB_n) > nr(A_n)r(B_n).
\]
Dividing by $r(A_n)r(B_n)$ yields operators $C_n$ and $D_n$ with
\[
r(C_nD_n) > n.
\]
Thus, at least one of the sequences $\{\|C_n\|\}$ and $\{\|D_n\|\}$ is unbounded. Assume, after replacing sequences by subsequences, that
\[
\|C_n\| \to \infty \quad \text{and} \quad \frac{C_n}{\|C_n\|} \to R.
\]
Now, $R \neq 0$ (because $\|R\| = 1$) and, by continuity of the spectral radius, $r(R) = 0$. Hence $R$ is nilpotent, so the semigroup has zero divisors, which is a contradiction. Thus there is a $k$ such that
\[
r(AB) \leq kr(A)r(B)
\]
for all $A$ and $B$ in the semigroup. Since the semigroup contains members $A$ with spectral radius one, and since $r(A^2) = r(A)^2$, it follows that $k \geq 1$. The fact that
\[
r(AB) \geq \frac{1}{k} r(A)r(B)
\]
then follows from Theorem 2 of [2].

Theorem 2.3. A transitive, homogeneous, closed semigroup that has no zero divisors is simultaneously similar to a group consisting of scalar multiples of unitary transformations.

Proof. Let $\mathcal{S}$ be such a semigroup.

By Lemma 3.1.6 of [3], since the semigroup is irreducible, there is an idempotent $E$ of minimal non-zero rank in $\mathcal{S}$ such that the restriction of $ESE \setminus \{0\}$ to the range of $E$ is simultaneously similar to a group of multiples of unitaries. If $E$ is the identity operator, there is nothing more to prove. Suppose otherwise; it must then be shown that $\mathcal{S}$ contains zero divisors.

Perform a similarity so that the restriction of $ESE$ to the range of $E$ consists of multiples of unitaries. Decompose members of $\mathcal{S}$ as block matrices with respect to the ranges of $E$ and $I - E$; each $(1,1)$ entry in this block decomposition is a multiple of a unitary operator.

Suppose there are no zero divisors in $\mathcal{S}$. By Theorem 2.2 above, there is then a $k$ such that
\[
r(AB) \leq kr(A)r(B)
\]
for all $A$ and $B$ in $\mathcal{S}$. We will show that the transitivity of $\mathcal{S}$ implies there is no such $k$.

Since $\mathcal{S}$ is transitive, there is a $T$ other than 0 in $\mathcal{S}$ whose $(1,2)$ block entry $T_{12}$ is not zero. Then $T$'s $(1,1)$ block $T_{11}$ is also different from zero, since otherwise $ET$
would be a non-trivial nilpotent member of $\mathcal{S}$. Multiply $T$ by an appropriate scalar so that $T_{11}$ is unitary, and call the resulting operator $R$. The operator $ER$ has the block form

$$
\begin{pmatrix}
R_{11} & R_{12} \\
0 & 0
\end{pmatrix}
$$

with $R_{11}$ unitary and $R_{12} \neq 0$. Choose a vector $g$ in the range of $I - E$ such that $R_{12}g \neq 0$.

Let $s$ and $t$ be positive numbers, to be specified later. Fix any unit vector $f$ in the range of $E$. By the transitivity of $\mathcal{S}$, we can find $S$ in $\mathcal{S}$ so that $Sf$ is within $t$ of $sf$. Then $\|S_{11}f\| < t$ and $\|S_{21}f - sg\| < t$.

Since $f$ is a unit vector and $S_{11}$ is a multiple of a unitary operator, it follows that $\|S_{11}\| < t$.

Let $A = SE$ and $B = ER$. Then $r(A) = r(S_{11}) < t$ and $r(B) = r(U_{11}) = 1$. Also,

$$
BA = ERSE = \begin{pmatrix}
R_{11}S_{11} + R_{12}S_{21} & 0 \\
0 & 0
\end{pmatrix}.
$$

Hence,

$$
BAf = (R_{11}S_{11} + R_{12}S_{21})f.
$$

By choosing $t$ sufficiently small, $BAf$ can be made as close to $sR_{12}g$ as desired (by the above inequalities). Thus the norm of $BA$ may be made arbitrarily large by choosing $s$ sufficiently large.

This leads to a contradiction, as follows: By the above, $r(A) < t$ and $r(B) = 1$. Note that $r(BA) = \|BA\|$, since the $(1,1)$ block of $BA$ is a multiple of a unitary operator and the other entries are 0. Thus choosing $s$ sufficiently large yields $r(BA) > kr(B)r(A)$.

**Theorem 2.4.** If $\mathcal{S}$ is a transitive semigroup of linear transformations on which the spectral radius is $k$-submultiplicative for some $k$, then there is an invertible transformation $A$ such that $\{A^{-1}SA : S \in \mathcal{S}\}$ consists of multiples of unitary transformations.

**Proof.** Let $\mathcal{J}$ be the closure of $\mathbb{C}\mathcal{S}$. Note that continuity of the spectral radius implies that $\mathcal{J}$ also satisfies the hypothesis. Assume that the spectral radius is not multiplicative. Then Theorem 2.3 together with this assumption implies that there are non-zero $A$ and $B$ in $\mathcal{S}$ with $AB = 0$. Thus the set $BSA$ consists of operators $X$ with $X^2 = 0$. Since $BSA \neq \{0\}$ (by the irreducibility of $\mathcal{S}$), the set $N$ of nilpotent members of $\mathcal{S}$ is different from $\{0\}$. Now the $k$-submultiplicativity of spectral radius on $\mathcal{S}$ implies that $N$ is a semigroup ideal of $\mathcal{S}$. Since $N$ is triangularizable by Levitzki’s Theorem ([1], Theorem 2.1.7 of [3]), and since every non-trivial ideal of an irreducible semigroup is irreducible (see Lemma 2.1.10 of [3]), this is the desired contradiction.

**Corollary 2.5.** If the spectral radius is $k$-submultiplicative on a transitive semigroup of linear transformations, then the spectral radius is multiplicative on the semigroup.

**Proof.** This follows immediately from Theorem 2.4.
As shown in [2], for a closed homogeneous semigroup \( S \) with no zero divisors and a number \( k \geq 1 \), the following statements are equivalent:

1. \( r(AB) \leq kr(A)r(B) \) for all \( A \) and \( B \) in \( S \),
2. \( r(AB) \geq \frac{1}{k}r(A)r(B) \) for all \( A \) and \( B \) in \( S \).

The following result shows that for every \( k \), there is a semigroup ‘sharply’ satisfying (1) and (2). (An example with \( k = 4 \) is contained in [2].)

**Proposition 2.6.** Let \( m \geq 1 \) be given. There is an irreducible semigroup \( S \) with no zero divisors that satisfies (1) and (2) above for \( k = m \) but does not satisfy those inequalities for any \( k < m \).

**Proof.** For a non-zero column vector

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

and a positive number \( v \),

we say that \( x \) is of variation \( v \) if \( \frac{x_i}{x_j} \leq v \) for all \( i \) and \( j \). (Note that this implies \( v \geq 1 \) and \( x_i \neq 0 \) for all \( i \).)

Let \( \chi_v \) be the set of all column vectors of variation \( v \). It is easily seen that \( \chi_v \) is a closed set in \( \mathbb{C}^n \) and \( \alpha x \in \chi_v \) if \( x \in \chi_v \) and \( 0 \neq \alpha \in \mathbb{C} \). Now define \( S_v \) to be the set of all matrices with exactly one column from \( \chi_v \) whose other columns are all zero.

Each member of \( S_v \) is of the form \( xe_i^* \), where \( x \in \chi_v \) and where \( \{e_j\} \) are the basis unit columns (with the \( j \)-th component of \( e_j \) equal to 1). If \( x \) and \( y \) are members of \( \chi_v \) and \( i, j \) are integers in \([1, n]\), then

\[
(xe_i^*)(ye_j^*) = (e_i^*y)xe_j^*.
\]

This shows that \( S_v \) is a semigroup. Since

\[
r(xe_i^*) = |x_i| = |e_i^*x|,
\]

where \( x_i \) is the \( i \)-th component of \( x \), we have

\[
r((xe_i^*)(ye_j^*)) = |e_i^*y| \cdot |x_j| = |y_i| \cdot |x_j|
\]

\[
= \left| \frac{y_j}{y_i} \right| \cdot \left| \frac{x_j}{x_i} \right| \cdot |x_i| \cdot |y_j|
\]

\[
\leq v^2 r(xe_i^*)r(ye_j^*),
\]

which shows that \( S \) satisfies (1) above with \( k = v^2 \). Thus (2) is also satisfied since \( S \) contains no zero divisors. It is easy to see that no \( k < v^2 \) would do: just consider

\[
A = \begin{pmatrix}
  1 & 0 & 0 & \ldots & 0 \\
  v & 0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

and \( B = \begin{pmatrix}
  0 & v & 0 & \ldots & 0 \\
  0 & 1 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 1 & 0 & \ldots & 0
\end{pmatrix}.\]

Then \( r(A) = r(B) = 1 \), and \( r(AB) = v^2 \). \( \square \)
3. Semigroups of compact operators

One of the above theorems can be extended to semigroups of compact operators on Hilbert spaces.

**Theorem 3.1.** If $S$ is a transitive semigroup of bounded linear operators on a Hilbert space, and if $S$ contains a compact operator other than 0, then the spectral radius is not $k$-submultiplicative on $S$ for any $k$.

**Proof.** Suppose there is a $k$ such that
\[
r(AB) \leq kr(A)r(B)
\]
for all $A$ and $B$ in $S$. Since a non-zero semigroup ideal of a transitive semigroup is transitive, we can and do assume that $S$ consists of compact operators. Turovski’s Theorem ([4], [3], Theorem 8.1.11) states that a semigroup of quasinilpotent compact operators has a non-trivial invariant subspace, so $S$ must contain operators that are not quasinilpotent.

Enlarge $S$ by including all scalar multiples of the members of $S$ and then taking the uniform closure; let $J$ be the enlarged semigroup. Since spectral radius is continuous on the collection of compact operators,
\[
r(AB) \leq kr(A)r(B)
\]
for $A, B \in J$. By Corollary 8.1.12 of [3], $J$ contains finite-rank operators other than 0, and Lemma 8.1.15 of [3] yields a finite-rank idempotent $E$ in $J$ of minimal non-zero rank.

It follows from the finite-dimensional case that the restriction of $JE \backslash \{0\}$ to the range of $E$ is simultaneously similar to a subgroup of multiples of unitary operators.

Now the proof proceeds along the lines of the last part of the proof of Theorem 2.3 above. Since $E$ is of finite rank, and thus is not the identity operator, the transitivity of $J$ contradicts the $k$-submultiplicativity of the spectral radius.

\[\square\]

**Corollary 3.2.** If $S$ is a semigroup of bounded linear operators on a Hilbert space that contains a compact operator other than 0, and if the spectral radius is $k$-submultiplicative on $S$ for some $k$, then there is a closed subset of the Hilbert space that is invariant under all the operators in $S$ and is different from $\{0\}$ and the entire space.

**Proof.** If $S$ did not have such a proper, closed, invariant set, then $S$ would be transitive, contradicting the previous theorem.

\[\square\]

**Proposition 3.3.** Let $k \geq 1$. There is an irreducible semigroup of finite-rank operators on $l^2$ which has no zero divisors and satisfies
\[
\frac{1}{k}r(A)r(B) \leq r(AB) \leq kr(A)r(B)
\]
for all $A$ and $B$ in the semigroup.

**Proof.** The example $S_v$ of Proposition 2.3 with $v = \sqrt{k}$ does not occur in infinite dimensions, since $l^2$ contains no vectors of any fixed variation. There are known semigroups of finite-rank operators, with no zero divisors on $l^2$, with multiplicative spectral radius. Let $J$ be such a semigroup (e.g. see Example 8.6.6 of [3]). Let $S_v$
be the example of Proposition 2.6 with \( v = \sqrt{k} \) on a space of dimension 2. Now let \( S = S_v \otimes J \) (on \( l^2 \otimes l^2 \), identified with \( l^2 \)).

The irreducibility of \( S \) and its lack of zero divisors follow from the same properties of \( S_v \) and \( J \). All that is left to prove is that if \( S_i, i = 1, 2 \) are semigroups satisfying

\[
\frac{1}{k_1} r(A) r(B) \leq r(AB) \leq k_1 r(A) r(B)
\]

for all \( A \) and \( B \) in \( S_i, i = 1, 2 \), then the members of \( S_1 \otimes S_2 \) satisfy

\[
\frac{1}{k_1 k_2} r(A) r(B) \leq r(AB) \leq k_1 k_2 r(A) r(B).
\]

But since \( r(S \otimes T) = r(S)r(T) \) for all \( S \) and \( T \), we have, for \( A_1, B_1 \) in \( S_1 \) and \( A_2, B_2 \) in \( S_2 \), that

\[
r((A_1 \otimes A_2)(B_1 \otimes B_2)) = r(A_1 B_1 \otimes A_2 B_2) = r(A_1 B_1) \cdot r(A_2 B_2)
\]

\[
\leq k_1 k_2 r(A_1) r(B_1) \cdot r(A_2) r(B_2)
\]

\[
= k_1 k_2 r(A_1) r(B_1) r(A_2) r(B_2)
\]

\[
= k_1 k_2 r(A_1 \otimes A_2) \cdot r(B_1 \otimes B_2).
\]

The inequality involving \( \frac{1}{k_1 k_2} \) can be similarly established. \( \square \)

References


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