DENSELY ALGEBRAIC BOUNDS FOR THE EXPONENTIAL FUNCTION

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Abstract. An upper bound for $e^x$ that implies the inequality between the arithmetic and geometric means is generalized with the introduction of a new parameter $n$. The new upper bound is smoothly and densely algebraic in $n$, and valid for $-b < x < 1$ for arbitrarily large positive $b$ provided that $n (> 1)$ is sufficiently close to 1. The range of its validity for negative $x$ is investigated through the study of a certain family of quadrinomials.

1. Introduction

In §4.2 of the classical treatise [1] the inequality between the arithmetic and geometric means is deduced from

$$1 + x \leq e^x.$$ 

This is the proof of “Pólya’s dream” [5]. With a change of variable this can be rewritten as

$$e^x \leq \frac{1}{1 - x}$$

for $x < 1$. In this paper, we shall establish the following generalization of which (1.1) is the case $n = 1$. For convenience, we let

$$U(n, x) = 1 - \frac{1}{n} + \frac{1}{n} \left( \frac{1 + \left( \frac{1}{n} \right) x}{1 - \frac{x}{n}} \right)^n.$$ 

Theorem 1.1. For real $n \geq 1$ and

$$-\frac{n}{n - 1} < x < n,$$

we have

$$e^x \leq U(n, x)$$

with equality if and only if $x = 0$. Moreover, for $0 \leq x < 1$ and $1 \leq n \leq 2$ we have

$$e^x \leq U(n, x) \leq \frac{1}{1 - x},$$

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and for $x < 0$ and $0 < n \leq 1$ we have

$$e^x \leq \frac{1}{1-x} \leq U(n, x).$$

Here $U(n, x)$ is smoothly and densely algebraic in $n$ in the sense that it is an algebraic function of $x$ whenever $n$ is rational and this algebraic function changes by arbitrarily small amounts on compact sets for sufficiently small rational changes in $n$.

Another upper bound for $e^x$ that generalizes (1.1) is Karamata’s [2]

$$e^x \leq \sum_{k=0}^{n-1} \frac{x^k}{k!} + \frac{x^n}{n!} \frac{n}{n-x}$$

provided $n$ is a positive integer and $0 \leq x < n$. This is tighter for $0 < x < n$ but often fails for $x < 0$. Also, since $n-1$ is the upper limit of the summation here, it is not smoothly algebraic in $n$. Another tighter bound for $e^x$ is Sewell’s [4]

$$e^x \leq \left(1 + \frac{x}{n}\right)^{n+(x/2)}$$

for $n$ a positive integer and $x \geq 0$. This is not algebraic in $x$, and can fail for $x < 0$.

The change of variable given by replacing $x$ with

$$\frac{n(x-1)}{n+x-1}$$

plays an important role here. In fact, it is immediate that the first part of Theorem 1.1 is equivalent to

**Theorem 1.2.** For real $n \geq 1$ and $x > 0$ we have

$$(1.2) \quad \exp\left(\frac{n(x-1)}{n+x-1}\right) \leq \frac{n-1+x^n}{n}$$

with equality if and only if $x = 1$.

Our proof for Theorem 1.2 will be in the spirit of §2.15 of [1] where some “fundamental inequalities” that also lead to the inequality between the arithmetic and geometric means, including

$$x^r - 1 > r(x-1), \quad r > 1, \quad x > 0, \quad x \neq 1,$$

are proved for all real $r > 1$ by first establishing them for rational numbers and then taking limits. Also the proof of Theorem 1.2 will rely on the polynomial

$$K(p, q, x) := q^2(x^{p+q} - x^p) + q(p-q)(x^p - x^{p-q}) + p(p-q)(x^{p-q} - 1),$$

where both $p$ and $q$ are integers and $p > q$. For our convenience, we write $K(x) = K(p, q, x)$.

For a survey of rational bounds for $e^x$ see pp. 266-270 of [3]. The examination of the inequalities between $(1-x)^{-1}$ and $U(n, x)$ from the point of view of their power series expansions leads to questions about a certain sequence of polynomials; see §3. We remark here that for $0 \leq x < 1$ the power series of $U(n, x) - (1-x)^{-1}$ about $x = 0$ is

$$\frac{(n-1)(n-2)}{2n} x^2 + \frac{1}{6} \left(2 + \frac{6}{n^2} - \frac{6}{n} - 3n + n^2\right) x^3 + \cdots.$$
So for $n > 2$ and small $x > 0$ we have $U(n, x) > (1 - x)^{-1}$. For $0 \leq x < 1$ and $n$ large some simple asymptotics (details omitted) show that $(1 - x)^{-1} < U(n, x)$ for $x \leq 1 - e^n$ for a fixed $c > 1/2$, while $U(n, x) < (1 - x)^{-1}$ for $x \geq 1 - e^n$ for a fixed $c \leq 1/2$.

2. Proofs

We begin with a lemma that leads to the inequalities between $(1 - x)^{-1}$ and $U(n, x)$, and then proceed to the inequalities between $e^x$ and $U(n, x)$. This latter inequality is of course immediate when $x < 1$ and $(1 - x)^{-1} \leq U(n, x)$.

**Lemma 2.1.** (a) Let $c = 1 - 1/n$, where $1 \leq n \leq 2$ and $0 \leq x < 1$. Then

$$l_1 := \frac{nc}{1 + cx} + \frac{1}{1 - \frac{x}{n}} \leq \frac{nc}{1 + ncx} + \frac{1}{1 - x} =: r_1.$$  

(b) Let $0 < a \leq b$, $1 \leq b$, and $x \geq 0$. Then

$$l_2 := \frac{a}{1 + \frac{a}{b}x} + \frac{b}{1 + bx} \leq \frac{a}{1 + ax} + \frac{b}{1 + x} =: r_2.$$  

**Proof.** For (a) we have

$$r_1 - l_1 = \frac{(n-1)(1 + cn)x(2 - n + cx(1 + n))}{(x - 1)(x - n)(1 + cx)(1 + cnx)} \geq 0,$$

while for (b) we have

$$r_2 - l_2 = \frac{(b-1)(b-a)x(a + b + a(1+b)x)}{(1 + x)(1 + ax)(b + ax)(bx + 1)} \geq 0.$$  

$\square$

We now proceed to the right side of the second part of Theorem 1.1. Observe that

$$n \log (1 + cx) - n \log \left(1 - \frac{x}{n}\right) \leq \log (1 + ncx) - \log (1 - x)$$

since there is equality when $x = 0$, and the corresponding inequality between the derivatives of each side follows from (a) of Lemma 2.1. Hence

$$\left(\frac{1 + cx}{1 - \frac{x}{n}}\right)^n \leq \frac{1 + ncx}{1 - x}$$

and we obtain $U(n, x) \leq 1/(1 - x)$. For the right side of the third part of Theorem 1.1 a similar argument using (b) of Lemma 2.1 yields

$$\left(\frac{1 + \frac{a}{b}x}{1 + x}\right)^b \leq \frac{1 + ax}{1 + bx}.$$  

Here we may take $b = \frac{1}{n}$ and $a = \frac{1}{n} - 1$ for $0 < n \leq 1$, so

$$\frac{1 - (n - 1)x}{1 + x} \leq \left(\frac{1 - (1 - \frac{1}{n})x}{1 + \frac{x}{n}}\right)^n.$$  

Upon replacing $x$ by $-x$ (so that $x \leq 0$), we obtain

$$\frac{1}{1 - x} \leq U(n, x).$$
For the proof of Theorem 1.2 (and hence of the remaining first part of Theorem 1.1) we observe that (1.2) is equivalent to

\[(2.1) \quad g(x) := \frac{n(x-1)}{n + x - 1} \leq \log \left( \frac{n - 1 + x^n}{n} \right) =: f(x). \]

Since both sides of (2.1) are zero when \(x = 1\) we may apply the following lemma (proof omitted) to reduce it to an inequality not involving transcendental functions.

**Lemma 2.2.** Let \(f(x)\) and \(g(x)\) be differentiable functions on a finite or infinite interval \(I\) containing 1 such that \(f(1) = g(1)\), and such that \(g'(x) \geq f'(x)\) for \(x < 1\) and \(g'(x) \leq f'(x)\) for \(x > 1\). Then \(g(x) \leq f(x)\).

Now

\[g'(x) = \frac{n^2}{(n - 1 + x)^2} \quad \text{and} \quad f'(x) = \frac{n x^{n-1}}{n - 1 + x}. \]

Replace \(n\) by \(p/q\) where both \(p\) and \(q\) are positive integers, \(p > q\). The change of variable \(x\) by \(x^s\) takes 1 to 1 and \((0, \infty)\) to \((0, \infty)\). To verify the hypothesis of Lemma 2.2 we need to show that \(H(x)\) has the same sign as \(x - 1\), where

\[H(x) := H(p, q, x) := f'(x) - g'(x) = \frac{p x^{b-q}}{p + q(x^p - 1)} - \frac{p^2}{(p + q(x^p - 1))^2} = \frac{p K(x)}{(p - q + qx^b)(p - q + qx^q)^2} \]

and

\[K(x) = q^2(x^{b+q} - x^p) + q(p - q)(x^p - x^{b-q}) + p(p - q)(x^{p-q} - 1). \]

Using the identity

\[(2.2) \quad x^s - x^t = \left( \frac{x^s - 1}{x - 1} - \frac{x^t - 1}{x - 1} \right) (x - 1) \]

for \(s \geq t \geq 0\) and the expansion of the terms in (2.2) into geometric series, we see that \(K(x)\) is the product of \(x - 1\) with polynomials in \(x\), all of whose coefficients are nonnegative. Hence \(K(x)\) has the same sign as \(x - 1\). The inequality of the theorem for rational \(n\) now follows from Lemma 2.2. For real \(n \geq 1\), it follows by letting \(p/q \to n\) where \(p\) and \(q\) run through sequences of integers such that \(p > q \geq 1\). The strict inequality for \(x \neq 1\) follows from the fact that the functions in \(x\)

\[\exp \left( \frac{n(x - 1)}{n + x - 1} \right) \quad \text{and} \quad \frac{n - 1 + x^n}{n} \]

are strictly increasing for \(x > 0\).

### 3. A Sequence of Polynomials

It is possible that the \(U(n, x) \leq (1 - x)^{-1}\) inequality for \(1 \leq n \leq 2\) can be strengthened to an inequality between the corresponding power series coefficients. In fact, we can make a stronger conjecture. Write

\[\frac{1}{n - 1} (U(n, x) - (1 - x)^{-1}) = \sum_{k=2}^{\infty} P_k(n)x^k. \]

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Here \( P_2(n) = n - 2 \), \( P_3(n) = n^3 - 2n^2 - 6 \) and
\[ P_4(n) = n^5 - 5n^4 + 18n^3 - 48n^2 + 12n - 24. \]
The unique real zeros of \( P_2 \), \( P_3 \) and \( P_4 \) are
\( n = 2 \), \( n = 2.7776 \cdots \) and \( n = 3.5934 \cdots \), respectively. We conjecture that each \( P_k(n) \) for \( k \geq 4 \) is a monic polynomial of degree \( 2k-3 \) whose coefficients alternate in sign, and has a unique real root \( r_k \) that exceeds the real part of every other root of \( P_k(n) \). Moreover,
\[ 0 < r_{k+1} - r_k < 1 \quad \text{and} \quad \lim_{k \to \infty} (r_{k+1} - r_k) = 1. \]
It also seems that \( k! \) divides \( P_k(2) \).

To describe qualitatively the conjectural zero distribution of \( P_k(n) \) we employ polar coordinates \( r \) and \( \phi \) to describe a certain curve \( \theta \). It is the cardioid \( H \) given by \( r = 1 + \cos \phi \) together with that part of the circle \( C \) defined by \( r = 1/4 \) that lies outside of \( H \). The left vertical tangent to \( C \), call it \( T \), is tangent to the cardioid at two points, and (much more crudely) the \( \theta \)-curve is topologically equivalent to the Greek letter \( \theta \). Note that the cusp of \( H \) is inside of the circle \( C \). The conjecture is that the zeros of \( P_k(n) \) for \( k \) large lie on or very close to a curve similar (in the non-technical sense) to \( \theta \) that has the imaginary axis as a line of triple tangency analogous to \( T \). Also, about \( 1/3 \) of the zeros lie on or inside the part of the \( \theta \)-curve that corresponds to the union of its \( C \) part with that part of \( H \) that lies inside of the full circle \( C \).

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