CONTINUITY OF THE MAXIMAL OPERATOR
IN SOBOLEV SPACES

HANNES LUIRO

(Communicated by Juha M. Heinonen)

Abstract. We establish the continuity of the Hardy-Littlewood maximal op-
erator on Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. As an auxiliary tool we
prove an explicit formula for the derivative of the maximal function.

1. Introduction

The classical Hardy-Littlewood maximal operator $M$ is defined on $L^1_{\text{loc}}(\mathbb{R}^n)$ by
setting for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$

\begin{align}
Mf(x) = \sup_{r > 0} \int_{B(x,r)} |f(y)| \, dy = \sup_{r > 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y)| \, dy,
\end{align}

for every $x \in \mathbb{R}^n$; here $m$ denotes the Lebesgue measure in $\mathbb{R}^n$ and $B_r = B(0, r)$. The
theorem of Hardy, Littlewood and Wiener asserts that $M$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. This theorem is one of the cornerstones of harmonic
analysis. Applications e.g. to the study of Sobolev-functions indicate that it is also
useful to know how it preserves differentiability properties of functions. Quite re-
cently, Kinnunen observed [K] that $M$ is bounded on the Sobolev-space $W^{1,p}(\mathbb{R}^n)$, for $1 < p \leq \infty$. Extensions and related results can be found from e.g. [KL], [Ko],
[KS], [HO].

Continuity of the maximal operator in $L^p(\mathbb{R}^n)$ follows from its sublinearity
and boundedness. Because of boundedness in $W^{1,p}(\mathbb{R}^n)$, it is very natural to
ask whether the maximal operator is continuous in $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, or not. This question was posed in [HO, Question 3] where it was attributed to T. Iwaniec. In general, bounded non-sublinear operators need not be continuous. An important example of this kind of phenomenon is the result of Almgren and Lieb [AL] who proved that the (known to be bounded) symmetric rearrangement $R : W^{1,p}(\mathbb{R}^n) \mapsto W^{1,p}(\mathbb{R}^n)$ is not continuous when $1 < p < n$ and $n > 1$. On Sobolev-spaces, $M$ is not sublinear and the issue of the continuity of $M$ is not
trivial even though we know the boundedness.

Our main result (Theorem 4.1 below) is the positive answer to the question of
Iwaniec. A central role in our proof is played by a careful analysis of the set $Rf(x)$

Received by the editors June 4, 2004 and, in revised form, August 8, 2005.
2000 Mathematics Subject Classification. Primary 42B25, 46E35, 47H99.
Key words and phrases. Maximal function, Sobolev spaces, continuity, regularity.
The author was supported by the Academy of Finland, project 201015.
©2006 American Mathematical Society
Reverts to public domain 28 years from publication

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(see [2.1] below), which consists of the radii \( r \) for which equality is achieved in [1]. As a useful auxiliary tool we establish in Theorem 3.1 an explicit formula for the derivative of the maximal function.

2. DEFINITIONS AND AUXILIARY RESULTS

Let us first introduce some notation. If \( A \subset \mathbb{R}^n \) and \( r \in \mathbb{R} \), we define
\[
d(r, A) := \inf_{a \in A} |r - a|, \quad \text{and} \quad A(\lambda) := \{ x \in \mathbb{R}^n : d(x, A) \leq \lambda \} \quad \text{for} \quad \lambda \geq 0.
\]

We endow \( W^{1,p}(\mathbb{R}^n) \) with the norm
\[
\|f\|_{1,p} = \|f\|_p + \|\nabla f\|_p,
\]
where \( \nabla f \) is the weak gradient of \( f \). Let us also denote by \( \|f\|_{p,A} \) the \( L^p \)-norm of \( \chi_A f \) for all measurable sets \( A \subset \mathbb{R}^n \).

The following new concept will be central in this work.

**Definition 2.1.** Let \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \). The set \( \mathcal{R}f(x) \) is defined as
\[
\mathcal{R}f(x) = \{ r \geq 0 : Mf(x) = \limsup_{r_k \to r} \int_{B(x,r_k)} |f(y)| \, dy, \text{ for some } r_k > 0 \}.
\]

**Remarks.** We comment on the above definition and the properties of the sets \( \mathcal{R}f(x) \). First, the definition clearly implies that \( \mathcal{R}f(x) \) is always closed. Moreover, for fixed \( x \in \mathbb{R}^n \) define \( u_x : [0, \infty) \to \mathbb{R} \) by
\[
u_x(0) = |f(x)| \text{ and } u_x(r) = \int_{B(x,r)} |f(y)| \, dy \text{ when } r \in (0, \infty).
\]

First of all, the functions \( u_x \) are continuous for almost all \( x \). The continuity on \((0, \infty)\) is clearly true for all \( x \) and at 0 it follows a.e., because almost every point \( x \in \mathbb{R}^n \) is a Lebesgue point for \( f \). Moreover, by H"older’s inequality we have
\[
u_x(r) \leq \|f\|_p (m(B_r))^{1/q - 1},
\]
where \( q \) is the conjugate exponent of \( p \), and hence \( \lim_{r \to \infty} u_x(r) = 0 \).

These facts together imply that, for almost all \( x \), the function \( u_x \) has at least one maximum point in \([0, \infty)\). Furthermore, they guarantee that for all \( x \in \mathbb{R}^n \) the set \( \mathcal{R}f(x) \) is nonempty and
\[
Mf(x) = \int_{B(x,r)} |f(y)| \, dy \text{ if } r \in \mathcal{R}f(x) \text{ and } r > 0, \forall x \in \mathbb{R}^n, \text{ and}
\]
\[
Mf(x) = |f(x)| \text{ for almost every } x \text{ such that } 0 \in \mathcal{R}f(x).
\]

Also, it is useful to observe that for every \( R > 0 \) (assuming \( f \neq 0 \)) it is true that
\[
sup\{r : r \in \mathcal{R}f(x), x \in B(0, R)\} < \infty.
\]

The following lemma tells us how the sets \( \mathcal{R}f(x) \) and \( \mathcal{R}g(x) \) are related to each other, especially when \( \|f - g\|_p \) is small.

**Lemma 2.2.** Let \( 1 \leq p < \infty \) and suppose \( f_j \rightharpoonup f \) in \( L^p(\mathbb{R}^n) \) when \( j \to \infty \). Then for all \( R > 0 \) and \( \lambda > 0 \) it holds that
\[
m(\{x \in B(0, R) : \mathcal{R}f_j(x) \nsubseteq \mathcal{R}f(x)(\lambda)\}) \to 0 \quad \text{if } j \to \infty.
\]
CONTINUITY OF THE MAXIMAL OPERATOR IN SOBOLEV SPACES 245

Proof: First we indicate why the above set is always Lebesgue-measurable when \( f \) and all the functions \( f_j \) are in \( L^p(\mathbb{R}^n) \). The continuity of the average functions \( u_x \), for almost every \( x \), is used as a main tool in the following argument. Let the set \( \mathcal{N} \) consist of those points which are not Lebesgue points of any of the functions \( f_j \) or \( f \), especially, \( m(\mathcal{N}) = 0 \). Moreover we denote by \( Q_+ \) the set of positive rationals. Now we can write

\[
\{ x : \mathcal{R}f_j(x) \not\in \mathcal{R}f(x)_\lambda \} \setminus \mathcal{N} = \bigcup_{i=1}^{\infty} \bigcap_{m=1}^{\infty} \{ x : \exists r > 0 \text{ s.t. } d(r, \mathcal{R}f(x)) > \lambda + \frac{1}{i} \text{ and } Mf_j(x) < \int_{B(x,r)} f_j + \frac{1}{m} \}
\]

\[
= \bigcup_{i=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{q \in Q_+} \left[ \{ x : d(q, \mathcal{R}f(x)) > \lambda + \frac{1}{i} \} \cap \{ x : Mf_j(x) < \int_{B(x,q)} f_j + \frac{1}{m} \} \right].
\]

From this we conclude that it is enough to prove that the set \( \{ x : d(q, \mathcal{R}f(x)) > \lambda \} \) is measurable for arbitrary \( q \) and \( \lambda \). Using the same reasoning as above, especially the continuity of the expression \( \int_{B(x,r)} |f| \) as a function of \( r \), we write that

\[
\{ x : d(q, \mathcal{R}f(x)) > \lambda \} = \bigcup_{k=1}^{\infty} \bigcap_{q' \in Q_+ \cap [q-\lambda,q+\lambda]} \{ x : Mf(x) > \int_{B(x,q')} f + \frac{1}{k} \}. \]

This implies the measurability.

Then we are ready to prove the lemma. It is sufficient to prove the claim in the case where both \( f \) and \( f_j \) are nonnegative, because \( \mathcal{R}f(x) = \mathcal{R}|f|(x) \). Observe that \( \mathcal{R}f(x) \in [0, \infty) \) for all \( x \) if \( f \equiv 0 \) a.e., whence this case is trivial. Let \( \lambda > 0, R > 0 \) and \( \varepsilon > 0 \). For almost every \( x \in B(0, R) \) there exists a natural number \( i(x) \in \mathbb{N} \) so that

\[
(3) \quad \int_{B(x,r)} f(y) dy < Mf(x) - \frac{1}{i(x)} \text{, when } d(r, \mathcal{R}f(x)) > \lambda.
\]

This can be seen in the following way: If the claim is not true there is a sequence of radii \( (r_k)_{k=1}^{\infty} \) so that

\[
\int_{B(x,r_k)} f(y) dy \to Mf(x) \text{ and } d(r_k, \mathcal{R}f(x)) > \lambda.
\]

By moving to a subsequence, if needed, we may assume that \( r_k \to r \) as \( k \to \infty \), because \( (2) \) implies that the sequence \( (r_k)_{k=1}^{\infty} \) must be bounded. It follows that \( r \in \mathcal{R}f(x) \). This is a contradiction, since obviously \( r \) satisfies \( d(r, \mathcal{R}f(x)) \geq \lambda \).

From \( (3) \) we conclude that there exists \( i \in \mathbb{N} \) so that

\[
B(0, R) \subset \{ x : \int_{B(x,r)} f(y) dy < Mf(x) - \frac{1}{i} \text{, if } d(r, \mathcal{R}f(x)) > \lambda \} \cup E =: A \cup E,
\]

where \( E \) is a measurable set with \( m(E) < \varepsilon \). The weak type \((1,1)\)-estimate for the maximal operator implies that there exists \( j_0 \in \mathbb{N} \) so that

\[
m(\{ x \in B(0, R) : |M(f_j - f)| \geq \frac{1}{4i} \}) < \varepsilon \text{ when } j \geq j_0.
\]
For all \(j\) we observe that
\[
A \subset \left\{ x : \int_{B(x,r)} f_j(y) \, dy < Mf(x) - \frac{1}{2i}, \text{ if } d(r, Rf(x)) > \lambda \right\}
\]
\[
\cup \left\{ x : \int_{B(x,r)} f(y) \, dy - \int_{B(x,r)} f_j(y) \, dy \geq \frac{1}{4i}, \text{ for some } r, d(r, Rf(x)) > \lambda \right\}
\]
=: \(A_j \cup B_j\).

Continuing the same reasoning, and using the fact that \(|Mf(x) - Mf_j(x)| \leq |M(f - f_j)(x)|\), we get
\[
A_j \subset \left\{ x : \int_{B(x,r)} f_j(y) \, dy < Mf_j(x) - \frac{1}{4i}, \text{ if } d(r, Rf(x)) > \lambda \right\}
\]
\[
\cup \left\{ x : |M(f - f_j)(x)| \geq \frac{1}{4i} \right\}
\]
=: \(C_j \cup D_j\).

Now
\[
C_j \subset \left\{ x : Rf_j(x) \subset Rf(x) \right\}.
\]

By combining the above observations, we conclude that for all \(j\)
\[
B(0, R) \subset \left\{ x : Rf_j(x) \subset Rf(x) \right\} \cup E \cup D_j \cup B_j.
\]

Observe finally that \(B_j \subset D_j\) and, by our choice of \(j_0\) we have \(m(D_j) < \varepsilon\) if \(j \geq j_0\), and therefore
\[
m(\{ x \in B(0, R) : Rf_j(x) \not\subset Rf(x) \}) < 2\varepsilon,
\]
if \(j \geq j_0\). \(\square\)

Let us introduce more notation. Assume that \(f \in L^p(\mathbb{R}^n), 1 \leq p < \infty\). Let \(e_i\) be one of the standard basevectors of \(\mathbb{R}^n\). For all \(h \in \mathbb{R}\), \(|h| > 0\), we define the functions \(f^i_h\) and \(f^i_{\tau(h)}\) by setting
\[
f^i_h(x) = \frac{f(x + he_i) - f(x)}{|h|} \quad \text{and} \quad f^i_{\tau(h)}(x) = f(x + he_i).
\]

Now we know that \(f^i_{\tau(h)} \to f\) in \(L^p(\mathbb{R}^n)\) when \(|h| \to 0\) and, if \(p > 1\), for functions \(f \in W^{1,p}(\mathbb{R}^n)\) we have (see [GFT, 7.11]) that \(f^i_h \to D_if\) in \(L^p(\mathbb{R}^n)\) when \(|h| \to 0\).

**Corollary 2.3.** Let \(f \in L^p(\mathbb{R}^n), 1 \leq p < \infty\). Then for all \(i, 1 \leq i \leq n\), \(R > 0, \lambda > 0\) one has
\[
m(\{ x \in B(0, R) : Rf(x) \not\subset Rf(x)_{(\lambda)} \} \quad \text{or} \quad Rf^i_{\tau(h)}(x) \not\subset Rf(x)_{(\lambda)}) \xrightarrow{h \to 0} 0.
\]

**Proof.** Now \(f^i_{\tau(h)} \to f\) and as a consequence of Lemma 2.2 it is clearly sufficient to prove that
\[
m(\{ x \in B(0, R) : Rf(x) \not\subset Rf^i_{\tau(h)}(x)_{(\lambda)} \} \to 0 \text{ as } h \to 0.
\]

But this also follows easily from Lemma 2.2 because for \(|h| < 1\) one has that
\[
\{ x \in B_R : Rf(x) \not\subset Rf^i_{\tau(h)}(x)_{(\lambda)} \}
\]
\[
= \{ x \in B_R : Rf^i_{\tau(-h)}(x + he_i) \not\subset Rf(x + he_i)_{(\lambda)} \}
\]
\[
\subset \{ y \in B_{R+1} : Rf^i_{\tau(-h)}(y) \not\subset Rf(y)_{(\lambda)} \} - he_i. \quad \square
\]
Remark. The previous corollary will become useful after the following observation.
Let us denote by

\[ \pi(A, B) := \inf\{\delta > 0 : A \subset B(\delta) \text{ and } B \subset A(\delta)\} \]

the Hausdorff distance of the sets \( A \) and \( B \). Let \( f \) be in \( L^p(\mathbb{R}^n) \). With the new notation, the corollary says that

\[ m(\{x \in B_R : \pi(\mathcal{R}f(x), \mathcal{R}f(x + he_i)) > \lambda\}) \rightarrow 0 \text{ when } h \rightarrow 0. \]

Therefore we easily infer that there is a sequence \((h_k)_{k=1}^{\infty}, h_k > 0 \) with \( h_k \rightarrow 0 \), and such that \( \pi(\mathcal{R}f(x), \mathcal{R}f(x + h_k e_i)) \rightarrow 0 \) as \( k \rightarrow \infty \) for almost every \( x \in B_R \). This is the decisive fact needed in the following section.

3. A formula for the derivative of the maximal function

Let us denote by \( D_i f(x) \) the partial derivative \( \frac{\partial f}{\partial x_i} \).

Theorem 3.1. Let \( f \in W^{1,p}(\mathbb{R}^n), 1 < p < \infty \). Then we have for almost all \( x \in \mathbb{R}^n \) that

\begin{align*}
(1) \quad & D_i Mf(x) = \int_{B(x,r)} D_i f(y) \, dy \quad \text{for all } r \in \mathcal{R}f(x), r > 0, \quad \text{and} \\
(2) \quad & D_i Mf(x) = D_i |f|(x) \quad \text{if } 0 \in \mathcal{R}f(x).
\end{align*}

Proof. It is sufficient to prove the claim for nonnegative functions, because \( Mf = M|f| \) and \( |f| \in W^{1,p}(\mathbb{R}^n) \) if \( f \in W^{1,p}(\mathbb{R}^n) \). Let \( R > 0 \). We start by choosing a sequence \((h_k)_{k=1}^{\infty}, h_k > 0 \) and \( h_k \rightarrow 0 \), so that \( \pi(\mathcal{R}f(x), \mathcal{R}f(x + h_k e_i)) \rightarrow 0 \) as \( k \rightarrow \infty \) for almost all \( x \in B_R \) (see the Remark after Corollary 2.3). Then we have

\begin{align*}
(i) \quad & \|D_i Mf - (Mf)_{h_k}\|_{p,B_R} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \\
(ii) \quad & \|D_i f - f^i_{h_k}\|_{p,B_R} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \\
(iii) \quad & \|M(D_i f - f^i_{h_k})\|_{p,B_R} \rightarrow 0 \quad \text{as } k \rightarrow \infty.
\end{align*}

Now, by extracting a subsequence if needed, we may assume that the convergences above are true pointwise almost everywhere as well. Moreover, we recall that the set

\[ \{x \in \mathbb{R}^n : \exists k \in \mathbb{N} \text{ s.t. } 0 \in \mathcal{R}f(x + h_k e_i) \text{ with } Mf(x + h_k e_i) \neq f(x + h_k e_i)\} \]

has measure zero as a countable union of the sets having measure zero. Let \( x \in B_R \) be a Lebesgue point of both \( f \) and \( D_i f \) outside the union of all these unwanted sets of measure zero (in particular, the pointwise analogies of (i)–(iii) hold at \( x \)) and let \( r \in \mathcal{R}f(x) \).
Now, because $\pi(Rf(x), Rf(x + h_k e_i)) \to 0$, we find $r_k \in Rf(x + h_k e_i)$ so that $r_k \to r$ when $k \to \infty$. If $r > 0$ we can estimate:

$$D_iMf(x) = \lim_{k \to \infty} \frac{1}{h_k} (Mf(x + h_k e_i) - Mf(x)) \leq \lim_{k \to \infty} \frac{1}{h_k} \left( \int_{B(x+h_k e_i, r_k)} f(y) \, dy - \int_{B(x, r_k)} f(y) \, dy \right) = \lim_{k \to \infty} \frac{1}{m(B(x, r_k))} \int_{B(x,r_k)} \frac{f(y + h_k e_i) - f(y)}{h_k} \, dy = \int_{B(x, r)} D_i f(y) \, dy.$$ 

The last equation holds, because $m(B_{r_k}) \to m(B_r)$ and

$$\chi_{B(x, r_k)} f_{h_k}^i \to \chi_{B(x, r)} D_i f \text{ in } L^1(\mathbb{R}^n) \text{ as } k \to \infty.$$ 

On the other hand, we get that

$$D_iMf(x) \geq \lim_{k \to \infty} \frac{1}{h_k} \left( \int_{B(x+h_k e_i, r)} f(y) \, dy - \int_{B(x, r)} f(y) \, dy \right) = \lim_{k \to \infty} \frac{1}{m(B(x, r))} \int_{B(x,r)} \frac{f(y + h_k e_i) - f(y)}{h_k} \, dy = \int_{B(x, r)} D_i f(y) \, dy.$$ 

Suppose instead that $r = 0$. The proof of the lower bound of $D_iMf(x)$ applies now, too, and we get that $D_iMf(x) \geq D_i f(x)$. If we have $r_k = 0$ for infinitely many $k$, we can decide straightforwardly that $D_iMf(x) = D_i f(x)$. If $r_k > 0$ starting from some $k_0$, we get by the same way as when studying the upper bound of $D_iMf(x)$ in the case $r > 0$ that

$$D_iMf(x) \leq \lim_{k \to \infty} \int_{B(x, r_k)} f_{h_k}^i(y) \, dy = D_i f(x),$$

because

$$\lim_{k \to \infty} \left| \int_{B(x, r_k)} f_{h_k}^i(y) \, dy - D_i f(x) \right| = \lim_{k \to \infty} \left| \int_{B(x, r_k)} (f_{h_k}^i(y) - D_i f(y)) \, dy \right| \leq \lim_{k \to \infty} M(f_{h_k}^i - D_i f)(x) = 0.$$ 

Now we have shown the claim in the ball $B(0, R)$. Since $R$ was arbitrary, this completes the proof. \qed

4. Continuity of the maximal operator in $W^{1,p}(\mathbb{R}^n)$

By using Theorem 3.1 and Lemma 2.2 we can establish quite easily our main result which verifies the continuity of the maximal operator in $W^{1,p}(\mathbb{R}^n)$.

**Theorem 4.1.** $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is continuous for all $1 < p < \infty$.

**Proof.** Let $f_j \to f$ in $W^{1,p}(\mathbb{R}^n)$ when $j \to \infty$. We have to show that $\|Mf_j - Mf\|_{1,p} \to 0$. Because we know the continuity of $M$ in $L^p(\mathbb{R}^n)$, it is sufficient to prove that
$\|D_i M f_j - D_i M f\|_p \to 0$ for all $i, 1 \leq i \leq n$. Also it is clear that we may assume the functions $f_j$ and $f$ to be nonnegative.

Let $\varepsilon > 0$ be fixed but arbitrary. We start by choosing $R > 0$ so that $\|2 M D_i f\|_{p, C_1} < \varepsilon$, where $C_1 = \mathbb{R}^n \setminus B(0, R)$. By absolute continuity we choose $\alpha > 0$ so that $\|2 M D_i f\|_{p, A} < \varepsilon$ always when $m(A) < \alpha$ and $A$ is a measurable subset of $B(0, R)$.

We let (compare with the remark after Definition 2.1) $u_x(r)$ stand for the average of $D_i f$ in the ball $B(x, r)$ and $u_x(0) = D_i f(x)$. As already observed, for almost every $x \in \mathbb{R}^n$ the functions $u_x$ are continuous on $[0, \infty)$ and converge to 0 when $r \to \infty$. Consequently for almost every $x$ the function $u_x$ is uniformly continuous on $[0, \infty)$ and therefore we can find $\delta(x) > 0$ such that $|u_x(r_1) - u_x(r_2)| < \frac{\varepsilon}{(m(B_R))^\frac{2}{p}}$ when $|r_1 - r_2| < \delta(x)$. Now we write that

$$B_R = \left( \bigcup_{i=1}^{\infty} \{ x \in B_R : \delta(x) > \frac{1}{i} \} \right) \cup \mathcal{N},$$

where $m(\mathcal{N}) = 0$. From that we infer that there exists $\delta > 0$ such that

$$m(\{ x \in B_R : |u_x(r_1) - u_x(r_2)| > \frac{\varepsilon}{(m(B_R))^\frac{2}{p}} \text{ for some } r_1, r_2, |r_1 - r_2| < \delta \}) =: m(C_2) < \frac{\alpha}{2}.$$

The set $C_2$ is easily shown to be measurable. Furthermore, Lemma 2.2 says that we can find $j_0$ so that

$$m(\{ x : \mathcal{R} f_j(x) \not\subseteq \mathcal{R} f(x)(\delta) \}) =: m(C') < \frac{\alpha}{2} \text{ when } j \geq j_0.$$

Then, let $j \geq j_0$ be fixed. It follows from Theorem 3.1 that almost everywhere in $\mathbb{R}^n$

$$|D_i M f_j(x) - D_i M f(x)| = \int_{B(x, r_1)} D_i f_j(y) dy - \int_{B(x, r_2)} D_i f(y) dy
\leq \int_{B(x, r_1)} D_i f(y) dy - \int_{B(x, r_2)} D_i f(y) dy
\leq M(D_i f_j - D_i f)(x) + \int_{B(x, r_1)} D_i f(y) dy - \int_{B(x, r_2)} D_i f(y) dy
\leq M(D_i f_j - D_i f)(x) + \int_{B(x, r_1)} D_i f(y) dy - \int_{B(x, r_2)} D_i f(y) dy
\leq M(D_i f_j - D_i f)(x) + \int_{B(x, 0)} D_i f(y) dy := D_i f(x).$$

This is obvious because for almost every $x$ it is true that $M f(x) \geq f(x)$, and by Theorem 3.1 $D_i M f(x) = D_i f(x)$ if $0 \in \mathcal{R} f(x)$. 

Now, if \( x \notin C_1 \cup C_2 \cup C_j \), we can pick \( r_1 \in \mathcal{R}f_j(x) \) and \( r_2 \in \mathcal{R}f(x) \) so that \( |r_1 - r_2| < \delta \). Our choice of \( \delta \) implies that

\[
    s := \left| \int_{B(x, r_1)} D_i f(y) \, dy - \int_{B(x, r_2)} D_i f(y) \, dy \right| < \frac{\varepsilon}{(m(B_R))^\frac{1}{p}}.
\]

If \( x \in C_1 \cup C_2 \cup C_j \), we estimate that \( s \leq 2MD_i f(x) \). Observe also that \( m(C_2 \cup C_j) < \alpha \).

Combining the above estimates it follows that

\[
    \|D_i Mf_j - D_i Mf\|_{p, \mathbb{R}^n} \leq \|M(D_i f_j - D_i f)\|_{p, \mathbb{R}^n} + \left\| \frac{\varepsilon}{(m(B_R))^\frac{1}{p}} \right\|_{p, B_R} + \|2MD_i f\|_{p, C_1} + \|2MD_i f\|_{p, C_2 \cup C_j}.
\]

The first term in the right-hand side of the inequality converges to zero when \( j \to \infty \). The rest of the terms are less than \( \varepsilon \), because of the choices of \( R \) and \( \alpha \). As \( \varepsilon \) was arbitrary we conclude that \( \|D_i Mf_j - D_i Mf\|_p \to 0 \) as \( j \to \infty \). The proof is complete.

**Remark.** One may ask, what kind of estimates we can find for the modulus of continuity of \( M \). Quite surprisingly, it turns out that there does not exist a function \( F : (0, \infty) \to (0, \infty) \) such that

\[
    \|Mf - Mg\|_{1, p} \leq F(\|f - g\|_{1, p}) \quad \text{for all } f, g \in W^{1, p}(\mathbb{R}^n).
\]

This is a consequence of the following two facts. First, \( M \) is not Lipschitz-continuous in \( W^{1, p}(\mathbb{R}^n) \), because this would imply that \( M \) is bounded in \( W^{2, p}(\mathbb{R}^n) \) which is not true (see for example \[KO\]). The philosophy of this phenomenon is that even the maximal function of a smooth positive function usually has angles in its graph. Second, the maximal operator is scale-invariant, thus \( M(cf) = cMf \) for all \( c > 0 \).

We thank Jani Onninen for pointing out the first fact.

**Acknowledgements**

I would like to thank Eero Saksman for the advice and support he has given me. I also thank Juha Kinnunen and Jani Onninen for their valuable comments on the manuscript.

**References**


Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), 40014 University of Jyväskylä, Finland
E-mail address: haluio@maths.jyu.fi