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A FEW UNCAUGHT UNIVERSAL HERMITIAN FORMS

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Abstract. We will complete the list of universal binary Hermitian forms over imaginary quadratic fields by investigating three Hermitian forms missed by previous researchers.

1. Introduction

The celebrated Four Square Theorem by Lagrange [4] states that any positive integer is a sum of four squares. This has been generalized by many mathematicians in many directions. In particular, Ramanujan [7] found all 54 positive definite diagonal quaternary quadratic forms representing all positive integers. We call a quadratic form *universal* if it represents all positive integers. Recently, several people studied the problem analogous to universal forms over the imaginary quadratic fields. Earnest and Khosravani [1] defined a universal Hermitian form as a positive definite one representing all positive integers, and they found 13 universal binary Hermitian forms over the imaginary quadratic fields of class number 1. More generally, Iwabuchi [2] investigated Hermitian lattices and found 9 lattices over the imaginary quadratic fields of class number bigger than 1.

We have discovered that they missed a few universal binary Hermitian forms. These are $x\bar{x}+3y\bar{y}$ in $\mathbb{Q}(\sqrt{-7})$ and $x\bar{x}+4y\bar{y}$ and $x\bar{x}+5y\bar{y}$ in $\mathbb{Q}(\sqrt{-2})$. These complete the Earnest-Khosravani-Iwabuchi list of binary universal Hermitian lattices over imaginary quadratic number fields.

2. Preliminaries

Let E be an imaginary quadratic field over $\mathbb{Q}$ and let $m > 0$ be a square-free integer for which $E = \mathbb{Q}(\sqrt{-m})$. We denote the $\mathbb{Q}$-involution by $^-$ and the ring of integers in E by $\mathcal{O}$.

Let V be an n-dimensional Hermitian space over E with nondegenerate Hermitian form H. A finitely generated $\mathcal{O}$-module L in V is called a *Hermitian lattice*.

In the case that E is a field of class number 1, $\mathcal{O}$ is a principal ideal domain and thus every Hermitian lattice is free. Then for a suitable basis $\{v_1, \cdots, v_n\}$ of L we can think of a Hermitian form as a function $f : \mathcal{O}^n \to \mathbb{Z}$ defined by $f(x_1, \cdots, x_n) = H(\sum x_i v_i) = \sum H(v_i, v_j) x_i \overline{x_j}$.

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We can regard (V,H) as a $2n$-dimensional quadratic space $(\tilde{V},B)$ over $\mathbb{Q}$ as defined in [3], where $B(x,y)=\frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(H(x,y))$. Similarly we can associate a quadratic lattice $\tilde{L}$ with the Hermitian lattice L. The ring $\mathcal{O}$ of integers in E has a basis $\{1,\omega\}$ as a $\mathbb{Z}$-module of rank 2, where $\omega=\frac{1+\sqrt{-m}}{2}$ if $m\equiv 3 \pmod 4$ and $\omega=\sqrt{-m}$ otherwise. Then $\tilde{f}(x_1,y_1,\cdots,x_n,y_n)=f(x_1+\omega y_1,\cdots,x_n+\omega y_n)$ is a quadratic form in $2n$ variables corresponding to the lattice $\tilde{L}$.

3. Main result

From now on we will consider only binary Hermitian forms and their corresponding quaternary quadratic forms. The binary Hermitian form f can be written as $f(x,y)=ax\bar{x}+bx\bar{y}+\bar{b}\bar{x}y+cy\bar{y}$ with $a,c\in\mathbb{N}$, $b=b_1+\omega b_2$, $b_1,b_2\in\mathbb{Z}$. We can reduce any binary Hermitian form to one whose coefficients satisfy:

$$a=\min f, -\frac{1}{2}a\le b_1\le\frac{1}{2}a, 0\le b_2\le\frac{1}{2}a, \text{and } a\le c.$$

Since any universal Hermitian form must represent 1, a must equal 1 and b must vanish for such a form. Thus we may consider only diagonal Hermitian forms $x\bar{x}+cy\bar{y}$.

In [1], the screening process was performed by using representation of the integers 1 through 5. But exactly three forms were missed and all of them are universal.

Theorem. *The binary Hermitian forms $x\bar{x}+3y\bar{y}$ in $\mathbb{Q}(\sqrt{-7})$ and $x\bar{x}+4y\bar{y}$ and $x\bar{x}+5y\bar{y}$ in $\mathbb{Q}(\sqrt{-2})$ are universal.*

Proof. The associated quaternary quadratic forms of $x\bar{x}+4y\bar{y}$ and $x\bar{x}+5y\bar{y}$ over $\mathbb{Q}(\sqrt{-2})$ are $\langle 1,2,4,8\rangle$ and $\langle 1,2,5,10\rangle$, respectively. Their universality has been shown by Ramanujan [7].

The genus of $f=x\bar{x}+3y\bar{y}$ over $\mathbb{Q}(\sqrt{-7})$ consists of the class of f and the class of $g=2x\bar{x}+x\bar{y}+\bar{x}y+2y\bar{y}$. Using the trace map, we get the quaternary quadratic forms corresponding to f and g, respectively:

$$\tilde{f}=x^2+2y^2+3z^2+6w^2+xy+3zw,$$
$$\tilde{g}=2x^2+2y^2+4z^2+4w^2+2xy+2xz+xw+yz+2yw+4zw.$$

We show that the genus of $\tilde{f}$ and $\tilde{g}$ is universal before the universality of f. The discriminant is $441=3^2\times 7^2$ in the sense of Nipp's table [5]. Thus if $p\ne 2,3,7$, the genus is locally universal by [6, 92:1b]. If $p=3$, then the Jordan splitting of $\tilde{f}_p$ and $\tilde{g}_p$ is $\langle 1,1\rangle\perp 3\langle 1,1\rangle$. The genus represents all 3-adic integers since $\langle 1,1\rangle$ represents all units by [6, 92:1b]. If $p=7$, then the Jordan splitting is $\langle 1,\Delta\rangle\perp 7\langle 1,\Delta\rangle$. Thus the genus represents all 7-adic integers. When $p=2$, $\tilde{f}_p\cong\tilde{g}_p\cong xy+zw$. Hence the genus is locally universal for $p=2$. Finally $\tilde{f}$ and $\tilde{g}$ are positive definite and so the genus is locally universal for any prime p.

Now suppose that n is a positive integer represented by $\tilde{g}$. There exist $x,y,z,w\in\mathbb{Z}$ such that

$$\tilde{g}(x,y,z,w)=n.$$

If x is even,

$$\tilde{f}(z+2w,\frac{x}{2}+y,z,\frac{x}{2})=\tilde{g}(x,y,z,w)=n.$$

If x is odd and y is even,

$$\tilde{f}(2z+w,x+\frac{y}{2},w,\frac{y}{2})=\tilde{g}(x,y,z,w)=n.$$

If x is odd and y is also odd,

$$\tilde{f}(z-w, \frac{x-y}{2}, z+w, \frac{x+y}{2}) = \tilde{g}(x,y,z,w) = n.$$

Hence f is universal. □

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