A UNIQUENESS RESULT OF KÄHLER RICCI FLOW WITH AN APPLICATION

XU-QIAN FAN

Abstract. In this paper, we will study the problem of uniqueness of Kähler Ricci flow on some complete noncompact Kähler manifolds and the convergence of the flow on $\mathbb{C}^n$ with the initial metric constructed by Wu and Zheng.

1. Introduction

It is well known that the Ricci flow was initially introduced by Hamilton [7, 8] and that the short time existence and uniqueness in the compact case were proved therein. Letting $(M^n, g_{\alpha \bar{\beta}}(x))$ be a complete noncompact Kähler manifold, we will consider the Kähler Ricci flow

$$\begin{align*}
\frac{\partial}{\partial t} g_{\alpha \bar{\beta}}(x, t) &= -R_{\alpha \bar{\beta}}(x, t), \\
g_{\alpha \bar{\beta}}(x, 0) &= g_{\alpha \bar{\beta}}(x),
\end{align*}$$

(1.1)

where $R_{\alpha \bar{\beta}}(x, t)$ is the Ricci curvature with respect to $g_{\alpha \bar{\beta}}(x, t)$. W.-X. Shi [13, 14] proved the following short time existence for the above system. See Theorem 1.1 in [13] and Theorems 2.1 and 5.1 in [14].

**Theorem 1.1.** Let $(M^n, g_{\alpha \bar{\beta}}(x))$ be a complete noncompact Kähler manifold with bounded Riemannian curvature tensor bounded by $K_0$. Then [14] has a solution $g_{\alpha \bar{\beta}}(x, t)$ on $M \times [0, T]$ for some $T(n, K_0)$ which is a family of Kähler metrics on $M$ satisfying

$$C^{-1} g_{\alpha \bar{\beta}}(x) \leq g_{\alpha \bar{\beta}}(x, t) \leq C g_{\alpha \bar{\beta}}(x)$$

for all $(x, t) \in M \times [0, T]$, where $C$ is a constant depending only on $n$, $K_0$ and $T$.

We want to apply a maximum principle to show that the above solution is unique if the Ricci tensor has a potential with respect to the initial metric. More precisely, we have the following:

**Theorem 1.2.** Let $(M^n, g_{\alpha \bar{\beta}}(x))$ be a complete noncompact Kähler manifold with bounded Riemannian curvature tensor. Suppose there is a smooth function $f(x)$ on $M$ such that $\sqrt{-1} \partial \bar{\partial} f = \text{Ric}$, where $\text{Ric}$ is the Ricci form of the metric $g_{\alpha \bar{\beta}}(x)$.
Suppose $g_{\alpha \beta}(x, t)$ and $\bar{g}_{\alpha \beta}(x, t)$ are two solutions on $M \times [0, T]$ to \text{\textbf{(1.1)}} with the same initial metric $g_{\alpha \beta}(x)$ satisfying

$$c^{-1}g_{\alpha \beta}(x) \leq g_{\alpha \beta}(x, t), \bar{g}_{\alpha \beta}(x, t) \leq cg_{\alpha \beta}(x)$$

for some constant $c > 0$ such that $g_{\alpha \beta}(x, t)$ and $\bar{g}_{\alpha \beta}(x, t)$ are Kähler for all $t \in [0, T]$. Then $g_{\alpha \beta}(x, t) = \bar{g}_{\alpha \beta}(x, t)$ on $M \times [0, T]$.

Recently, Chen and Zhu \cite{5} proved independently the uniqueness result for the Ricci flow up to some time $T$ assuming only the curvature is bounded at each time $t \in [0, T]$.

This paper is organized as follows. In Section 2, we will prove Theorem \text{\textbf{1.2}}. In Section 3, we will apply the result to study the convergence of Kähler Ricci flows on $\mathbb{C}^n$ with the initial metrics constructed by Wu and Zheng \cite{10}.

\section{2. The Proof of Theorem 1.2}

We need the following result about exhaustion functions, which is due to W.-X. Shi; see Theorem 3.6 in \cite{14}.

\textbf{Lemma 2.1.} Suppose $(M^n, g_{ij}(x))$ is an $n$-dimensional complete noncompact Riemannian manifold and its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfies

$$|R_{ijkl}|^2 \leq k_0$$

on $M$ for some constant $k_0$. Then there is a constant $C_1(n, k_0)$ such that for any fixed point $x_0 \in M$, there is a smooth function $\psi(x) \in C^\infty(M)$ satisfying

$$\begin{cases}
C_1^{-1}[1 + r(x, x_0)] \leq \psi(x) \leq C_1[1 + r(x, x_0)], \\
|\nabla \psi(x)| \leq C_1, \\
|\nabla_j \nabla_i \psi(x)| \leq C_1
\end{cases}$$

for all $x \in M$.

Now we will prove Theorem \text{\textbf{1.2}}.

\textbf{Proof of Theorem 1.2.} For any fixed point $x \in M$, let

$$u(x, t) = \int_0^t \left( \log \frac{\det(g_{\alpha \beta}(x, \tau))}{\det(g_{\alpha \beta}(x))} - f(x) \right) d\tau$$

for all $t \in [0, T]$. One has

$$\begin{cases}
\frac{\partial}{\partial t}u(x, t) = \log \frac{\det(g_{\alpha \beta}(x, t))}{\det(g_{\alpha \beta}(x))} - f(x), \\
u(x, 0) = 0.
\end{cases}$$

Noting that $R_{\alpha \beta}(x) = f_{\alpha \beta}(x)$ and $R_{\alpha \beta}(x, t) = -(\log(\det(g_{ij}(x, t))))_{\alpha \beta}$, one has

$$u_t(x, t) = -R_{\alpha \beta}(x, t) + R_{\alpha \beta}(x) - f_{\alpha \beta}(x) = -R_{\alpha \beta}(x, t).$$

As in \cite{2}, let

$$S_{\alpha \beta}(x, t) = g_{\alpha \beta}(x, t) - g_{\alpha \beta}(x) - u_{\alpha \beta}(x, t).$$

From $\frac{\partial}{\partial t}g_{\alpha \beta}(x, t) = -R_{\alpha \beta}(x, t)$, one has

$$\frac{\partial}{\partial t}S_{\alpha \beta}(x, t) = \frac{\partial}{\partial t}g_{\alpha \beta}(x, t) - \frac{\partial}{\partial t}u_{\alpha \beta}(x, t) = 0.$$
So
\begin{equation}
\frac{\partial}{\partial t} S_{\alpha\beta}(x, t) = 0 \text{ with } S_{\alpha\beta}(x, 0) = 0.
\end{equation}
Hence $S_{\alpha\beta}(x, t) = 0$. That is, $g_{\alpha\beta}(x, t) = g_{\alpha\beta}(x) + u_{\alpha\beta}(x, t)$.

Similarly, letting
\begin{equation}
v(x, t) = \int_0^t \left( \log \frac{\det(\tilde{g}_{\alpha\beta}(x, \tau))}{\det(g_{\alpha\beta}(x))} - f(x) \right) d\tau,
\end{equation}
ones has
\begin{equation}
\frac{\partial}{\partial t} v(x, t) = \log \frac{\det(\tilde{g}_{\alpha\beta}(x, t))}{\det(g_{\alpha\beta}(x))} - f(x),
\end{equation}
with $\tilde{g}_{\alpha\beta}(x, t) = g_{\alpha\beta}(x) + v_{\alpha\beta}(x, t)$. Setting $w = u - v$, one has
\begin{equation}
w(x, t) = \int_0^t \left( \log \frac{\det(g_{\alpha\beta}(x, \tau))}{\det(\tilde{g}_{\alpha\beta}(x, \tau))} \right) d\tau \leq 2nT \log c
\end{equation}
for all $(x, t) \in M \times [0, T]$, where $c$ is the same as in [1,3], and
\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} w(x, t) = \log \frac{\det(g_{\alpha\beta}(x) + u_{\alpha\beta}(x, t))}{\det(g_{\alpha\beta}(x))}, \\
w(x, 0) = 0.
\end{cases}
\end{equation}

As in [3] (see the proof of Lemma 6.2 in [3]), one has
\begin{equation}
\log \frac{\det(g_{\alpha\beta}(x) + u_{\alpha\beta}(x, t))}{\det(g_{\alpha\beta}(x) + v_{\alpha\beta}(x, t))} = \log \det(g_{\alpha\beta}(x) + u_{\alpha\beta}(x, t)) - \log \det(g_{\alpha\beta}(x) + v_{\alpha\beta}(x, t))
\end{equation}
\begin{equation}
= \int_0^1 \frac{\partial}{\partial s} \log \det(g_{\alpha\beta}(x) + su_{\alpha\beta}(x, t)) + (1 - s)v_{\alpha\beta}(x, t)ds
\end{equation}
\begin{equation}
= \int_0^1 \Delta_{(t, s)} wds,
\end{equation}
where $\Delta_{(t, s)}$ is the Laplacian operator with respect to the Kähler metric
\begin{equation}
h_{\alpha\beta}(x, t, s) = g_{\alpha\beta}(x, t) + su_{\alpha\beta}(x, t) + (1 - s)v_{\alpha\beta}(x, t)
\end{equation}
on $M \times [0, T] \times [0, 1]$. That is, $\Delta_{(t, s)} w(x, t) = h^{\alpha\beta}(x, t, s) w_{\alpha\beta}(x, t)$. So
\begin{equation}
\frac{\partial}{\partial t} w(x, t) - \int_0^1 \Delta_{(t, s)} wds = 0
\end{equation}
for all $(x, t, s) \in M \times [0, T] \times [0, 1]$. From [1,3], one has for any $s \in [0, 1]$,
\begin{equation}
c^{-1} g_{\alpha\beta}(x) \leq s g_{\alpha\beta}(x, t) + (1 - s) \tilde{g}_{\alpha\beta}(x, t) \leq c g_{\alpha\beta}(x)
\end{equation}
for all $(x, t) \in M \times [0, T] \times [0, 1]$. That is equivalent to
\begin{equation}
c^{-1} g_{\alpha\beta}(x) \leq h_{\alpha\beta}(x, t, s) \leq c g_{\alpha\beta}(x)
\end{equation}
for all $(x, t, s) \in M \times [0, T] \times [0, 1]$. Since the Riemannian curvature tensor with respect to $g_{\alpha\beta}(x)$ is bounded, by Lemma [2,1] there is a real function $\psi(x) \in C^\infty(M)$ such that
\begin{equation}
\begin{cases}
\mu^{-1}(1 + r(x)) \leq \psi(x) \leq \mu(1 + r(x)), \\
|\nabla \psi| \leq \mu, \\
|\nabla_1 \nabla_2 \psi| \leq \mu
\end{cases}
\end{equation}
for some constant $\mu > 1$, where $\nabla \psi$ and $\nabla_i \nabla_j \psi$ are the gradient and Hessian of $\psi$ with respect to $g_{ij}(x)$. Since $\psi_{\alpha ij}$ is a Hermitian symmetric $(1,1)$ tensor, by the last inequality of (2.11), one has that at any point $x \in M$

\begin{equation}
(2.12) \quad \left( \inf_{0 \neq \nu \in T^1_x M} \frac{\psi_{\alpha \beta}(x)(\nu, \nu)}{h(x, t, s)(\nu, \nu)} \right)^2 + \left( \sup_{0 \neq \nu \in T^2_x M} \frac{\psi_{\alpha \beta}(x)(\nu, \nu)}{g_{\alpha \beta}(x)(\nu, \nu)} \right)^2 \leq C(n) \mu^2
\end{equation}

for some constant $C(n)$ depending only on $n$. By (2.10), one has for any $0 \neq \nu \in T^1_x M$,

\[ \frac{\psi_{\alpha \beta}(x)(\nu, \nu)}{h(x, t, s)(\nu, \nu)} = \frac{\psi_{\alpha \beta}(x)(\nu, \nu)}{g_{\alpha \beta}(x)(\nu, \nu)} \leq C \frac{\psi_{\alpha \beta}(x)(\nu, \nu)}{g_{\alpha \beta}(x)(\nu, \nu)}, \]

where $c$ is the same as in (2.10). So there is another constant $C(n, \mu, C)$ such that

\begin{equation}
(2.13) \quad \begin{cases}
C^{-1}(1 + r(t, s)(x)) \leq \psi(x) \leq C(1 + r(t, s)(x)), \\
|\nabla(t, s)\psi|(t, s) \leq C,
\end{cases}
\end{equation}

where $r(t, s)(x)$ and $\nabla(t, s)\psi$ are taken with respect to $h_{\alpha \beta}(x, t, s)$ for all $(t, s) \in [0, T] \times [0, 1]$. Let $W(x, t) = e^{\lambda t} \psi(x)$, where $\lambda$ will be chosen later. We have

\[ W_t = \lambda e^{\lambda t} \psi(x) \text{ and } \Delta(t, s)W(x, t) = e^{\lambda t} \Delta(t, s)\psi \]

and (2.13) implies that

\[ \Delta(t, s)W(x, t) \leq Ce^{\lambda t}, \]

where $C$ is the constant in (2.13). Setting $\lambda = C^2 + 1$ for example, one has

\begin{equation}
(2.14) \quad W_t - \Delta(t, s)W(x, t) > 0
\end{equation}

for all $(x, t, s) \in M \times [0, T] \times [0, 1]$. For any $\epsilon > 0$, setting

\[ \theta_\epsilon(x, t) = w(x, t) - \epsilon W(x, t), \]

one has $\theta_\epsilon(x, t) < 0$ for $x$ tending to infinity because $w$ is bounded for all $(x, t) \in M \times [0, T]$ by (2.7). Hence if $\theta_\epsilon(x, t) > 0$ somewhere, then $\theta_\epsilon(x, t)$ has a maximal value at some point $(x_1, t_1) \in M \times [0, T]$. By the maximum principle, at $(x_1, t_1)$,

\[ 0 \leq \left( \frac{\partial}{\partial t} - \Delta(t, s) \right) \theta_\epsilon(x, t). \]

So by (2.9) and (2.14), one has

\[ 0 \leq \frac{\partial}{\partial t} \theta_\epsilon(x, t) - \int_0^1 \Delta(t, s)\theta_\epsilon(x, t)ds = -\epsilon \left( W_t - \int_0^1 \Delta(t, s)W(x, t)ds \right) \]

\[ < 0. \]

This is a contradiction. Hence $w(x, t) \leq \epsilon W(x, t)$. Letting $\epsilon \to 0$, we have $w(x, t) \leq 0$, i.e., $u \leq v$. By symmetry, we have $u(x, t) = v(x, t)$. Therefore $g_{\alpha \beta}(x, t) = \tilde{g}_{\alpha \beta}(x, t)$. □
The existence of the potential to the Ricci tensor is related to solving the corresponding Poincaré-Lelong equation \( \sqrt{-1}\partial\bar{\partial}u = \text{Ric} \). This question was extensively studied (see [9] [11] [10] [15] [6]). In particular, from Theorem 6.1 in [11] (see also Theorem 5.1 in [10] and Theorem 3.3 in [6]), one has that if \((M^n, g_{\alpha\beta}(x))\) is a complete noncompact Kähler manifold of complex dimension \( n \) with nonnegative holomorphic bisectional curvature and bounded scalar curvature \( \mathcal{R} \), then the Poincaré-Lelong equation has a solution, provided either

(i) \( \int_0^\infty f_{B_z}(t) \mathcal{R} dt < \infty \), or

(ii) the Ricci curvature \( \text{Ricci}(x) \geq \frac{a \ln(10 + r(x))}{(1 + r^2(x)) \ln(10 + r(x))} \) for some \( a > 268(n + 2)^2 \) and all group homomorphisms from \( \pi_1(M) \) to \( \mathbb{R} \) are trivial; moreover the universal covering space \( \tilde{M} \) of \( M \) with the covering metric has no compact factors.

3. CONVERGENCE OF SOME KÄHLER RICCI FLOWS

As a consequence of Theorem [1.2] we will show that the rotational symmetry is preserved under the Kähler-Ricci flow under a certain assumption. More precisely, we have

**Proposition 3.1.** Let \( g_{\alpha\beta}(x) \) be a complete Kähler metric on \( \mathbb{C}^n \) which is rotationally symmetric, i.e., \( U(n) \)-invariant, satisfying that the Riemannian curvature tensor with respect to \( g_{\alpha\beta}(x) \) is bounded. Suppose a family of Kähler metrics \( g_{\alpha\beta}(x, t) \) on \( \mathbb{C}^n \) for \( t \in [0, T] \) is a solution to (1.1) with the initial value \( g_{\alpha\beta}(x) \) such that (1.2) holds for some constant \( C \) and for all \((x, t) \in \mathbb{C}^n \times [0, T]\). Then \( g_{\alpha\beta}(x, t) \) is also rotationally symmetric for all \( t \in [0, T] \).

**Proof.** Let \((z^1, \cdots, z^n)\) be the standard coordinate on \( \mathbb{C}^n \). Then

\[
g_{\alpha\beta}(z, t) = \langle \frac{\partial}{\partial z^\alpha}(z), \frac{\partial}{\partial \bar{z}^\beta}(z) \rangle_t,
\]

For any \( \phi \in U(n) \), setting \((y^1, \cdots, y^n) = \phi^{-1}(z^1, \cdots, z^n)\) and \( \tilde{g} = \phi^*g \), which is the pulled-back of \( g \), one has

\[
\tilde{R}_{\alpha\beta}(y, t) = \phi^*\text{Ricci} \left( \frac{\partial}{\partial y^\alpha}(y), \frac{\partial}{\partial \bar{y}^\beta}(y) \right),
\]

where \( \tilde{R}_{\alpha\beta}(y, t) \) is the Ricci curvature with respect to \( \tilde{g}_{\alpha\beta}(y, t) \). Since \( \frac{\partial}{\partial z^\alpha} g_{\alpha\beta} = -R_{\alpha\beta} \), one has \( \frac{\partial}{\partial \bar{z}^\alpha} g_{\alpha\beta} = -\tilde{R}_{\alpha\beta} \). Since the initial metric \( g \) is rotationally symmetric, i.e., \( g = \phi^*g \), one has that there is a function \( \varphi(z, \bar{z}) \) such that \( g_{\alpha\beta} = \varphi_{\alpha\bar{\beta}} \), where \( z \in \mathbb{C}^n \), and \( \varphi(z, \bar{z}) = w(|z|^2) \) for some smooth function \( w \). See, for example, page 4 in [1]. So \( -\log \frac{\det(\delta_{\alpha\beta})}{\det(\delta_{\alpha\bar{\beta}})} \) is a potential of Ricci tensor, where \( \delta_{\alpha\beta} \) is the standard metric on \( \mathbb{C}^n \). Since \( g_{\alpha\beta}(x, t) \) is uniformly equivalent to \( g_{\alpha\beta}(x) \) and \( g_{\alpha\beta}(x) \) is \( U(n) \)-invariant, one has that (1.2) is also true for \( \tilde{g} \). By Theorem 1.2 we have \( g(t) = \phi^*g(t) \). That is, the solution \( g(x, t) \) is rotationally symmetric. \( \square \)

From now until the end of the paper, we will study the Kähler Ricci flow with the initial metric on \( \mathbb{C}^n \) constructed by Wu and Zheng [16] and \( g_{\alpha\beta}(z, t) \) will always denote the metric that will be defined in [3.1].

For convenience, let us recall some part of the results in [16]. See Example 2 in [16]. Let \( z = (z^1, \cdots, z^n) \) be the standard coordinate on \( \mathbb{C}^n \), and

\[
f = \frac{R^{1-\epsilon} - 1}{(1 - c)r},
\]
where \( R = 1 + r, \ r(z) = |z|^2 \) and \( 0 < c < 1 \). Then for any fixed \( 0 < c < 1 \), the Kähler metric
\[
g_{ij}(z;c) = f(r)\delta_{ij} + f'(r)\bar{z}_i z_j
\]
on \mathbb{C}^n \) is complete with positive bisectional curvature. Moreover it has maximal volume growth and quadratic curvature decay. The scalar curvature function \( \mathcal{R}(z) \) is given by
\[
\mathcal{R}(z) = A + 2(n - 1)B + \frac{1}{2}n(n - 1)C,
\]
where
\[
A = \frac{c}{R^{2-c}}, \quad B = \frac{1}{f^2 R r}(1 - f), \quad C = \frac{2}{f^2 r}(f - R^{-c}).
\]
The distance function to the origin is given by
\[
s(z) = \int_0^r \frac{1}{2\sqrt{(1 + \tau)^c}} d\tau.
\]
Let \( B_0(s) \) be the geodesic ball centered at the origin with radius \( s \) with respect to the metric \( g_{ij}(z;c) \), i.e., with radius \( \sqrt{R} \) with respect to the standard metric. Then the volume of it with respect to \( g_{ij}(z;c) \) is given by \( V_o(s) = \omega_n(r f)^n \), where \( \omega_n \) is the volume of unit ball in \( \mathbb{C}^n \) with respect to the standard metric. In [17], Zheng pointed out \( \mathcal{R}s^2(z) \leq cQ \) for \( 0 < c < 1/2 \), where \( Q \) is a constant depending only on \( n, V_o(s)/\omega_n s^{2n} \geq (1 - c)^n \) for \( 0 < c < 1 \), with a sketch of the proof for these facts. For convenience, we will modify slightly the statement and explain his idea in greater detail.

**Lemma 3.2.** With the above notation, \( V_o(s) \geq \omega_n s^{2n}(1 - c)^n \) for \( 0 < c < 1 \), and \( \mathcal{R}R^{1-c} \leq (2n^2 - n)c \) for \( 0 < c \leq 1/2 \).

**Proof.** We will check the volume growth first. Setting \( u(r) = \frac{r f}{\sqrt{R}} \), one has
\[
u' = s^{-3}(1 - c)^{-1}R^{-c}u_1,
\]
where
\[
u_1 = s(1 - c) - r^{-1/2}R^{1-c} + r^{-1/2}R^c/2.
\]
We want to show \( \nu_1 \leq 0 \). First \( \nu_1(0) = \lim_{r \to 0} \frac{R^{c/2} - R^{1-c}/2}{r^{1/2}} = 0 \),
\[
u'_1 = r^{-c}u_2,
\]
where \( u_2 = \frac{1}{2}R^{-\frac{c}{2}} + \left( -\frac{1}{2} + \frac{c}{2} \right)R^c - \frac{c}{2}R^{c-1} \). Noting that \( u_2(0) = 0 \) and
\[
u'_2 = cR^{-\frac{c}{2}} - \frac{c}{2}R^{c-1} + \frac{1}{4}R + \left( -\frac{1}{2} + \frac{c}{4} \right)R\frac{c}{2} - \left( \frac{c}{4} - \frac{1}{2} \right)R^c
\]
while \( u_1(0) = 0 \) and \( u_1'(0) < 0 \) with \( u_1'' < 0 \), one can get \( u_1 \leq 0 \) and \( u \) is non-increasing. By (3.3), one has \( s \leq \frac{1}{2} \int_0^r \frac{1}{\sqrt{(1 + \tau)^c}} d\tau \), as \( 0 < c < 1 \). So
\[
u_1 \geq \lim_{r \to \infty} \frac{r}{2} \geq 1 - c.
\]
Now we will check the curvature decay. Clearly \( A \times R^{1-c} \leq c \). Since \( R^{-c} \leq f \), one has
\[
B \times R^{1-c} = f^{-2}R^{-2c}R^r(1 - f)
\]
\[
\leq R^{r-1}(1 - f).
\]
We want to show that $R^c r^{-1}(1 - f)$ is nonincreasing in $r \in (0, \infty)$ as $0 < c \leq 1/2$. Letting $B_1 = R^c r^{-1} (1 - f) = \frac{(1 - c)R^c - fR^c}{(1 - c)r^3}$, one has

$$B'_1 = \frac{- (1 - c)R^c + c(1 - c)R^c - 1 - c + cR^c - 1 - 2R^c}{(1 - c)r^3} = \frac{B_2}{(1 - c)r^3}$$

with $B_2(0) = 0$:

$$B'_2 = (-1 + c^2 - c^3)R^c + (-c - 2c^2 + 2c^3)R^c - 1 + (c^2 - c^3)R^{-2} + 1$$

with $B'_2(0) = 0$;

$$B''_2 = R^c - 3\left[(-c + c^2 + c^3 - c^4)R^c + (c + c^2 - 4c^3 + 2c^4)R^c - (2c^2 + 3c^3 - c^4)\right]$$

with $B''_2(0) = 0$;

So $2(n - 1)BR^{1-c} \leq (n - 1)c$, because $B_1(0) = \lim_{r \to 0} \frac{(1 - c)R^c - 1 - 2R^c}{(1 - c)r^3} = \frac{c}{2}$. Clearly

$$\frac{1}{2}n(n - 1)CR^{1-c} = (n - 1)f^{-2}R^c - (f - R^{-c})R^{1+c} \leq (n - 1)r^{-1}(f - R^{-c}).$$

Letting $C_1 = R^{-1}R^{1+c}(f - R^{-c})$, one has

$$C'_1 = \frac{(1 - c)R^{1+c} - (1 + c)R^c - 1 + c}{(1 - c)r^3} = \frac{C_2}{(1 - c)r^3}$$

with $C_2(0) = 0$, and

$$C'_2 = (1 - c^2)R^c - (1 + c) + c(1 + c)R^{-1}$$

with $C'_2(0) = 0$, while

$$C''_2 = c(1 - c^2)R^{1-c} - c(1 - c^2)R^{-2} \geq 0.$$ 

So $C_1$ is nondecreasing in $r$. Since $C_1(\infty) = \frac{c}{1-c}$, we have

$$\frac{1}{2}n(n - 1)CR^{1-c} \leq n(n - 1)\frac{c}{1-c}.\tag{3.5}$$

By (3.2), (3.4) and (3.5), one can get the second part of the lemma. \hfill \Box

We will claim that the averages of the scalar curvature on the geodesics on any fixed center are uniformly quadratic decay in the radii of the geodesic balls as $n \geq 2$. Since $(1 - c)s^2 \leq rf = \frac{R^{1-c}}{1-c}$, one has

$$\frac{(1 - c)^2}{R^{1-c}} \leq \frac{1}{1 + s^2}$$

for $0 < c < 1$, and then

$$\mathcal{R} \leq \frac{c}{1 + s^2}.$$
for \(0 < c \leq 1/2\), where \(c_1 = \frac{(2n^2-n)c}{(1-c)^2}\). Since \(V_\alpha(s) \geq \omega_n s^{2n}(1-c)^n\), by volume comparison, one has, for \(n \geq 2\),

\[
\frac{1}{V_\alpha(s)} \int_{B_\alpha(s)} R = \frac{1}{V_\alpha(s)} \int_0^s \left( \int_{\partial B_\alpha(t)} R \right) dt \\
\leq \frac{1}{V_\alpha(s)} \int_0^s \int_{\partial B_\alpha(t)} \frac{c_1}{1 + t^2} dt \\
\leq \frac{c_1}{1 + s^2} + \frac{2c_1}{s^{2n}(1-c)^n} \int_0^s \frac{t^{2n+1}}{(1+t^2)^2} dt \\
\leq \frac{c_2}{1 + s^2},
\]

(3.6)

where \(c_2 = (1 + \frac{1}{(n-1)(1-c)}) c_1\). Now we will show that, for \(n \geq 2\),

\[
\frac{1}{V_\alpha(s)} \int_{B_\alpha(s)} R \leq \frac{\tilde{c}}{1 + s^2}
\]

(3.7)

for \(0 < c \leq 1/2\), \(x \in \mathbb{C}^n\), where \(\tilde{c} = c_2^{2n}(1-c)^n\). For \(s \geq r(x)/2\), by Lemma 3.2 and (3.0), one has

\[
\frac{1}{V_\alpha(s)} \int_{B_\alpha(s)} R \leq \frac{V_\alpha(s + r)}{V_\alpha(s)} \cdot \frac{1}{V_\alpha(s + r)} \int_{B_\alpha(s+r)} R \\
\leq \frac{\omega_n(s+r)^{2n}}{\omega_n s^{2n}(1-c)^n} \cdot \frac{1}{V_\alpha(s + r)} \int_{B_\alpha(s+r)} R \\
\leq \frac{\tilde{c}}{1 + s^2}.
\]

For \(s < r(x)/2\), it is easy to see that (3.7) holds.

In [12], Ni and Tam proved the following result (see Theorem 1.3 in [12]).

**Theorem 3.3.** Let \((M^n, g_{\alpha\beta}(x))\) be a complete noncompact manifold with nonnegative holomorphic bisectional curvature such that its scalar curvature \(R_0\) is bounded and satisfies

\[
\int_0^\infty \int_{B_\alpha(s)} R_0 ds \leq C_1
\]

(3.8)

for some constant \(C_1\) and for all \(x\). Then (1.1) has a long-time solution \(g_{\alpha\beta}(x, t)\), which has nonnegative holomorphic bisectional curvature for any \(t > 0\). Moreover, there is a function \(u(x, t)\) such that

\[
\left\{ \begin{array}{l}
\sqrt{-1} \partial \bar{\partial} u(x, t) = \text{Ric}(g_{\alpha\beta}(x, t)), \\
|\nabla u| \leq C(n)C_1, \\
R(x, t) + |\nabla u|^2(x, t) \leq \sup_M R_0 + (C(n)C_1)^2
\end{array} \right.
\]

for all \((x, t)\).

So for any fixed \(T > 0\), \(g_{\alpha\beta}(x, t)\) is uniformly equivalent to \(g_{\alpha\beta}(x)\) for all \((x, t) \in M^n \times [0, T]\). Hence by Proposition 3.1, (3.7) and Theorem 3.3 one has

**Proposition 3.4.** For any \(0 < c \leq 1/2\), the Kähler Ricci flow with the initial metric \(g_{\alpha\beta}(z; c)\) has a long-time solution \(g_{\alpha\beta}(z; t; c)\) which is rotationally symmetric for \(t \geq 0\), and the scalar curvature has uniformly upper bound for all time \(t\), moreover, the potential of the Ricci tensor with respect to \(g_{\alpha\beta}(z; t; c)\) is of at most linear growth.
The following theorem is due to Chau and Tam (see Theorem 1.1 or Theorem 4.3 in [4]).

**Theorem 3.5.** There exists a constant $\zeta$ depending only on $n$ such that, if $M^n$ is a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature satisfying

(i) $$\frac{1}{V_x(r)} \int_{B_x(r)} R \leq \frac{\zeta}{1 + r^2}$$

for all $x \in M$ and for all $r > 0$; and

(ii) there exist a point $p \in M$ and a sequence $t_k \to \infty$ such that $\frac{1}{|v_p|^2} g(p, t_k)$ are uniformly equivalent to $g(p, 0)$ at $p$, where $v_p$ is a fixed vector in $T^1,0_p(M)$ with $|v_p|^2 = 1$.

then the metrics $\frac{1}{|v_p|^2} g(x, t_k)$ subconverge uniformly in the $C^\infty$ topology in compact sets to a complete Kähler flat metric on $M$. In particular, the universal covering space of $M$ is biholomorphic to $\mathbb{C}^n$.

From Theorem 3.5, we have

**Proposition 3.6.** For $0 < c \leq 1/2$, $n \geq 2$ and $\tilde{c} \leq \zeta$, where $\tilde{c}$, $\zeta$ are the same as in (3.7) and Theorem 3.5 respectively, there is a sequence $\{\mu_k\}$ such that $\mu_k g(z, t_k; c)$ converges uniformly in the $C^\infty$ topology in compact sets to the standard metric on $\mathbb{C}^n$ up to a constant factor.

**Proof.** By Proposition 3.4 and Theorem 3.5 for $0 < c \leq 1/2$, $n \geq 2$ and $\tilde{c} \leq \zeta$, one has that there is a sequence $\{\mu_k\}$ such that $\mu_k g(z, t_k; c)$ converges uniformly in the $C^\infty$ topology in compact sets to a complete Kähler flat metric on $\mathbb{C}^n$. Let $g(z; c) = \lim_{k \to \infty} \mu_k g(z, t_k; c)$. Since $\mu_k g(z, t_k; c)$ is rotationally symmetric, it follows that $g(z; c)$ will also be. So as in [11], for any chosen $c$, there is a smooth function $w(r; c)$ on $[0, \infty)$ such that

$$g_{ij}(z; c) = w_{ij}(|z|^2; c).$$

Letting $s = \log |z|^2$ and $u(s) = u(e^s)$, one has

$$g_{ij}(z; c) = e^{-s} u'(s) \delta_{ij} + e^{-2s} \bar{z}_i z_j (u''(s) - u'(s))$$

and

$$\det(g_{ij}(z; c)) = e^{-ns} (u'(s))^{n-1} u''(s).$$

Setting $\varphi(s) = -\log(\det(g_{ij}(z; c)))$, one has

$$R_{ij} = \partial_i \partial_j \varphi(s) = e^{-s} \varphi'(s) \delta_{ij} + e^{-2s} \bar{z}_i z_j (\varphi''(s) - \varphi'(s)).$$

So at point $z = (z_1, 0, \cdots, 0)$, one has

$$R_{ij} = \begin{cases} e^{-s} \varphi''(s), & i = j = 1, \\ e^{-s} \varphi'(s), & i = j \geq 2, \\ 0, & i \neq j. \end{cases}$$

Since $g(z; c)$ is flat, one has $e^{-s} \varphi'(s) = e^{-s} \varphi'(s) = 0$. So from (3.10),

$$\varphi(s) = -\log(\det(g_{ij}(z; c))) = \text{constant}.$$

This is equivalent to $e^{-ns} (u'(s))^{n-1} u''(s) = \text{constant}$ and then $(u'(s))^n = n \lambda c^{ns}$ for some constant $\lambda > 0$. So $(u'(s))^n = \lambda (e^{ns} - \lambda_1)$, where $\lambda_1$ is a constant. By (3.9),
considering $i = j = 2$, one has $g_{22}(z; c) = e^{-s}u'(s)$. Since $g_{22}(z; c)$ is well defined at the origin, one has $\lim_{s \to -\infty} u'(s) = 0$. So $-\lambda_1 = \lim_{s \to -\infty} (u'(s))^n = 0$, and then $\lambda_1 = 0$. Hence $(u')^n = \lambda e^{ns}$, i.e., $u' = \lambda^{1/n}e^s$. Therefore $g_{ij}(z; c) = \lambda^{1/n}\delta_{ij}$. This completes the proof of the proposition. □

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Department of Mathematics, Jinan University, Guangzhou, 510632 People’s Republic of China

E-mail address: xqfan@hotmail.com