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FENCHEL DUALITY, FITZPATRICK FUNCTIONS AND THE EXTENSION OF FIRMLY NONEXPANSIVE MAPPINGS

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Abstract. Recently, S. Reich and S. Simons provided a novel proof of the Kirszbraun-Valentine extension theorem using Fenchel duality and Fitzpatrick functions. In the same spirit, we provide a new proof of an extension result for firmly nonexpansive mappings with an optimally localized range.

Throughout this paper, we assume that X is a real Hilbert space, with inner product $p = \langle \cdot \mid \cdot \rangle$ and induced norm $\| \cdot \|$, and we denote the identity mapping on X by Id. A mapping T from a subset D of X to X is called *firmly nonexpansive* if

$$(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2; \tag{1}$$

equivalently [13, 14], if $2T - \mathrm{Id}$ is *nonexpansive* (Lipschitz continuous with constant 1), i.e.,

$$(\forall x \in D)(\forall y \in D) \quad \|(2T - \mathrm{Id})x - (2T - \mathrm{Id})y\| \leq \|x - y\| \tag{2}$$

or if

$$(\forall x \in D)(\forall y \in D) \quad 0 \leq \langle Tx - Ty \mid (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \rangle . \tag{3}$$

Firmly nonexpansive mappings play an important role in various contexts; see, e.g., [1, 2, 3, 7, 8, 9, 10, 15, 17, 21, 22, 25]. The Kirszbraun-Valentine theorem (see, e.g., [5, 13, 16, 20, 26]) states that any nonexpansive mapping can be extended to a nonexpansive mapping defined on the whole space. A beautiful proof of this result, based on Fenchel duality and Fitzpatrick functions, was recently provided by Reich and Simons [23]. (For further applications of Fitzpatrick functions, see, e.g., [4, 24].) In this note, we refine their technique to obtain a new proof of an extension theorem for firmly nonexpansive mappings where the range of the extension is optimally localized. This extension theorem easily implies the Kirszbraun-Valentine result. Notation not explicitly defined in the following is standard in convex analysis; see, e.g., [27].

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Definition 1. Let D be a nonempty subset of X and let $T\colon D \to X$ be firmly nonexpansive. Then the associated *Fitzpatrick function* [12] $\phi = \phi_T$ is

$$X \times X \to \left]-\infty, +\infty\right] : (x, y) \mapsto \sup_{d \in D} \langle x \,|\, d - Td\rangle + \langle y \,|\, Td\rangle - \langle Td \,|\, d - Td\rangle, \tag{4}$$

and we also set $G = G_T = \big\{(d - Td, Td) \mid d \in D\big\}$.

Proposition 2. *Let D be a nonempty subset of X, let $T\colon D \to X$ be firmly nonexpansive, and let x and y be in X. Then:*

(i) $\phi = (\iota_G + p)^*$.
(ii) *The extension $\widetilde{T}\colon D \cup \{y\} \to X$ of T which maps y to x is still firmly nonexpansive if and only if $\phi(y - x, x) \leq p(y - x, x)$.*
(iii) *ϕ is convex, lower semicontinuous and proper.*
(iv) $\operatorname{conv} G \subset \operatorname{dom} \phi^* \subset \overline{\operatorname{conv}}\, G \subset \overline{\operatorname{conv}}\,(\operatorname{Id} - T)(D) \times \overline{\operatorname{conv}}\, T(D)$.
(v) $p \leq \phi^*$.

Proof. Fix x and y in X. (i): For every $d \in D$, we have

$$\langle x \,|\, d - Td\rangle + \langle y \,|\, Td\rangle - \langle d - Td \,|\, Td\rangle = \langle (d - Td, Td) \,|\, (x, y)\rangle - (p + \iota_G)(d - Td, Td), \tag{5}$$

from which the identity follows by supremizing over $d \in D$. (ii): This is a consequence of (3). (iii): By (ii), $\phi \leq p$ on G. Hence ϕ is proper. The function ϕ is convex and lower semicontinuous, as it is a Fenchel conjugate by (i). (iv): This is clear, since $G = \operatorname{dom}(\iota_G + p)$ and $\phi = (\iota_G + p)^*$. (v): In view of (3), $p(G - G) \subset [0, +\infty[$. Suppose that $(x, y) \in \operatorname{conv} G$, say it is a finite convex combination $(x, y) = \sum_{i \in I} \lambda_i (x_i, y_i)$ of elements in G. Then $\sum_{i \in I} \lambda_i p(x_i, y_i) = p(x, y) + \frac{1}{2} \sum_{i,j \in I} p((x_i, y_i) - (x_j, y_j)) \geq p(x, y)$, hence $p \leq \operatorname{conv}(\iota_G + p)$. Since p is continuous, it follows that $p \leq \overline{\operatorname{conv}}\,(\iota_G + p) = (\iota_G + p)^{**} = \phi^*$. □

Fact 3 (Fenchel duality)**.** *Let Y be a real Hilbert space and let $L\colon Y \to X$ be linear and continuous. Let $f\colon Y \to \left]-\infty, +\infty\right]$ and $g\colon X \to \left]-\infty, +\infty\right]$ be convex, lower semicontinuous, and proper such that g is continuous and finite at some point in $L \operatorname{dom} f$. Then*

$$\inf_{y \in Y} \big(f(y) + g(Ly)\big) = -\min_{x \in X} \big(f^*(-L^* x) + g^*(x)\big). \tag{6}$$

Proof. See, e.g., [27, Corollary 2.8.5]. □

Theorem 4. *Let D be a nonempty subset of X, let $T\colon D \to X$ be firmly nonexpansive, and let $y \in X$. Then T has a firmly nonexpansive extension $\widetilde{T}\colon D \cup \{y\} \to \overline{\operatorname{conv}}\, T(D)$.*

Proof. Set $\phi = \phi_T$ and $C = \overline{\operatorname{conv}}\, T(D)$, and assume first that $y = 0$. In view of Proposition 2(ii), we must show that

$$\min_{x \in X} \phi(x, -x) + \|x\|^2 + \iota_C(x) \leq 0. \tag{7}$$

Set $f\colon X \times X \to \left]-\infty, +\infty\right] : (x^*, y^*) \mapsto \frac{1}{2}\phi^*(2x^*, 2y^*)$ so that $f^* = \frac{1}{2}\phi$, and let $j = \frac{1}{2}\|\cdot\|^2$. Now set $g = (j + \iota_C)^*$ and observe (using [19]) that $g = j \Box \iota_C^* = j - (j \Box \iota_C) = j - \frac{1}{2} d_C^2$, where $\Box$ denotes the infimal convolution and d_C the distance function. Further set $L\colon X \times X \to X : (x^*, y^*) \mapsto y^* - x^*$. We claim that $\inf_{(x^*, y^*) \in X \times X} f(x^*, y^*) + g(L(x^*, y^*)) \geq 0$. Indeed, pick $(x^*, y^*) \in \operatorname{dom} f$. By

Proposition 2(iv), $(2x^*, 2y^*) \in \operatorname{dom}\phi^* \subset X \times C$ and hence $2y^* \in C$. Using Proposition 2(v), we deduce that

$$0 = 4\,\langle x^* \mid y^*\rangle + \|y^* - x^*\|^2 - \|x^* + y^*\|^2 \tag{8}$$

$$= p(2x^*, 2y^*) + \|y^* - x^*\|^2 - \|(y^* - x^*) - 2y^*\|^2 \tag{9}$$

$$\leq \phi^*(2x^*, 2y^*) + \|y^* - x^*\|^2 - d_C^2(y^* - x^*) \tag{10}$$

$$= 2\big(f(x^*, y^*) + g(y^* - x^*)\big) \tag{11}$$

$$= 2\big(f(x^*, y^*) + g(L(x^*, y^*))\big). \tag{12}$$

Hence $\inf(f + gL)(X \times X) \geq 0$ and, since $\operatorname{dom} g = X$, Fact 3 now implies that

$$\min_{x \in X} f^*(-L^*x) + g^*(x) \leq 0. \tag{13}$$

Since $f^* = \frac{1}{2}\phi$, $g^* = j + \iota_C$, and $L^*\colon X \to X \times X\colon x \mapsto (-x, x)$, we see that (13) clearly yields (7).

Now assume that $y \neq 0$. Let $E = D - y$ and define $U\colon E \to X\colon z \mapsto T(z + y)$. Then U is firmly nonexpansive and $U(E) = T(D)$. By what we just proved, there exists an extension $\widetilde{U}\colon E \cup \{0\} \to \overline{\operatorname{conv}}\, U(E) = \overline{\operatorname{conv}}\, T(D)$. Therefore, $\widetilde{T}\colon D \cup \{y\} \to \overline{\operatorname{conv}}\, T(D)\colon z \mapsto \widetilde{U}(z - y)$ is as required. □

Corollary 5. *Let D be a nonempty subset of X and let $T\colon D \to X$ be firmly nonexpansive. Then T has a firmly nonexpansive extension $\widetilde{T}\colon X \to \overline{\operatorname{conv}}\, T(D)$.*

Proof. Let $\mathcal{M}$ be the set of all pairs (U, E), where $D \subset E \subset X$ and $U\colon E \to \overline{\operatorname{conv}}\, T(D)$ is a firmly nonexpansive extension of T. Partially order $\mathcal{M}$ via $(U_1, E_1) \preceq (U_2, E_2)$ if $E_1 \subset E_2$ and U_2 extends U_1. Zorn's lemma guarantees the existence of a maximal element $(\widetilde{T}, \widetilde{D})$. Now Theorem 4 shows that $\widetilde{D} = X$. □

Remark 6 (range localization is optimal). The conclusion that the range of the extension $\widetilde{T}$ lies in the *closed convex hull* of $T(D)$ cannot be improved upon in general. Indeed, let D be a nonempty subset of X, let T be $\operatorname{Id}|_D$, and let $\widehat{T}\colon X \to X$ be any firmly nonexpansive extension of T. Then $D = \operatorname{Fix} T \subset \operatorname{Fix} \widehat{T}$, and the last set is closed and convex [13, 14]. Hence $C = \overline{\operatorname{conv}}\, T(D) = \overline{\operatorname{conv}}\, D \subset \operatorname{Fix} \widehat{T} \subset \widehat{T}(X)$. In particular, let $\widetilde{T}\colon X \to C$ be any firmly nonexpansive extension of T as in Corollary 5. Then $\widetilde{T}(X) = C$ and $\widetilde{T}|C = \operatorname{Id}|_C$; therefore, $\widetilde{T}$ is the projector onto C.

Corollary 7 (Kirszbraun-Valentine)**.** *Let D be a nonempty subset of X and let $N\colon D \to X$ be nonexpansive. Then N has a nonexpansive extension $\widetilde{N}\colon X \to \overline{\operatorname{conv}}\, N(D)$.*

Proof. (See also [13, 16, 20, 26] for different proofs and related results.) Let $T = \frac{1}{2}\operatorname{Id}|_D + \frac{1}{2}N$, which is firmly nonexpansive. Corollary 5 guarantees a firmly nonexpansive extension $\widetilde{T}\colon X \to \overline{\operatorname{conv}}\, T(D)$. Let P be the (firmly) nonexpansive projector onto $\overline{\operatorname{conv}}\, N(D)$. Then $\widetilde{N} = P \circ (2\widetilde{T} - \operatorname{Id})$ is as required. □

Remark 8. We do not know whether it is possible to deduce Corollary 5 from Corollary 7. The following technique of going back and forth between firmly nonexpansive and nonexpansive mappings, utilized in the proof of Corollary 7, does not work in reverse. Let D be a nonempty subset of X and $T\colon D \to X$ be firmly nonexpansive. Then $N = 2T - \operatorname{Id}|_D\colon D \to X$ is nonexpansive, and hence (by

Corollary 7) it has an extension $\widetilde{N}\colon X \to \overline{\operatorname{conv}}\, N(D)$. It is tempting to conjecture that $\widetilde{T} = \frac{1}{2}\operatorname{Id} + \frac{1}{2}\widetilde{N}$ would be an extension of T as in Corollary 5. However, let us concretely consider $D = \{0\} \subset X$ and $T\colon D \to X : 0 \mapsto 0$. Then $N = 2T - \operatorname{Id}_D = T$ and so $\widetilde{N} \equiv 0$. Hence $\widetilde{T} = \frac{1}{2}\operatorname{Id}$, which does not satisfy $\widetilde{T}(X) \subset \overline{\operatorname{conv}}\, T(D) = \{0\}$.

Remark 9. The correspondence revealed by Minty [18] between (maximal) monotone operators and firmly nonexpansive mappings (with full domain) provides a reformulation of Theorem 4 in terms of monotone operators (see, e.g., [6, Theorem 2.1]), which in turn relates to the work of Debrunner and Flor [11]. The new proof presented here provides a convex-analytical handle on these results (see also [4]). Furthermore, in the present Hilbert space setting, Reich ([21, Lemma 2.1]) showed that Corollary 5 is equivalent to the following result. *Let A be a monotone operator on X with a nonempty graph. Then A has a maximal monotone extension $\widetilde{A}$ such that* $\overline{\operatorname{conv}}\operatorname{dom} A = \overline{\operatorname{conv}}\operatorname{dom}\widetilde{A}$. (In fact, his result is about *accretive* operators in general Banach spaces.) Using [21, Proposition 2.2], it follows that Corollary 5 actually characterizes Hilbert spaces among all Banach spaces of dimension not less than three.

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References

[1] H. H. Bauschke and J. M. Borwein, "On projection algorithms for solving convex feasibility problems," *SIAM Review,* vol. 38, pp. 367–426, 1996. MR1409591 (98f:90045)

[2] H. H. Bauschke and P. L. Combettes, "A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces," *Mathematics of Operations Research,* vol. 26, pp. 248–264, 2001. MR1895827 (2003f:65101)

[3] H. H. Bauschke, P. L. Combettes, and S. Reich, "The asymptotic behavior of the composition of two resolvents," *Nonlinear Analysis: Theory, Methods, and Applications,* vol. 56, pp. 283–301, 2005. MR2101879 (2006d:47088)

[4] J. M. Borwein, "Maximal monotonicity via convex analysis," to appear in *Journal of Convex Analysis.*

[5] J. M. Borwein and Q. J. Zhu, *Techniques of Variational Analysis,* Springer-Verlag, 2005. MR2144010

[6] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert,* North-Holland, 1973. MR0348562 (50:1060)

[7] R. E. Bruck and S. Reich, "Nonexpansive projections and resolvents of accretive operators in Banach spaces," *Houston Journal of Mathematics,* vol. 3, pp. 459–470, 1977. MR0470761 (57:10507)

[8] Y. Censor and S. Reich, "Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization," *Optimization,* vol. 37, pp. 323–339, 1996. MR1402641 (98j:47161)

[9] P. L. Combettes, "Construction d'un point fixe commun à une famille de contractions fermes," *Comptes Rendus des Séances de l'Académie des Sciences, Série I, Mathématique,* vol. 320, pp. 1385–1390, 1995. MR1338291 (96c:47087)

[10] P. L. Combettes, "Solving monotone inclusions via compositions of nonexpansive averaged operators," *Optimization,* vol. 53, pp. 475–504, 2004. MR2115266 (2005i:47088)

[11] H. Debrunner and P. Flor, "Ein Erweiterungssatz für monotone Mengen," *Archiv der Mathematik,* vol. 15, pp. 445–447, 1964. MR0170189 (30:428)

[12] S. Fitzpatrick, "Representing monotone operators by convex functions," *Workshop/Miniconference on Functional Analysis and Optimization (Canberra 1988)*, Proceedings of the Centre for Mathematical Analysis, Australian National University vol. 20, Canberra, Australia, pp. 59–65, 1988. MR1009594 (90i:47054)

[13] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge University Press, 1990. MR1074005 (92c:47070)

[14] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings,* Marcel Dekker, 1984. MR0744194 (86d:58012)

[15] W. Kaczor, "Fixed points of asymptotically regular nonexpansive mappings on nonconvex sets," *Abstract and Applied Analysis,* vol. 2003, pp. 83–91, 2003. MR1960139 (2004a:47066)

[16] M. D. Kirszbraun, "Über die zusammenziehende und Lipschitzsche Transformationen," *Fundamenta Mathematicae,* vol. 22, pp. 77–108, 1934.

[17] K. C. Kiwiel and B. Łopuch, "Surrogate projection methods for finding fixed points of firmly nonexpansive mappings," *SIAM Journal on Optimization,* vol. 7, pp. 1084–1102, 1997. MR1479616 (98j:90049)

[18] G. J. Minty, 'Monotone (nonlinear) operators in Hilbert space," *Duke Mathematical Journal,* vol. 29, pp. 341–346, 1962. MR0169064 (29:6319)

[19] J.-J. Moreau, "Proximité et dualité dans un espace hilbertien," *Bulletin de la Société Mathématique de France,* vol. 93, pp. 273–299, 1965. MR0201952 (34:1829)

[20] S. Park, "Generalized Kirszbraun-Minty type inequalities," in *Fixed point theory and applications,* pp. 197–203, Nova Science Publishers, 2002. MR2083504 (2005d:49017)

[21] S. Reich, "Extension problems for accretive sets in Banach spaces," *Journal of Functional Analysis,* vol. 26, pp. 378–395, 1977. MR0477893 (57:17393)

[22] S. Reich and I. Shafrir, "The asymptotic behavior of firmly nonexpansive mappings," *Proceedings of the American Mathematical Society,* vol. 101, pp. 245–250, 1987. MR0902536 (88i:47030)

[23] S. Reich and S. Simons, "Fenchel duality, Fitzpatrick functions and the Kirszbraun-Valentine extension theorem," *Proceedings of the American Mathematical Society,* vol. 133, pp. 2657–2660, 2005. MR2146211 (2006d:46025)

[24] S. Simons and C. Zălinescu, "Fenchel duality, Fitzpatrick functions and maximal monotonicity," *Journal of Nonlinear and Convex Analysis,* vol. 6, pp. 1–22, 2005. MR2138099 (2005k:49102)

[25] P. Tseng, "On the convergence of the products of firmly nonexpansive mappings," *SIAM Journal on Optimization,* vol. 2, pp. 425–434, 1992. MR1172499 (93f:90161)

[26] F. A. Valentine, "On the extension of a vector function so as to preserve a Lipschitz condition," *Bulletin of the American Mathematical Society,* vol. 49, pp. 100–108, 1943. MR0008251 (4:269d)

[27] C. Zălinescu, *Convex Analysis in General Vector Spaces,* World Scientific Publishing, 2002. MR1921556 (2003k:49003)

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