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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL

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ABSTRACT. Let Ω be an open bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, $0 \in \Omega$. We are concerned with the asymptotic behavior of solutions for the elliptic problem:

$$(*) \qquad -\Delta u - \frac{\mu u}{|x|^2} = f(x,u), \qquad u \in H_0^1(\Omega),$$

where $0 \leq \mu < \left(\frac{N-2}{2}\right)^2$ and $f(x,u)$ satisfies suitable growth conditions. By Moser iteration, we characterize the asymptotic behavior of nontrivial solutions for problem $(*)$. In particular, we point out that the proof of Proposition 2.1 in Proc. Amer. Math. Soc. **132** (2004), 3225–3229, is wrong.

1. Introduction and main results

Let Ω be an open bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, $0 \in \Omega$. We are concerned with the problem

$$(1.1) \qquad \begin{cases} -\Delta u - \frac{\mu u}{|x|^2} = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \leq \mu < \left(\frac{N-2}{2}\right)^2$ and $f(x,u)$ satisfies suitable growth conditions.

In recent years, much attention has been paid to the existence of nontrivial solutions for problem (1.1) with $f(x,u) = \lambda u + |u|^{2^*-2}u$ $(2^* = \frac{2N}{N-2})$. E. Jannelli [11] proved that if $0 < \mu \leq \bar{\mu} - 1$, then problem (1.1) admits a positive solution for all $\lambda \in (0, \lambda_1(\mu))$; if $\bar{\mu} - 1 < \mu < \bar{\mu}$, and $\Omega = B_1(0)$, then there exists $\lambda^* \in (0, \lambda_1(\mu))$ such that problem (1.1) admits a positive solution if and only if $\lambda \in (\lambda^*, \lambda_1(\mu))$, where $\lambda_1(\mu)$ is the first eigenvalue of the positive operator $-\Delta - \frac{\mu}{|x|^2}$ $(0 \leq \mu < \bar{\mu})$ with Dirichlet boundary condition. D. Cao and P. Han [4] proved that if $\mu \in [0, \bar{\mu} - (\frac{N+2}{N})^2)$, then problem (1.1) admits a nontrivial solution for all $\lambda > 0$. A. Ferrero and F. Gazzola [8] also obtained some results for problem (1.1). For other relevant papers see [1, 3, 6, 7, 10, 12], and the references therein.

In this paper, we suppose that

(H) *$f(x,t)$ is a Caratheodory function. Moreover,* $|f(x,t)| \leq C(|t| + |t|^{2^*-1})$, $\forall t \in \mathbb{R}$.

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In a recent paper [2], J. Chen considered the asymptotic behavior of positive solutions for (1.1) at zero and proved the following.

Suppose that hypothesis (H) *holds. If* $u \in H_0^1(\Omega)$ *is a positive solution of problem* (1.1)*, then there exist positive constants* $M_1,\ M_2$ *such that*

$$M_1|x|^{-(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} \le u(x) \le M_2|x|^{-(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} \quad \text{for any} \ \ x \in B_r(0)\backslash\{0\} \tag{1.2}$$

holds for r *sufficiently small.*

Unfortunately, the proof of Proposition 2.1 given in [2] is wrong, which is crucial to the proof of the main result (Theorem 1.1) in [2].

In this note, based on Theorem 2.3 in [13], we, by means of Moser iteration, give a correct proof on the asymptotic behavior for solutions of (1.1):

Theorem 1.1. *Assume that* (H) *holds. Then any solution* $u \in H_0^1(\Omega)$ *satisfies*

$$|u(x)| \le C|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} \quad \forall x \in B_{\rho_0}(0)\backslash\{0\}, \tag{1.3}$$

where $\rho_0 > 0$ *is suitably small.*

Remark 1.2. If we also suppose that $f(x,t) \ge 0$ for $t \ge 0$, then any solution u of (1.1) is positive. Moreover,

$$u(x) \ge M_1|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} \quad \forall x \in B_{\rho_0}(0)\backslash\{0\}.$$

Its proof is the same as that of Proposition 2.2 in [2].

Throughout this paper, we denote the norm of $L^l(\Omega)$ $(1 \le l < \infty)$ by $\|u\|_{L^l(\Omega)} = (\int_\Omega |u|^l dx)^{\frac{1}{l}}$ and positive constants (possibly different) by C.

2. Proof of the main result

Before giving the proof of Theorem 1.1, we introduce one preliminary lemma, which can be found in [13].

Lemma 2.1. *Let* Ω *be a bounded neighborhood of* 0 *in* $\mathbb{R}^N$ $(N \ge 3)$*. Assume that* $V \in L^{\frac{N}{2}}(\Omega)$ *and* $g \in L^q(\Omega),\ q \ge 2$*. If* $u \in H_0^1(\Omega)$ *is a weak solution of*

$$-\Delta u - \frac{\mu u}{|x|^2} - V(x)u + \nu u = g \quad in \ \ \Omega,$$

where ν *is such that the linear operator on the left-hand side is positive, then*

$$u \in \bigcap_{p<p_{lim}} L^p(\Omega) \quad with \ \ p_{lim} = 2^* \min\left\{\frac{q}{2},\ \frac{\sqrt{\bar{\mu}}}{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\right\}.$$

Remark 2.2. Let $u \in H_0^1(\Omega)$ be a solution of (1.1). In view of Lemma 2.1, we have

$$u \in L^p(\Omega), \ \ \forall p < p_{lim} = \frac{2N}{N-2-2\sqrt{\bar{\mu}-\mu}}. \tag{2.1}$$

Proof of Theorem 1.1. Set

$$v(x) = |x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}u(x).$$

We claim that $v \in H_0^1(\Omega, |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dx)$. Indeed, by Hardy's inequality, we obtain

$$\begin{aligned}
&\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|\nabla v|^2 dx\\
= &\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\\
&\times\left||x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}\nabla u - (\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})|x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}-2}xu\right|^2 dx\\
\le\ & 2\int_\Omega \left(|\nabla u|^2 + (\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})^2\frac{|u|^2}{|x|^2}\right)dx\\
\le\ & \frac{2(\bar\mu + (\sqrt{\bar\mu}-\sqrt{\bar\mu})^2)}{\bar\mu}\int_\Omega |\nabla u|^2 dx\\
\le\ & C.
\end{aligned}$$

After a direct calculation, we deduce that for any $x \in \Omega\backslash\{0\}$, v satisfies

$$-div(|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla v) = |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}f(x, |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}v). \tag{2.2}$$

By the elliptic regularity theory (see [9]), $v \in C^2(\Omega\backslash\{0\}) \cap C^1(\overline{\Omega}\backslash\{0\})$. Let $\rho > 0$ be small enough such that $B_\rho(0) \subset\subset \Omega$. Choose

$$\begin{aligned}
&\varphi = \eta^2 v v_l^{2(s-1)} \in H_0^1(\Omega, |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dx),\\
&s, l > 1, \quad v_l = \min\{|v|, l\}, \quad \eta \in C_0^\infty(B_\rho(0)),
\end{aligned}$$

with the properties $0 \le \eta \le 1$, $\eta = 1$ in $B_r(0)$, $r < \rho$ and $|\nabla\eta| \le \frac{4}{\rho - r}$. Then from (2.2), we have

$$\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla v\cdot\nabla\varphi dx = \int_\Omega |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}f(x, |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}v)\varphi. \tag{2.3}$$

Furthermore, (2.3) can be rewritten as:

$$\begin{aligned}
&\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\\
&\times\big(2\eta v v_l^{2(s-1)}\nabla\eta\cdot\nabla v + \eta^2 v_l^{2(s-1)}|\nabla v|^2 + 2(s-1)\eta^2 v_l^{2(s-1)}|\nabla v_l|^2\big)dx\\
= &\int_\Omega |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}f(x, |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}v)\eta^2 v v_l^{2(s-1)}dx.
\end{aligned} \tag{2.4}$$

Observe that for any $\epsilon > 0$ small,

$$\begin{aligned}
&\left|2\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\eta v v_l^{2(s-1)}\nabla\eta\cdot\nabla v\right|\\
\le\ &\epsilon\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\eta^2 v_l^{2(s-1)}|\nabla v|^2 dx\\
&+C(\epsilon)\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|\nabla\eta|^2|v|^2 v_l^{2(s-1)}dx,
\end{aligned} \tag{2.5}$$

and by the assumption of (H),

$$\begin{aligned}
&\Big|\int_\Omega |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} f(x,|x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}v)\eta^2 v v_l^{2(s-1)}dx\Big|\\
\le\ & C\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\eta^2|v|^2 v_l^{2(s-1)}dx\\
&+C\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\eta^2|v|^{2^*} v_l^{2(s-1)}dx.
\end{aligned}\tag{2.6}$$

Inserting (2.5), (2.6) into (2.4), we get

$$\begin{aligned}
&\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(\eta^2 v_l^{2(s-1)}|\nabla v|^2+2(s-1)\eta^2 v_l^{2(s-1)}|\nabla v_l|^2\big)dx\\
\le\ & C\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(\eta^2+|\nabla\eta|^2\big)|v|^2 v_l^{2(s-1)}dx\\
&+C\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\eta^2|v|^{2^*} v_l^{2(s-1)}dx.
\end{aligned}\tag{2.7}$$

Now we recall the Caffarelli-Kohn-Nirenberg inequality (see [5]):

$$\Big(\int_\Omega |x|^{-bp}|w|^p dx\Big)^{\frac{2}{p}} \le C_{a,b}\int_\Omega |x|^{-2a}|\nabla w|^2 dx \quad \forall w\in H_0^1(\Omega,|x|^{-2a}dx),\tag{2.8}$$

where $-\infty < a < \frac{N-2}{2}$, $a\le b\le a+1, p=\frac{2N}{N-2+2(b-a)}$ and $C_{a,b}$ is a positive constant depending on a,b.

In the sequel we take $a=b=\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}<\frac{N-2}{2}$ in (2.8); then $p=2^*$. Choosing $w=\eta v v_l^{s-1}$ in (2.8), together with (2.7), we derive

$$\begin{aligned}
&\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}}\\
\le\ & C_{a,a}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|\nabla(\eta v v_l^{s-1})|^2dx\\
\le\ & 2C_{a,a}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\\
&\times\Big(|\nabla\eta|^2|v|^2v_l^{2(s-1)}+\eta^2v_l^{2(s-1)}|\nabla v|^2dx+(s-1)^2\eta^2v_l^{2(s-1)}|\nabla v_l|^2\Big)dx\\
\le\ & Cs\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(\eta^2+|\nabla\eta|^2\big)|v|^2 v_l^{2(s-1)}dx\\
&+Cs\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\eta^2|v|^{2^*} v_l^{2(s-1)}dx.
\end{aligned}\tag{2.9}$$

Since $u\in L^p(\Omega)$, $\forall p<\frac{2N}{N-2-2\sqrt{\bar\mu-\mu}}$ (see Remark 2.2), we may choose

$$\frac{N}{2}<q<\frac{N(N-2)}{2\big(N-2-2\sqrt{\bar\mu-\mu}\big)}.$$

Then

$$(2^*-2)q<\frac{2N}{N-2-2\sqrt{\bar\mu-\mu}} \qquad \text{and} \qquad 2<\frac{2q}{q-1}<2^*.$$

Therefore we deduce that for any $\epsilon > 0$,

$$
\begin{aligned}
&\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\eta^2|v|^{2^*}v_l^{2(s-1)}dx \\
=\ &\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|u|^{2^*-2}|\eta v v_l^{s-1}|^2dx \\
\le\ &\|u\|^{2^*-2}_{L^{(2^*-2)q}(\Omega)}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}\eta v v_l^{s-1}\|^2_{L^{\frac{2q}{q-1}}(\Omega)} \\
\le\ &\|u\|^{2^*-2}_{L^{(2^*-2)q}(\Omega)}\times\big(\epsilon\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}\eta v v_l^{s-1}\|_{L^{2^*}(\Omega)} \\
&+C(N,q)\epsilon^{-\frac{N}{2q-N}}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}\eta v v_l^{s-1}\|_{L^2(\Omega)}\big)^2 \\
\le\ &C\epsilon^2\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}} \\
&+C\epsilon^{-\frac{2N}{2q-N}}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^2dx.
\end{aligned} \tag{2.10}
$$

Inserting (2.10) into (2.9), we obtain

$$
\begin{aligned}
&\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}} \\
\le\ &Cs\epsilon^2\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}} \\
&+Cs\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(\eta^2+|\nabla\eta|^2\big)|v|^2v_l^{2(s-1)}dx \\
&+Cs\epsilon^{-\frac{2N}{2q-N}}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^2dx.
\end{aligned} \tag{2.11}
$$

Taking $\epsilon = \frac{1}{\sqrt{2Cs}}$ in (2.11), we conclude

$$
\begin{aligned}
&\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}} \\
\le\ &Cs^\alpha\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(\eta^2+|\nabla\eta|^2\big)|v|^2v_l^{2(s-1)}dx,
\end{aligned} \tag{2.12}
$$

where $\alpha = \frac{2q}{2q-N} > 0$.

Note that

$$
\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|\eta|^{2^*}|v|^2v_l^{2^*s-2}dx \le \int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|\eta v v_l^{s-1}\big|^{2^*}dx,
$$

so from (2.12), we get

$$
\begin{aligned}
&\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|\eta|^{2^*}|v|^2v_l^{2^*s-2}dx\Big)^{\frac{2}{2^*}} \\
\le\ &Cs^\alpha\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(\eta^2+|\nabla\eta|^2\big)|v|^2v_l^{2(s-1)}dx \\
\le\ &Cs^\alpha\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(\eta^2+|\nabla\eta|^2\big)|v|^2v_l^{2(s-1)}dx.
\end{aligned} \tag{2.13}
$$

Using the choice of the cut-off function η, we deduce

$$\begin{aligned}&\left(\int_{B_r(0)}|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2^*s-2}dx\right)^{\frac{2}{2^*}}\\ \le\ &\frac{Cs^\alpha}{(\rho-r)^2}\int_{B_\rho(0)}|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2s-2}dx.\end{aligned}\tag{2.14}$$

Choosing $s^*>0$ such that

$$\frac{N}{N-2}<s^*<\frac{N}{N-2-2\sqrt{\bar\mu-\mu}},$$

we define the sequence

$$s_j=s^*\left(\frac{2^*}{2}\right)^j,\quad j=0,1,2,\cdots,$$

and take $s=s_j$ in (2.14). Then

$$\begin{aligned}&\left(\int_{B_r(0)}|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2s_{j+1}-2}dx\right)^{\frac{1}{2s_{j+1}}}\\ \le\ &\left(\frac{Cs_j^\alpha}{(\rho-r)^2}\right)^{\frac{1}{2s_j}}\left(\int_{B_\rho(0)}|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2s_j-2}dx\right)^{\frac{1}{2s_j}}.\end{aligned}\tag{2.15}$$

Let $\rho_0>0$ be small enough such that $B_{2\rho_0}(0)\subset\subset\Omega$, and $r_j=\rho_0(1+\rho_0^j)$, $j=0,1,2,\cdots$. Taking $\rho=r_j$ and $r=r_{j+1}$ in (2.15), we derive

$$\begin{aligned}&\left(\int_{B_{r_{j+1}}(0)}|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2s_{j+1}-2}dx\right)^{\frac{1}{2s_{j+1}}}\\ \le\ &\left(\frac{Cs_j^\alpha}{(\rho_0-\rho_0^2)\rho_0^j}\right)^{\frac{1}{2s_j}}\left(\int_{B_{r_j}(0)}|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2s_j-2}dx\right)^{\frac{1}{2s_j}}\\ \le\ &\cdots\le\left(\frac{C}{(1-\rho_0)\rho_0}\right)^{\sum_{j=0}^{\infty}\frac{1}{2s_j}}\rho_0^{-\sum_{j=0}^{\infty}\frac{j}{2s_j}}\\ &\times\prod_{j=0}^{\infty}s_j^{\frac{\alpha}{2s_j}}\left(\int_{B_{r_0}(0)}|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2s^*-2}dx\right)^{\frac{1}{2s^*}}.\end{aligned}\tag{2.16}$$

It is not difficult to verify

$$\sum_{j=0}^{\infty}\frac{1}{2s_j}=\frac{1}{2s^*}\sum_{j=0}^{\infty}\left(\frac{2}{2^*}\right)^j\le C,\qquad \sum_{j=0}^{\infty}\frac{j}{2s_j}=\frac{1}{2s^*}\sum_{j=0}^{\infty}j\left(\frac{2}{2^*}\right)^j\le C,\tag{2.17}$$

$$\prod_{j=0}^{\infty}s_j^{\frac{\alpha}{2s_j}}=(s^*)^{\frac{\alpha}{2s^*}\sum_{j=0}^{\infty}(\frac{2}{2^*})^j}\left(\frac{2^*}{2}\right)^{\frac{\alpha}{2s^*}\sum_{j=0}^{\infty}j(\frac{2}{2^*})^j}\le C.\tag{2.18}$$

By the choice of s^*, we deduce

$$2^*<2s^*<\frac{2N}{N-2-2\sqrt{\bar\mu-\mu}},$$

and then

$$\begin{aligned}
&\int_{B_{r_0}(0)} |x|^{-2^*(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}|v|^2 v_l^{2s^*-2}dx \\
\le\ & \int_{B_{r_0}(0)} |x|^{(2s^*-2^*)(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}|u|^{2s^*}dx \\
\le\ & (diam\Omega)^{(2s^*-2^*)(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}\int_{\Omega}|u|^{2s^*}dx \\
\le\ & C.
\end{aligned} \tag{2.19}$$

Inserting (2.17)-(2.19) into (2.16), we obtain

$$\begin{aligned}
& \|v_l\|_{L^{2s_{j+1}}(B_{\rho_0}(0))} \\
\le\ & \|v_l\|_{L^{2s_{j+1}}(B_{r_{j+1}}(0))} \\
\le\ & (diam\Omega)^{\frac{2^*(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}{2s_{j+1}}}\left(\int_{B_{r_{j+1}}(0)}|x|^{-2^*(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}v^2 v_l^{2s_{j+1}-2}dx\right)^{\frac{1}{2s_{j+1}}} \\
\le\ & C.
\end{aligned}$$

Note that $s_{j+1} \longrightarrow \infty$ as $j \longrightarrow \infty$. So let $j \longrightarrow \infty$ in the above inequality, and infer

$$\|v_l\|_{L^\infty(B_{\rho_0}(0))} \le C,$$

and (1.3) can be obtained by taking $l \longrightarrow +\infty$. □

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