ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL

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Abstract. Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^N (N \geq 3) \) with smooth boundary \( \partial \Omega \), \( 0 \in \Omega \). We are concerned with the asymptotic behavior of solutions for the elliptic problem:

\[
- \Delta u - \frac{\mu u}{|x|^2} = f(x, u), \quad u \in H^1_0(\Omega),
\]

where \( 0 \leq \mu < \left( \frac{N-2}{2} \right)^2 \) and \( f(x, u) \) satisfies suitable growth conditions. By Moser iteration, we characterize the asymptotic behavior of nontrivial solutions for problem (\( \ast \)). In particular, we point out that the proof of Proposition 2.1 in Proc. Amer. Math. Soc. 132 (2004), 3225–3229, is wrong.

1. Introduction and main results

Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^N (N \geq 3) \) with smooth boundary \( \partial \Omega \), \( 0 \in \Omega \). We are concerned with the problem

\[
\begin{cases}
- \Delta u - \frac{\mu u}{|x|^2} = f(x, u) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( 0 \leq \mu < \left( \frac{N-2}{2} \right)^2 \) and \( f(x, u) \) satisfies suitable growth conditions.

In recent years, much attention has been paid to the existence of nontrivial solutions for problem (1.1) with \( f(x, u) = \lambda u + |u|^{2^* - 2} u \) \( (2^* = \frac{2N}{N-2}) \). E. Jannelli [11] proved that if \( 0 < \mu \leq \bar{\mu} - 1 \), then problem (1.1) admits a positive solution for all \( \lambda \in (0, \lambda_1(\mu)) \); if \( \bar{\mu} - 1 < \mu < \bar{\mu} \), and \( \Omega = B_1(0) \), then there exists \( \lambda^* \in (0, \lambda_1(\mu)) \) such that problem (1.1) admits a positive solution if and only if \( \lambda \in (\lambda^*, \lambda_1(\mu)) \), where \( \lambda_1(\mu) \) is the first eigenvalue of the positive operator \( -\Delta - \frac{\mu}{|x|^2} \) \( (0 \leq \mu < \bar{\mu}) \) with Dirichlet boundary condition. D. Cao and P. Han [4] proved that if \( \mu \in [0, \bar{\mu} - (\frac{N+2}{N})^2) \), then problem (1.1) admits a nontrivial solution for all \( \lambda > 0 \). A. Ferrero and F. Gazzola [8] also obtained some results for problem (1.1). For other relevant papers see [1, 3, 6, 7, 10, 12], and the references therein.

In this paper, we suppose that

\( f(x, t) \) is a Caratheodory function.

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In a recent paper [2], J. Chen considered the asymptotic behavior of positive solutions for (1.1) at zero and proved the following.

Suppose that hypothesis (H) holds. If \( u \in H^1_0(\Omega) \) is a positive solution of problem (1.1), then there exist positive constants \( M_1, M_2 \) such that

\[
M_1 |x|^{-\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}} \leq u(x) \leq M_2 |x|^{-\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}} \quad \text{for any} \quad x \in B_r(0) \setminus \{0\}
\]

holds for \( r \) sufficiently small.

Unfortunately, the proof of Proposition 2.1 given in [2] is wrong, which is crucial to the proof of the main result (Theorem 1.1) in [2].

In this note, based on Theorem 2.3 in [13], we, by means of Moser iteration, give a correct proof on the asymptotic behavior for solutions of (1.1):

**Theorem 1.1.** Assume that (H) holds. Then any solution \( u \in H^1_0(\Omega) \) satisfies

\[
|u(x)| \leq C |x|^{-\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}} \quad \forall x \in B_{\rho_0}(0) \setminus \{0\},
\]

where \( \rho_0 > 0 \) is suitably small.

**Remark 1.2.** If we also suppose that \( f(x,t) \geq 0 \) for \( t \geq 0 \), then any solution \( u \) of (1.1) is positive. Moreover,

\[
u(x) \geq M_1 |x|^{-\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}} \quad \forall x \in B_{\rho_0}(0) \setminus \{0\}.
\]

Its proof is the same as that of Proposition 2.2 in [2].

Throughout this paper, we denote the norm of \( L^l(\Omega) \) (\( 1 \leq l < \infty \)) by \( ||u||_{L^l(\Omega)} = (\int_{\Omega} |u|^l \, dx)^{1/l} \) and positive constants (possibly different) by \( C \).

2. PROOF OF THE MAIN RESULT

Before giving the proof of Theorem 1.1, we introduce one preliminary lemma, which can be found in [13].

**Lemma 2.1.** Let \( \Omega \) be a bounded neighborhood of \( 0 \) in \( \mathbb{R}^N \) (\( N \geq 3 \)). Assume that \( V \in L^{2\bar{\mu}}(\Omega) \) and \( g \in L^q(\Omega), \ q \geq 2 \). If \( u \in H^1_0(\Omega) \) is a weak solution of

\[
-\Delta u - \frac{\mu u}{|x|^2} - V(x)u + vu = g \quad \text{in} \quad \Omega,
\]

where \( \nu \) is such that the linear operator on the left-hand side is positive, then

\[
u \in \bigcap_{p < \rho_{\lim}} L^p(\Omega) \quad \text{with} \quad \rho_{\lim} = 2^* \min \left\{ \frac{q}{2}, \frac{\sqrt{\bar{\mu}}}{\sqrt{\bar{\mu} - \sqrt{\bar{\mu} - \mu}}} \right\}.
\]

**Remark 2.2.** Let \( u \in H^1_0(\Omega) \) be a solution of (1.1). In view of Lemma 2.1, we have

\[
u \in L^p(\Omega), \quad \forall p < \rho_{\lim} = \frac{2N}{N - 2 - 2\sqrt{\bar{\mu} - \mu}}.
\]

**Proof of Theorem 1.1.** Set

\[
\nu(x) = |x|^{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}} u(x).
\]
We claim that \( v \in H^1_0(\Omega, |x|^{-2(\sqrt{\mu} - \sqrt{\nu})}dx) \). Indeed, by Hardy’s inequality, we obtain

\[
\int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu})} |\nabla v|^2 dx
= \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu})} \\
\times |x|\sqrt{\mu - \sqrt{\nu} - \mu} \nabla u - (\sqrt{\mu} - \sqrt{\nu} - \mu)|x|\sqrt{\mu - \sqrt{\nu} - \mu} - 2xu|^2 dx
\leq 2 \int_{\Omega} \left( |\nabla u|^2 + (\sqrt{\mu} - \sqrt{\nu} - \mu)^2 |u|^2 \right) dx
\leq \frac{2(\mu + (\sqrt{\mu} - \sqrt{\nu})^2)}{\rho} \int_{\Omega} |\nabla u|^2 dx
\leq C.
\]

After a direct calculation, we deduce that for any \( x \in \Omega \setminus \{0\}, v \) satisfies

\[
-\text{div}(|x|^{-2(\sqrt{\mu} - \sqrt{\nu})} \nabla v) = |x|^{-\sqrt{\mu} + \sqrt{\nu} - \mu} f(x, |x|^{-\sqrt{\mu} + \sqrt{\nu} - \mu} v).
\]

By the elliptic regularity theory (see [9]), \( v \in C^2(\Omega \setminus \{0\}) \cap C^1(\overline{\Omega} \setminus \{0\}) \). Let \( \rho > 0 \) be small enough such that \( B_\rho(0) \subset \subset \Omega \). Choose

\[
\varphi = \eta^2 v_i^{2(s-1)} \in H^1_0(\Omega, |x|^{-2(\sqrt{\mu} - \sqrt{\nu})}dx),
\]

\[
s, l > 1, \ v_l = \min\{|v|, l\}, \ \eta \in C^\infty_0(B_\rho(0)),
\]

with the properties \( 0 \leq \eta \leq 1, \ \eta = 1 \) in \( B_r(0), \ r < \rho \) and \( |\nabla \eta| \leq \frac{1}{\rho - r} \). Then from (2.2), we have

\[
\int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu})} \nabla \cdot \nabla \varphi dx = \int_{\Omega} |x|^{-\sqrt{\mu} + \sqrt{\nu} - \mu} f(x, |x|^{-\sqrt{\mu} + \sqrt{\nu} - \mu} v) \varphi.
\]

Furthermore, (2.3) can be rewritten as:

\[
\int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu})} \\
\times \left( 2\eta v_i^{2(s-1)} \nabla \eta \cdot \nabla v + \eta^2 v_i^{2(s-1)} |\nabla v|^2 + 2(s-1)\eta^2 v_i^{2(s-1)} |\nabla v_i|^2 \right) dx
= \int_{\Omega} |x|^{-\sqrt{\mu} + \sqrt{\nu} - \mu} f(x, |x|^{-\sqrt{\mu} + \sqrt{\nu} - \mu} v) \eta^2 v_i^{2(s-1)} dx.
\]

Observe that for any \( \epsilon > 0 \) small,

\[
2 \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu})} \eta v_i^{2(s-1)} \nabla \eta \cdot \nabla v
\leq \epsilon \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu})} \eta^2 v_i^{2(s-1)} |\nabla v|^2 dx
+ C(\epsilon) \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu})} |\nabla \eta|^2 |v|^2 v_i^{2(s-1)} dx.
\]
and by the assumption of \((H)\),
\[
\left| \int_{\Omega} |x|^{-\sqrt{\mu} + \sqrt{\nu - \mu}} f(x, |x|^{-\sqrt{\mu} + \sqrt{\nu - \mu}}) \varphi \, dx \right| \leq C \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} (\eta^2 v_i^2 \varphi) \, dx
\]
(2.6)
\[
\leq C \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} \eta^2 |v_i|^{2(s-1)} \, dx + C \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} \eta^2 |v_i|^{2(s-1)} \, dx.
\]
Inserting (2.5), (2.6) into (2.4), we get
\[
\int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} (\eta^2 v_i^2 \varphi) \, dx
\]
(2.7)
\[
\leq C \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} (\eta^2 + \nabla \eta^2) |v_i|^2 \, dx + C \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} \eta^2 |v_i|^{2(s-1)} \, dx.
\]
Now we recall the Caffarelli-Kohn-Nirenberg inequality (see [5]):
\[
\left( \int_{\Omega} |x|^{-2b} |w|^p \, dx \right)^{\frac{2}{p}} \leq C_{a,b} \int_{\Omega} |x|^{-2a} |\nabla w|^2 \, dx \quad \forall w \in H_0^1(\Omega, |x|^{-2a} \, dx),
\]
(2.8)
where \(-\infty < a < \frac{N-2}{2}, a \leq b \leq a+1, p = \frac{2N}{N-2+2(b-a)}\) and \(C_{a,b}\) is a positive constant depending on \(a, b\).

In the sequel we take \(a = b = \sqrt{\mu} - \sqrt{\nu - \mu} < \frac{N-2}{2}\) in (2.8); then \(p = 2^*\).
Choosing \(w = \eta v_i^2 \varphi^{-1}\) in (2.8), together with (2.7), we derive
\[
\left( \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} |\eta v_i^{s-1}|^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C_{a,b} \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} |\nabla (\eta v_i^{s-1})|^2 \, dx
\]
(2.9)
\[
\leq 2C_{a,b} \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} \times \left( |\nabla \eta|^2 |v_i|^{2(s-1)} + \eta^2 v_i^{2(s-1)} |\nabla \eta|^2 \, dx + (s-1)^2 \eta^2 v_i^{2(s-1)} |\nabla \eta|^2 \right) \, dx
\]
\[
\leq Cs \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} (\eta^2 + |\nabla \eta|^2) |v_i|^{2(s-1)} \, dx + C \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\nu - \mu})} \eta^2 |v_i|^{2(s-1)} \, dx.
\]
Since \(u \in L^p(\Omega), \forall p < \frac{2N}{N-2+2(\sqrt{\mu} - \mu)}\) (see Remark 2.2), we may choose
\[
\frac{N}{2} < q < \frac{N(N-2)}{2(N-2+2\sqrt{\mu} - \mu)}.
\]
Then
\[
(2^* - 2)q < \frac{2N}{N-2+2\sqrt{\mu} - \mu} \quad \text{and} \quad 2 < \frac{2q}{q-1} < 2^*.
\]
Therefore we deduce that for any $\epsilon > 0$,

\[
\int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |v|^{2^{*}v_1^{2(s-1)}} dx \\
= \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |u|^{2^{*}} |\eta v v_1^{s-1}|^2 dx \\
\leq ||u||_{L^{2^{*}-(2s)}(\Omega)} ||x|^{-\sqrt{\mu} + \sqrt{\mu - \mu_0}} |\eta v v_1^{s-1}|^2_{L^{2s-1}(\Omega)} \\
(2.10)
\]

\[
\leq ||u||_{L^{2^{*}-(2s)}(\Omega)} \times (\epsilon ||x|^{-\sqrt{\mu} + \sqrt{\mu - \mu_0}} |\eta v v_1^{s-1}|_{L^{2s}(\Omega)}^2 \\
+ C(N,q)\epsilon^{-\frac{N}{2q}} ||x|^{-\sqrt{\mu} + \sqrt{\mu - \mu_0}} |\eta v v_1^{s-1}|_{L^{2}(\Omega)}^2)^2 \\
\leq C\epsilon^2 \left( \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta v v_1^{s-1}|^2 dx \right)^{\frac{2}{2^*}} \\
+ C\epsilon^{-\frac{2N}{2q-N}} \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta v v_1^{s-1}|^2 dx.
\]

Inserting (2.10) into (2.9), we obtain

\[
\left( \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta v v_1^{s-1}|^2 dx \right)^{\frac{2}{2^*}} \\
\leq C\epsilon^2 \left( \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta v v_1^{s-1}|^2 dx \right)^{\frac{2}{2^*}} \\
+ C\epsilon \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} (\eta^2 + |\nabla \eta|^2) |v|^{2^{*}v_1^{2(s-1)}} dx \\
+ C\epsilon^{-\frac{2N}{2q-N}} \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta v v_1^{s-1}|^2 dx.
\]

Taking $\epsilon = \frac{1}{\sqrt{2^{*}s}}$ in (2.11), we conclude

\[
\left( \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta v v_1^{s-1}|^2 dx \right)^{\frac{2}{2^*}} \\
\leq C\epsilon^2 \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} (\eta^2 + |\nabla \eta|^2) |v|^{2^{*}v_1^{2(s-1)}} dx,
\]

where $\alpha = \frac{2q}{2q - N} > 0$.

Note that

\[
\int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta|^2 |v|^{2^{*}v_1^{2(s-2)}} dx \leq \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta v v_1^{s-1}|^2 dx,
\]

so from (2.12), we get

\[
\left( \int_{\Omega} |x|^{-2^{*}(\sqrt{\mu} - \sqrt{\mu - \mu_0})} |\eta|^2 |v|^{2^{*}v_1^{2(s-2)}} dx \right)^{\frac{2}{2^*}} \\
\leq C\epsilon^2 \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\mu - \mu_0})} (\eta^2 + |\nabla \eta|^2) |v|^{2^{*}v_1^{2(s-1)}} dx \\
\leq C\epsilon^2 \int_{\Omega} |x|^{-2(\sqrt{\mu} - \sqrt{\mu - \mu_0})} (\eta^2 + |\nabla \eta|^2) |v|^{2^{*}v_1^{2(s-1)}} dx.
\]
Using the choice of the cut-off function $\eta$, we deduce

\[
\left( \int_{B_r(0)} |x|^{-2^*(\sqrt{\mu} - \sqrt{\mu - \mu}) |v|^{2^* s - 2} dx \right)^{\frac{1}{s^*}} \leq \frac{C s^*_0}{(\rho - r)^2} \int_{B_r(0)} |x|^{-2^*(\sqrt{\mu} - \sqrt{\mu - \mu}) |v|^{2^* s - 2} dx.
\]

(2.14)

Choosing $s^* > 0$ such that

\[
\frac{N}{N - 2} < s^* < \frac{N}{N - 2 - 2\sqrt{\mu - \mu}},
\]

we define the sequence

\[
s_j = s^* \left( \frac{2^*}{2} \right)^j, \quad j = 0, 1, 2, \ldots
\]

and take $s = s_j$ in (2.14). Then

\[
\left( \int_{B_{r_{j+1}}(0)} |x|^{-2^*(\sqrt{\mu} - \sqrt{\mu - \mu}) |v|^{2^* s_{j+1} - 2} dx \right)^{\frac{1}{s_{j+1}}} \leq \frac{C s^*_0}{(\rho_0 - \rho_j)^2} \left( \int_{B_{r_j}(0)} |x|^{-2^*(\sqrt{\mu} - \sqrt{\mu - \mu}) |v|^{2^* s_j - 2} dx \right)^{\frac{1}{s_j}}.
\]

(2.15)

Let $\rho_0 > 0$ be small enough such that $B_{2\rho_0}(0) \subset \subset \Omega$, and $r_j = \rho_0(1 + \rho_j^2)$, $j = 0, 1, 2, \ldots$. Taking $\rho = r_j$ and $r = r_{j+1}$ in (2.15), we derive

\[
\left( \int_{B_{r_{j+1}}(0)} |x|^{-2^*(\sqrt{\mu} - \sqrt{\mu - \mu}) |v|^{2^* s_{j+1} - 2} dx \right)^{\frac{1}{s_{j+1}}} \leq \frac{C s^*_0}{(1 - \rho_0) \rho_0} \sum_{j=0}^{\infty} \frac{1}{s_j} - \sum_{j=0}^{\infty} \frac{1}{s_j} \times \prod_{j=0}^{\infty} \frac{s^*_0}{s_j} \left( \int_{B_{r_0}(0)} |x|^{-2^*(\sqrt{\mu} - \sqrt{\mu - \mu}) |v|^{2^* 2s_j - 2} dx \right)^{\frac{1}{s_j}}.
\]

(2.16)

It is not difficult to verify

\[
\sum_{j=0}^{\infty} \frac{1}{2s_j} = \frac{1}{2s^*} \sum_{j=0}^{\infty} \left( \frac{2}{2^*} \right)^j \leq C, \quad \sum_{j=0}^{\infty} \frac{j}{2s_j} = \frac{1}{2s^*} \sum_{j=0}^{\infty} j \left( \frac{2}{2^*} \right)^j \leq C,
\]

(2.17)

\[
\prod_{j=0}^{\infty} \frac{s^*_0}{s_j} = (s^*)^{\frac{s^*_0}{s_j} \sum_{j=0}^{\infty} \left( \frac{2}{2^*} \right)^j} \left( \frac{2^*}{2} \right)^{\frac{s^*_0}{s_j} \sum_{j=0}^{\infty} j \left( \frac{2}{2^*} \right)^j} \leq C.
\]

(2.18)

By the choice of $s^*$, we deduce

\[
2^* < 2s^* < \frac{2N}{N - 2 - 2\sqrt{\mu - \mu}}.
\]
and then
\[ \int_{B_{r_0}(0)} |x|^{-2^*((\sqrt{\bar{\mu}}-\sqrt{\mu-\mu})+2v^2 u^2 s^{s-2} - 2l^2 s^{s-2} \int_{\Omega} |u|^{2s} \, dx \leq (diam\Omega)^{2s^* - 2^*} \int_{\Omega} |u|^{2s} \, dx \leq C. \]

Inserting (2.17)-(2.19) into (2.16), we obtain
\[ \|v_l\|_{L^{2s^*+1}(B_{\rho_0}(0))} \leq \|v_l\|_{L^{2s^*+1}(B_{r_j+1}(0))} \leq (diam\Omega)^{2s^* - 2^*} \int_{B_{r_j+1}(0)} |x|^{-2^*((\sqrt{\bar{\mu}}-\sqrt{\mu-\mu})+2v^2 u^2 s^{s-2} - 2l^2 s^{s-2} \int_{\Omega} |u|^{2s} \, dx \leq C. \]

Note that \( s_{j+1} \rightarrow \infty \) as \( j \rightarrow \infty \). So let \( j \rightarrow \infty \) in the above inequality, and infer
\[ \|v_l\|_{L^\infty(B_{\rho_0}(0))} \leq C, \]
and (1.3) can be obtained by taking \( l \rightarrow +\infty \).

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