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WHEN ARE FREDHOLM TRIPLES OPERATOR HOMOTOPIC?

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Abstract. Fredholm triples are used in the study of Kasparov's KK-groups, and in Connes's noncommutative geometry. We define an absorption property for Fredholm triples, and give an if and only if condition for a Fredholm triple to be absorbing. We study the interaction of the absorption property with several of the more common equivalence relations for Fredholm triples. In general these relations are coarser than homotopy in the norm topology. We give simple conditions for an equivalence of triples to be implemented by an operator homotopy (*i.e.* a homotopy with respect to the norm topology). This can be expected to have applications in index theory, as we illustrate by proving two theorems of Pimsner-Popa-Voiculescu type. We show that there is some relationship with the interesting Toms–Winter characterization of $\mathcal{D}$-absorbing algebras, recently obtained as part of Elliott's classification program.

1. Introduction

Kasparov's group $KK^1(A, B)$ can be defined in terms of so-called Fredholm triples. Fredholm triples are of fundamental importance in index theory, noncommutative geometry [8], and of increasing importance in C^*-algebra theory. The notion of Fredholm triple is motivated by the classical Fredholm alternative, see for example chapters 14 and 17 of [24]. We address the following question: if two Fredholm triples are homotopic in the weak sense defined by Kasparov, can the homotopy be improved to a norm-continuous homotopy? This is of considerable interest in index theory, since an index, however defined, is likely to remain unchanged under norm-continuous homotopies, but is less likely to be invariant under Kasparov homotopy (homotopy with respect to the strict topology). Situations where it would be desirable to replace a homotopy in the strict topology by a homotopy in the finer norm topology also arise in C^*-algebra theory.

One of our results proves a generalization of a conjecture in a preprint by Leichtnam and Piazza (compare Corollary 1.8 and the conjecture at the end of section 2.4 of the preprint [19]).

We first define the class of Fredholm modules that we will work with.

Definition. Let B be a σ-unital stable C^*-algebra, and let A be a separable C^*-algebra. A stabilized ungraded Fredholm cycle in $KK^1(A, B)$ is a triple $(\mathcal{M}(B), \phi, F)$

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where ϕ is a homomorphism from A to $\mathcal{M}(B)$, and the self-adjoint operator $F \in \mathcal{M}(B)$ is such that $(F^2 - F)\phi(a) \in B$, $[F, \phi(a)] \in B$, and $\phi(a)(F - F^*) \in B$. A cycle is *degenerate* if these three expressions are zero.

Thus, Fredholm triples in KK^1 are specified by a homomorphism ϕ and an operator that is an approximate projection. There is a similar description of the even degree group $KK^0(A, B)$ involving unitaries instead of projections. For more information on the various pictures of KK-theory, see chapter 17 of [2].

There are a number of apparently quite different equivalence relations on Fredholm triples that however all give the same group. In general, these equivalence relations are motivated by index theory. For example, the so-called *cp* equivalence relation is given by perturbation of the operator F by the "almost-compact" operators $J_\phi := \{m \in \mathcal{M}(B) : \ \phi(a)m \in B, \quad \forall\, a \in A\}$, addition of degenerate cycles, and unitary equivalence by multiplier unitaries. This relation interacts very nicely with, for example, the index functor defined in [24, ch. 17]. The *oh* equivalence relation is given by norm-continuous homotopy of the operator F, unitary equivalence, and addition of degenerate cycles. Kasparov's h equivalence relation is basically given by a strict topology homotopy. In working with these equivalence relations, one finds that relations involving addition of degenerate cycles are inconvenient since there are many degenerate cycles, and it is hard to tell what the effect of adding a degenerate cycle is.

Since a Kasparov homotopy in the strict topology is sufficient to imply that two triples are equivalent, our problem of improving homotopies leads to the following two questions, which we call the *cp*-equivalence problem and the *oh*-equivalence problem, respectively:

i) Given two equivalent Fredholm triples $\mathcal{T}_i := (E, \phi, F)$, when is it true that equivalence of the triples can be implemented by compact perturbation and unitary equivalence (without adding degenerate cycles)?
ii) Given two equivalent Fredholm triples $\mathcal{T}_i := (E, \phi, F)$, when is it true that equivalence of the triples can be implemented by norm-continuous homotopy and unitary equivalence (without adding degenerate cycles)?

One solution to the first problem is simply to require that $\mathcal{T}_1$ and $\mathcal{T}_2$ be *cp*-absorbing, in the sense that $\mathcal{T}_i + \mathcal{D}$ is equivalent to $\mathcal{T}_i$ for all degenerate cycles $\mathcal{D}$, under compact perturbation and unitary equivalence. There is a corresponding notion of *oh*-absorption. It is clear that assuming absorption will simplify the above equivalence relations, but we have just replaced one hard problem by another, since it is not yet clear which cycles are absorbing in either sense.

In order to keep our proofs short, we shall use the abstract Weyl-von Neumann theorem due to Elliott and Kucerovsky [9] as a lemma. This theorem is directly applicable to the Ext picture of KK-theory, so our initial result (Theorem 1.4, which generalizes the main result of [9]) will be phrased in terms of extensions. We now give some relevant definitions.

Definition. The $\mathrm{Ext}(A, B)$ group is given by injective completely positive weakly nuclear maps $\tau : A \longrightarrow \mathcal{M}(B \otimes \mathcal{K})$ that become homomorphisms modulo $B \otimes \mathcal{K}$, with the equivalence relation given by unitary equivalence by multiplier unitaries and addition of trivial extensions. Trivial extensions are those that are homomorphisms into $\mathcal{M}(B \otimes \mathcal{K})$.

We say that a map τ as above is weakly nuclear if $a \mapsto b\tau(a)b^*$ is always a nuclear map, for each $b \in B$. A weakly nuclear extension π is said to be absorbing if the BDF sum of π with a weakly nuclear trivial extension ϕ is unitarily equivalent to π.

Recalling [3] that an extension $\tau : A \longrightarrow \mathcal{M}(B)$ can be equivalently given as a (semisplit) short exact sequence $0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$, we say that

Lemma 1.1 ([9])**.** *Let A and B be separable C^*-algebras, with B stable, and A unital. Let $0 \longrightarrow B \longrightarrow C \longrightarrow A \to 0$ be an essential (unital) extension with a weakly nuclear completely positive splitting map. Then the following are equivalent:*

i) *The extension absorbs all trivial weakly nuclear (unital) extensions.*
ii) *The extension is* purely large*, meaning that the extension algebra C has the property that, for every positive element $c \in C^+$ that is not in B, the hereditary subalgebra $\overline{cBc}$ contains a stable subalgebra that generates B as an ideal.*

There is one more nice fact about (extension) algebras $C \subset \mathcal{M}(B)$ that have the purely large property: they satisfy an abstract Weyl-von Neumann theorem [9].

Lemma 1.2. *Let B be a separable stable C^*-algebra, and let C be a separable subalgebra of $\mathcal{M}(B)$, containing B, and having the purely large property (that $\overline{cBc}$ contains a stable subalgebra that is full in B, for each positive c in C that is not in B). If $\phi : C \longrightarrow \mathcal{M}(B)$ is a unital, weakly nuclear, completely positive map that is zero on B, then there is a multiplier isometry V such that $\phi(c) = V^*cV$ mod B.*

We now need a routine C^*-algebraic lemma, and we will then be ready to prove a necessary generalization of Lemma 1.1. We say that a positive element a is properly infinite if $a \succsim b+c$, where b and c are orthogonal positive elements that are Murray-von Neumann equivalent to a, where $\succsim$ is Rørdam's comparison operator [13].

Lemma 1.3. *Let A be a unital C^*-algebra that contains a unital copy of O_2. A positive element $a \in A$ is properly infinite and full if and only for some $x \in A$ we have $xax^* = 1$. Moreover, xx^* is full and properly infinite if and only if x^*x is.*

The following theorem improves our earlier results and will be needed for the applications. The interest of the theorem comes from the fact that, like an index theorem, it shows that a purely topological property, namely absorption, is equivalent to certain algebraic properties.

Theorem 1.4. *Let $\tau : A \longrightarrow \mathcal{M}(B)/B$ be an extension of separable algebras, and let B be stable. The following are equivalent:*

i) *The extension τ absorbs trivial nuclear extensions.*
ii) *Every positive element of the image of τ is full and properly infinite.*
iii) *Every positive element of the extension algebra of τ that is not in the canonical ideal is properly infinite and full in the multipliers.*
iv) *Every positive element c of the extension algebra that is not in the canonical ideal, B, has the property that $\overline{cBc}$ is stable.*

Remark 1.5. If we consider the special case of algebras having the property that every full extension is absorbing, then the present theorem shows that this topological property is equivalent to the algebraic property that every full positive corona element is properly infinite. This is already known, since it can be obtained by combining Theorem 4.2 of [17] and Lemma 4.4 of [14].

Proof. That iv) implies i) is an immediate consequence of the main result of Elliott and Kucerovsky [9].

We now show that i) implies ii). Since an absorbing extension will absorb a trivial full nuclear extension (and such an extension exists), the absorbing extension is necessarily itself full, in the sense that the image has trivial intersection with ideals of the corona.

Hence we need only demonstrate that elements of the image are properly infinite. Choosing some arbitrary positive element $a \in A$, we restrict the map τ to $\tau_a : C^*(a) \longrightarrow \mathcal{M}(B)/B$. We also identify a with its image in the corona, so that τ_a is a map from a subalgebra of the corona to the corona. Since $C^*(a)$ is commutative, it is nuclear, and hence we can apply the Choi-Effros lifting theorem [7] to obtain a completely positive map $\chi : C^*(a) \longrightarrow \mathcal{M}(B)$ lifting $\tau_a + \tau_a$. Let $\tilde{a}$ denote some positive lifting of a to the multipliers. Then, the map $\chi \circ \pi : C^*(\tilde{a}, B) \longrightarrow \mathcal{M}(B)$ is zero on B, and moreover the source algebra is purely large in the sense of Elliott and Kucerovsky (since this algebra is a subalgebra of the extension algebra of the absorbing extension τ). If the algebra $C^*(\tilde{a}, B)$ is not unital, we unitize it, and it is known that the unitization of a purely large algebra is still purely large [9, §16]. The unitization of a completely positive map is completely positive, so we now have a map $\chi \circ \pi : C^*(\tilde{a}, B)\tilde{} \longrightarrow \mathcal{M}(B)$ that is zero on B, unital, completely positive, and has a purely large algebra as domain. We can apply the abstract Weyl-von Neumann theorem to this map. There is therefore a multiplier isometry W such that in the corona, $\chi \circ \pi(c) = W^*cW$. In the corona, $\chi \circ \pi$ is effectively equal to the map $\tau_a + \tau_a$. The sum is a BDF sum, so $\tau_a + \tau_a$ is really $v_1\tau_a(\cdot)v_1^* + v_2\tau_a(\cdot)v_2^*$, where the v_i generate a copy of O_2 in the multipliers. We conclude that $W^*f(a)W = v_1\tau_a(f(a))v_1^* + v_2\tau_a(f(a))v_2^*$ in the corona. We see that $W^*f(a)W = b + c$, where b and c are orthogonal and equivalent to $f(a)$. This certainly implies that $f(a) \succsim f(a) \oplus f(a)$, where $\succsim$ is Rørdam's comparison operator, and therefore $f(a)$ is properly infinite in the corona, as claimed.

We next show that ii) implies iii). Note that if c from the extension algebra is positive, then the image in the corona, c/B, is properly infinite and full. Hence, there is an r_0 such that $r_0\pi(c)r_0^*$ is approximately equal to $1_{\mathcal{M}(B)/B}$, hence invertible. Applying the inverse from left and right, and lifting, we find that there is an $\tilde{r}$ such that $\tilde{r}c\tilde{r}^* = 1_{\mathcal{M}(B)} + b$. The element b belongs to the canonical ideal B, which is stable, so there is a sequence of isometries, v_i, coming from stability, such that $v_i^*bv_i$ goes to zero in norm. We thus have that for some suitably large index i, the expression $v_i^*\tilde{r}c\tilde{r}^*v_i = 1 + v_i^*bv_i$ is close enough to 1 to be invertible, and thus there is an $r' \in \mathcal{M}(B)$ such that $r'cr'^* = 1_{\mathcal{M}(B)}$.

Finally, we prove that iii) implies iv). To see this, we use the partial converse to Brown's stable isomorphism theorem [5]. This converse states that if a σ-unital C^*-algebra A contains a stable full hereditary subalgebra generated by a multiplier projection, then the algebra A is necessarily stable. Consider the algebra $A := \overline{cBc}$, where c is a full positive element of the multipliers. By hypothesis, $r'cr'^* = 1_{\mathcal{M}(B)}$, meaning that $V := c^{1/2}r'^*$ is an isometry. Since $VV^* \leq c\|r'\|^2$, the stable hereditary subalgebra generated by VV^* within B is contained in A. It follows moreover that the projection VV^* actually multiplies A into itself. We thus have exactly the situation that Brown's second theorem applies to, and A is stable, as claimed. $\square$

We say that triples satisfying the conditions of the next proposition are *large.*

Corollary 1.6. *A stabilized Fredholm triple* $(\mathcal{M}(B), \phi, F) \in KK^1(A, B)$ *is cp-absorbing if and only if* $\pi(F\phi(a)F^*)$ *is properly infinite and full in the corona for all positive* $a \in A$.

Proof. There is a natural isomorphism

$$\begin{aligned} KK^1(A, B) &\longrightarrow \mathrm{Ext}(A, B), \\ (\phi, F) &\mapsto \pi(F\phi F^*), \end{aligned}$$

and, as noted by Kasparov, the natural equivalence relation on Ext corresponds exactly to the stabilized *cp* equivalence relation in KK^1, meaning equivalence modulo perturbation by J_ϕ, unitary equivalence, and addition of stabilized degenerate cycles [2, §17.6.4]. Addition of stabilized degenerate cycles corresponds to the addition of trivial extensions in Ext, so we see that for absorbing cycles (ϕ_1, F_1) and (ϕ_2, F_2), we have that $(\phi_1, F_1) \sim_{cp} (\phi_2, F_2)$ in KK^1 if and only if $\pi(F_1\phi_1F_1^*)$ is unitarily equivalent to $\pi(F_2\phi_2F_2^*)$. □

We next consider the *oh*-equivalence problem. Our main results are:

i) An extension is large if and only if it is *oh*-absorbing.
ii) If two triples are operator homotopic, and one is large, then the entire homotopy can be taken to lie in the set of large triples.
iii) If two large stabilized triples are Kasparov homotopic, then they are actually operator homotopic.

It is quite possible for a large triple to be Kasparov homotopic to a triple that is not large, so the two results above are the most we can expect in general. It is quite interesting that *oh*-absorption is equivalent to *cp*-absorption.

Proposition 1.7. *Let* $(\mathcal{M}(B), \phi, F_0)$ *and* $(\mathcal{M}(B), \phi, F_1)$ *be two operator homotopic Fredholm triples. If one of them is large, they both are.*

Proof. Let F_t be the operator homotopy linking the two cycles. Defining $I_\phi := \{m \in \mathcal{M}(B) : [\phi(a), m] \in B, \ \forall\, a \in A\}$ and $J_\phi := \{m \in \mathcal{M}(B) : \phi(a)m \in B, \forall\, a \in A\}$, we note that F_t is a norm-continuous path of projections in the C^*-algebra I_ϕ/J_ϕ. Since norm-continuous homotopy of projections implies unitary equivalence of projections, it follows that there is a norm continuous path u_t of unitaries in I_ϕ/J_ϕ such that $\pi_{J_\phi}(F_t) = u_t\pi_{J_\phi}(F_0)u_t^*$. We note that ϕ multiplies I_ϕ/J_ϕ into the corona, $\mathcal{M}(B)/B$. Applying the map from $KK^1(A, B)$ to $\mathrm{Ext}(A, B)$, we have that $(\mathcal{M}(B), \phi, F_t)$ goes to the extension $\pi_B(F_t\phi(\cdot)F_t)$, but u_t commutes with $\phi(a)$ modulo B, so that $u_t^*\phi(a)u_t = \phi(a)$. Hence we see that the given operator homotopy goes to the extension $a \mapsto u_tF_0\phi(a)F_0u_t^*$, where u_t is a unitary in I_ϕ/J_ϕ. The proof is finished if we can show that $u_tF_0\phi(a)F_0u_t^*$ is properly infinite and full (in the corona) when $F_0\phi(a)F_0$ is properly infinite and full. This would be obvious if u_t were in the corona, but that is not the case, so we resort to a factoring trick. By Lemma 1.3, if $F_0\phi(a)F_0$ is properly infinite and full, there is an r such that $1 = rF_0\phi(a)F_0r^*$. But then $r' := r\phi(a_n)u_t^*$ is an element of the corona, where a_n comes from an approximate unit for A. Moreover, choosing a_n appropriately, we can insure that r' is approximately a quasi-inverse for $u_tF_0\phi(a)F_0u_t^*$, in the sense that $r'u_tF_0\phi(a)F_0u_t^*r'^*$ is close to 1 in norm, and hence is invertible. □

The next corollary shows that if two large cycles define the same element of the KK-group (in which case they are Kasparov homotopic), then in fact the equivalence can be implemented using only unitary equivalence and operator homotopy. As such, it is one of our main results.

Corollary 1.8. *A stabilized Fredholm triple $(\mathcal{M}(B), \phi, F) \in KK^1(A, B)$ is oh-absorbing if and only if it is large.*

Proof. Let us first assume *oh*-absorption. Thus, the given cycle $\mathcal{T}$ is *oh*-homotopic to $\mathcal{T} + \mathcal{D}$, where $\mathcal{D}$ is a degenerate cycle. Let $\mathcal{D} =: (\mathcal{M}(B), \psi, G)$ be such that $a \mapsto \pi[G\psi(a)G]$ is properly infinite and full in the corona for all positive $a \in A$. (For example, pick $G = 1$ and ψ to be Kasparov's generalized GNS representation [11]). Then, if we consider $\mathcal{T} + \mathcal{D} = (M_2(\mathcal{M}(B)), \phi \oplus \psi, F \oplus 1)$, we note that if a is positive, then $\pi[(F \oplus 1)(\phi(a) \oplus \psi(a))(F \oplus 1)]$ majorizes the properly infinite full element $\pi[0 \oplus \psi(a)]$. Therefore, the cycle $\mathcal{T} + \mathcal{D}$ is large. By Proposition 1.7, it follows that the cycle $\mathcal{T}$ is itself large, as was to be shown.

Conversely, if $\mathcal{T}$ is large, then by Corollary 1.6, the sum $\mathcal{T} + \mathcal{D}$ with a degenerate cycle is unitarily equivalent, modulo compact perturbation, to $\mathcal{T}$. However, a compact perturbation can of course be regarded as an operator homotopy (by a straight-line homotopy). □

2. Remarks on full extensions (the corona factorization property)

In [17] we gave a simple algebraic criterion that determines exactly when all full extensions of A by B are absorbing, and (see also [15]) we have moreover characterized this property in terms of the existence of copies of W in certain relative commutants, where W is the abelian subalgebra of $O_\infty := C^*(v_i)$ generated by the projections $v_i v_i^*$. We push this characterization one step further in the following proposition, derived from Lin's absorption theorem, where it seems convenient to rephrase in terms of copies of O_2 instead.

Proposition 2.1. *Let A and B be separable, with A unital and B stable. Suppose moreover that B is of the special form $B' \otimes \mathcal{K}$ with B' unital. Then, the following are equivalent:*

i) *All full unital extensions of A by B are absorbing, and*
ii) *for every full positive element c of the corona, there exists an approximately commuting sequence of unital copies of O_2 (in the corona).*

Proof. Supposing that ii) holds, we thus have a sequence of copies of O_2, with generators a_n and b_n such that a_n and b_n asymptotically commute with the given element c. Recall that Kirchberg and Rørdam defined [13] a positive element c in the corona Q to be properly infinite if there exists a sequence of elements R_n in $M_2(\mathrm{Q})$ such that $R_n \left(\begin{smallmatrix} c & 0 \\ 0 & 0 \end{smallmatrix}\right) R_n^*$ goes in norm to $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$. It is straightforward to check that $R_n := \left(\begin{smallmatrix} a_n & 0 \\ b_n & 0 \end{smallmatrix}\right)$ will do the job. Thus, every positive element of the corona is properly infinite, which implies that full elements of the corona have the quasi-invertibility property of Theorem 1.4, so that if every positive element of the image of some extension is either full in the corona or zero, it follows that this extension is absorbing.

For the converse, note that it is sufficient to find a copy of O_2 in $\ell_\infty(Q)$, where Q is the corona algebra of B, that commutes with the constant sequence (c) in the

quotient $\ell_\infty(Q)/c_0(Q)$. In fact, since O_2 has stable relations, it is semiprojective, and thus [21] a homomorphism $\phi : O_2 \longrightarrow \ell_\infty(Q)/c_0(Q)$ lifts to a homomorphism into $\ell_\infty(Q)$. It is therefore sufficient to find a copy of O_2 in $\ell_\infty(Q)/c_0(Q)$ that commutes with $(c) \in \ell_\infty(Q)/c_0(Q)$. To do this, we first of all note that Remark 1.5 showed that if all full extensions are absorbing, then all full positive multiplier elements have the quasi-invertibility property $r'cr'^* = 1_{\mathcal{M}(B)}$. This of course implies (and is actually equivalent to) the same property for full corona elements. Lemma 6.5 in [20] states that if we have this property for full corona elements, and also two other properties that hold in the corona of a stable algebra, then there is a unital copy of O_∞ in the relative commutant of the map $\psi : C^*(c) \longrightarrow \ell_\infty(Q)/c_0(Q)$ defined by $f(c) \mapsto (f(c), f(c), f(c), \cdots)$. The unit in a copy of O_∞ is properly infinite, thus giving the required copy of O_2. □

The above result is likely to have a generalization to the case of individual extensions.

It is easy to show that all full extensions of a purely infinite (possibly nonsimple) stable separable algebra are absorbing. The first part of the above proof gives an interesting generalization of this fact. As background, recall that a C^*-algebra B is said to be $\mathcal{D}$-absorbing if $\mathcal{D} \otimes B \cong B$, and $\mathcal{D}$ is said to be self-absorbing if $\mathcal{D} \otimes \mathcal{D} \cong \mathcal{D}$. Toms and Winter [23, Thm. 2.2] have characterized this property, for algebras $\mathcal{D}$, as follows.

Theorem 2.2 (Toms–Winter)**.** *Let B and $\mathcal{D}$ be separable C^*-algebras with $\mathcal{D}$ unital and approximately self-absorbing. Then B is $\mathcal{D}$-absorbing if and only if there is a central embedding*

$$\iota : \mathcal{D} \longrightarrow \ell_\infty(\mathcal{M}(B))/c_0(\mathcal{M}(B)).$$

The main examples of simple, unital, and approximately self-absorbing algebras are the Jiang and Su algebra $\mathcal{Z}$, the Glimm algebras, and the Cuntz algebras O_2 and O_∞.

We now show that if an algebra is $\mathcal{D}$-absorbing for a $\mathcal{D}$ with a properly infinite unit, then every full extension of that algebra is absorbing as an extension (which is equivalent to the algebra having the so-called corona factorization property).

Proposition 2.3. *If B is separable, stable, and $\mathcal{D}$-absorbing for a $\mathcal{D}$ with a properly infinite unit, then every full extension $\tau : A \longrightarrow \mathcal{M}(B)/B$ is absorbing as an extension.*

Proof. Consider the embedding given by the previous theorem. The image of a properly infinite element under a homomorphism is still properly infinite, so in $\ell_\infty(\mathcal{M}(B))/c_0(\mathcal{M}(B))$ we have a unital and central copy of O_2. We proceed exactly as in the first part of the proof of Proposition 2.1. □

In fact, it is true that $\mathcal{Z}$-stability implies that all full extensions are absorbing, even though the unit of $\mathcal{Z}$ is not infinite. This will be discussed elsewhere.

We now return to the setting of Fredholm triples.

3. $C(X)$-algebras and PPV-type theorems

Recall that a $C(X)$-algebra is simply a C^*-algebra B with a map $s : C(X) \longrightarrow B$ into the center of B. For each compact Hausdorff topological space X one can

consider the $C(X)$-extensions

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$

where triviality, the Busby construction, and addition of extensions are all defined precisely as in the C^*-algebra case. There is a slight difference when nuclearity is considered: a $C(X)$-algebra is $C(X)$-nuclear if and only it is a continuous field of algebras over X, and the fiber algebras are nuclear in the usual sense [1, Thm. 7.2].

It is moreover technically preferable to define invertibility of an extension τ as the existence of an extension σ such that $\tau \oplus \sigma$ is trivial. One other, more significant difference between the $C(X)$ case and the C^* case is that there appears to be no UCT-type theorem for the $C(X)$ counterpart of KK-theory. The group $\mathrm{Ext}(X, A, B) = RKK^1(X, A, B)$ can be defined to be the quotient of the invertible $C(X)$-extensions by the usual equivalence relation: unitary equivalence (in the corona) by multiplier unitaries. The absence of a theorem of UCT type seems to indicate that it can be difficult to explicitly construct an $RKK^1(X, A, B)$ cycle. It may therefore be of interest to find theorems giving conditions under which an extension is absorbing in $RKK^1(X, A, B)$. Since the definition of addition and of equivalence is the same as in the C^*-algebra case, it follows that the appropriate definition of absorption for $C(X)$-extensions is just the same as for C^*-extensions. Given a $C(X)$-extension $0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$, there is for each $x \in X$ a fiberwise C^*-extension $0 \longrightarrow B_x \longrightarrow C_x \longrightarrow A_x \longrightarrow 0$, so that we may pose the following question: is a $C(X)$-extension absorbing if all the fiberwise extensions are? This is false in general (a counterexample can be deduced from our earlier paper [18]). We shall give two positive results.

The first one is deduced from a recently announced deformation of projections result of Blanchard.

Proposition 3.1. *If p_x is an infinite projection for each x in $X \subseteq [0,1]^d$, with d finite and X compact (and Hausdorff), then p is an infinite projection.*

The point of the proposition is somewhat subtle: a projection may generally be infinite in the fibers but not infinite globally; however, if the base space is sufficiently nice, this problem does not appear. We also need the following related lemma, which can be proven either from Brown's theorem or the Kasparov stabilization theorem.

Lemma 3.2 ([17, Prop. 3.2])**.** *Let B be a stable σ-unital C^*-algebra and let ℓ be a nonzero positive element of $\mathcal{M}(B)$. The hereditary subalgebra, $\overline{\ell B \ell}$, generated by ℓ of the multipliers $\mathcal{M}(B)$, is isomorphic to a hereditary subalgebra generated by a multiplier projection P. Moreover, if ℓ is a norm-full element of $\mathcal{M}(B)$, then P is also a norm-full element of $\mathcal{M}(B)$.*

The next theorem is closely related to the Pimsner-Popa-Voiculescu result [22] that gives conditions for an extension of $C(X) \otimes \mathcal{K}$ to be absorbing.

Theorem 3.3. *Let $[\tau] := [0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0]$ be an essential $C(X)$-extension with $X := \mathrm{Prim} B$ a closed subset of $[0,1]^d$, with d finite, and B $C(X)$-nuclear. If the extension satisfies a homogeneity condition, namely that $c \in C^+$ is in B if and only if c_x is in B_x for some $x \in X$, then:*

i) *The extension τ is absorbing if and only if every τ_x is absorbing.*

Proof. As in the C^*-algebraic case, given an essential extension as above, there is an injection of C into $\mathcal{M}(B)$ and an injection of A into $\mathcal{M}(B)/B$. If the extension

is absorbing, then by Theorem 1.4, positive elements of $C \setminus B$ are properly infinite (and full), thus the same is true for positive elements $C_x \setminus B_x$. This proves one direction.

For the other direction, suppose that the extensions $0 \longrightarrow B_x \longrightarrow C_x \longrightarrow A_x \longrightarrow 0$ are absorbing, so that for each positive $d \in C_x$ that is not in B_x, the algebra $\overline{dB_xd}$ is stable. By homogeneity, we have that an element $c \in C$ is either in B_x for all x or it is never in B_x for any x. We must show that in this latter case, $\overline{cBc}$ at least contains a stable full subalgebra. By Lemma 3.2, $\overline{cBc}$ is isomorphic to $\overline{pBp}$ for some (full) multiplier projection $p \in \mathcal{M}(B)$. Since the unitary in the construction of Lemma 3.2 commutes with the action of $C(X)$, it follows that the construction respects the fiber structure of the algebra, and hence $\overline{c_xB_xc_x}$ is isomorphic to $\overline{p_xB_xp_x}$. By the absorption hypothesis and Theorem 1.4, the algebras $\overline{p_xB_xp_x}$ are all stable, implying that p_x is properly infinite. By Proposition 3.1, the projection p is infinite, and hence is Murray-von Neumann equivalent to a proper subprojection q. Let v be the partial isometry implementing the equivalence. The abstract Toeplitz algebra $C^*(v)$ generated by v contains a stable subalgebra, S. We claim that $\overline{SBS}$ is a stable full subalgebra of $\overline{pBp}$. Since $X = \mathrm{Prim}B$ is Hausdorff, the fibers B_x are actually simple, implying that to show that $\overline{SBS}$ is full, we need only show that S does not vanish in any fiber. However, S_x is zero if and only if $p_x = q_x$, but of course p_x is infinite. □

Our second result is a generalization of the main result of [18], and is included primarily for purposes of comparison with the above. In addition, we wish to illustrate that Theorem 1.4 allows us to remove one of the technical obstacles in [18].

Recall that in [18], we first obtained the following result on deformation of projections. This result says, roughly speaking, that a strictly continuous family of properly infinite projections is, under suitable conditions, properly infinite.

Theorem 3.4. *Let B be a separable, σ_p-unital, and stable C^*-algebra. Let X be a paracompact and finite-dimensional topological space. If $P \in \mathcal{M}(B \otimes C(X))$ is such that for each $x \in X$, the projections P_x and $1 - P_x$ are both properly infinite and full in $\mathcal{M}(B)$, then P and $1 - P$ are properly infinite and full in $\mathcal{M}(B \otimes C(X))$.*

From this result we had obtained a theorem similar to the following, but originally only for the case of trivial extensions τ.

Theorem 3.5. *Let B_0 be a stable, separable, C^*-algebra with an approximate unit consisting of projections. Let A be unital and separable. Let X be paracompact, second-countable, and finite dimensional*

Consider a nuclear C^*-extension $\hat{\tau} : A \to \mathcal{M}(C(X) \otimes B_0)/(C(X) \otimes B_0)$. If $\hat{\tau}$ is unital and $\hat{\tau}_x$ is absorbing at every point $x \in X$, then $\hat{\tau}$ is absorbing.

The above theorem can be proven much like Theorem 3.3, except that instead of using simplicity of the B_x to show that $\overline{SBS}$ is full, we use the fact that p is properly infinite (and full) to deduce that S is full in $\mathcal{M}(B)$ (and full in B).

Comparing Theorems 3.3 and 3.5, the first theorem allows the fibers to vary, but on the other hand, the extension must be a $C(X)$-extension, and the primitive ideal space of the canonical ideal must be Hausdorff. In the second theorem, the fibers are constant, but there is no requirement that the primitive ideal space of $B_0 \otimes C(X)$ be Hausdorff. Theorem 3.5 has an interesting corollary for Fredholm

triplets (the corollary generalizes, as we may replace the unit interval I by any well-behaved finite-dimensional topological space):

Corollary 3.6. *If $(\mathcal{M}(B), \phi_t, F_t)$ is a Kasparov homotopy via large (stabilized) Fredholm triples, then the cycle $(\mathcal{M}(IB), \phi, F) \in KK^1(A, IB)$ defining the homotopy is itself absorbing.*

This, together with our other results, shows that most of the constructions we may wish to perform with Kasparov triples can be done inside the class of large triples. One interesting open question, which has partially motivated our work, is if there is a simple formula or construction for the Kasparov product of large triples.

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