ROOT CLOSED FUNCTION ALGEBRAS ON COMPACTA OF LARGE DIMENSION

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Abstract. Let \( X \) be a Hausdorff compact space and let \( C(X) \) be the algebra of all continuous complex-valued functions on \( X \), endowed with the supremum norm. We say that \( C(X) \) is (approximately) \( n \)-th root closed if any function from \( C(X) \) is (approximately) equal to the \( n \)-th power of another function. We characterize the approximate \( n \)-th root closedness of \( C(X) \) in terms of \( n \)-divisibility of the first Čech cohomology groups of closed subsets of \( X \). Next, for each positive integer \( m \) we construct an \( m \)-dimensional metrizable compactum \( X \) such that \( C(X) \) is approximately \( n \)-th root closed for any \( n \). Also, for each positive integer \( m \) we construct an \( m \)-dimensional compact Hausdorff space \( X \) such that \( C(X) \) is \( n \)-th root closed for any \( n \).

1. Introduction

Relations between algebraic closedness of the algebra of continuous bounded complex-valued functions \( C(X) \) on a space \( X \) and topological properties of \( X \) have been studied since the 1960s [5]. Recall that the algebra \( C(X) \) is called algebraically closed if each monic polynomial with coefficients in \( C(X) \) has a root in \( C(X) \). For a locally connected compact Hausdorff space, the algebra \( C(X) \) is algebraically closed if and only if \( \dim X \leq 1 \) and \( H^1(X; \mathbb{Z}) = 0 \) [8], [13], where \( H^1(X; \mathbb{Z}) \) denotes the first Čech cohomology group of \( X \) with the integer coefficient (see section 2). It is proved in [13] that for a first-countable compact Hausdorff space \( X \), algebraic closedness of \( C(X) \) is equivalent to a weaker property of square root closedness. The latter means that every function from \( C(X) \) is a square of another function. It should be noted that this property appears in the study of subalgebras of \( C(X) \) [4].

An even weaker property of approximate square root closedness was introduced by Miura [12] and was proved to be equivalent to the square root closedness when the underlying compact Hausdorff space \( X \) is locally connected.

There is a nice characterization of algebraic closedness of \( C(X) \) when \( X \) is a metrizable continuum. Namely, in this case \( C(X) \) is algebraically closed if and only if \( X \) is a dendrite (i.e., a Peano continuum containing no simple closed curves) [10], [13].

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The approximate $n$-th root closedness of $C(X)$ was studied by Kawamura and Miura and was proved to be equivalent to $n$-divisibility of $H^1(X;\mathbb{Z})$ under the additional assumption $\dim X \leq 1$. The universal space for metrizable compacta with the approximately $n$-th root closed $C(X)$ is constructed in [3].

In this paper we characterize the approximate $n$-th root closedness of $C(X)$ for any Hausdorff paracompact space $X$. Namely, $C(X)$ is approximately $n$-th root closed if and only if the group $H^1(A;\mathbb{Z})$ is $n$-divisible for every closed subset $A$ of $X$. If $\dim X \leq 1$, then the $n$-divisibility of $H^1(X;\mathbb{Z})$ implies the $n$-divisibility of $H^1(A;\mathbb{Z})$, so this generalizes Theorem 1.3 of [10]. Further, for each positive integer $m$ we construct an $m$-dimensional metrizable compactum $X$ such that $C(X)$ is approximately $n$-th root closed for any $n$. Note that such examples were known in dimension 1 only. Also, for each positive integer $m$ we construct an $m$-dimensional compact Hausdorff space $X$ such that $C(X)$ is $n$-th root closed for any $n$. This example solves the problem posed in [10]: for a compact Hausdorff space $X$, does square root closedness of $C(X)$ imply $\dim X \leq 1$?

2. Notations, definitions, and ideas of constructions

All maps considered in this paper are continuous. For spaces $X$ and $Y$, we denote the set of all maps from $X$ to $Y$ by $C(X,Y)$. As usual, by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ we denote the integers, the rational numbers, and the complex numbers, respectively. We let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the multiplicative subgroup of $\mathbb{C}$. An inverse spectrum over a directed partially ordered set $(\mathcal{A},<)$ consisting of spaces $X_\alpha$, $\alpha \in \mathcal{A}$, and projections $p_\alpha^\beta : X_\beta \to X_\alpha$, $\alpha,\beta \in \mathcal{A}, \beta > \alpha$, is denoted by $\{X_\alpha,p_\alpha^\beta,\mathcal{A}\}$. Throughout this section $n > 1$ denotes an integer.

By $H^k(X;G)$ we denote the $k$-th Čech cohomology group of the space $X$ with an abelian coefficient group $G$. Note that in the case when $X$ is a Hausdorff paracompact space the Čech cohomologies are naturally isomorphic to the Alexander-Spanier cohomologies [14, p. 334]. Note also that due to the Huber's theorem [9] for a Hausdorff paracompact space $X$ there exists a natural isomorphism between the group of all homotopy classes of maps from $X$ to $K(G,k)$ and the group $H^k(X;G)$ if $G$ is countable. Here $K(G,k)$ denotes the Eilenberg-MacLane complex.

For a space $X$, by $C(X)$ we denote the algebra of all bounded complex-valued functions on $X$, endowed with the supremum norm. We say that $C(X)$ is approximately $n$-th root closed if for every $f \in C(X)$ and every $\varepsilon > 0$ there exists $g \in C(X)$ such that $||f - g^n|| < \varepsilon$. The algebra $C(X)$ is said to be $n$-th root closed if any $f \in C(X)$ has an $n$-th root, which means that there exists $g \in C(X)$ such that $f = g^n$. Note that if $C(X)$ is (approximately) $n$-th root closed, then $C(A)$ is also (approximately) $n$-th root closed for any closed subset $A$ of $X$.

We consider $C(X,\mathbb{C}^*)$ as a multiplicative subgroup of $C(X)$ with a metric inherited from $C(X)$. We say that $C(X,\mathbb{C}^*)$ is (approximately) $n$-th root closed if any $f \in C(X,\mathbb{C}^*)$ has an (approximate) $n$-th root in $C(X,\mathbb{C}^*)$.

The basic idea explored in this paper — the construction of a projective $n$-th root resolution — is outlined as follows. The simplest case has been known in the theory of uniform algebra, and it is called the Cole construction (cf. [15, Chapter 3, §19, pp. 194-197]).

Given a space $X$ and a function $f : X \to \mathbb{C}$ it is not always possible to solve, even approximately, the problem of finding an $n$-th root of $f$ (consider for instance any homotopically non-trivial map from a circle $S^1$ to $\mathbb{C}^*$). Nevertheless, it is
always possible to solve the n-th root problem projectively in the following sense. There exists a space denoted $R_n(X, f)$ and a map $\pi^f: R_n(X, f) \to X$ such that the composition $f \circ \pi^f$ has an n-th root. The space $R_n(X, f)$ is simply the graph of the (multivalued) n-th root of $f$,
$$R_n(X, f) = \{(x, z) \mid f(x) = z^n\} \subset X \times \mathbb{C},$$
and the map $\pi^f$ is the natural projection on $X$. Obviously, the projection of $R_n(X, f)$ to $\mathbb{C}$ is an n-th root of the composition $f \circ \pi^f$. We say that the space $R_n(X, f)$ together with the map $\pi^f$ resolve the n-th root problem for $f$ projectively.

Given any family of maps $\mathcal{M} \subset C(X)$ we can projectively resolve the n-th root problem for all maps from $\mathcal{M}$ simultaneously by using the space
$$R_n(X, \mathcal{M}) = \{(x, (z_f)_{f \in \mathcal{M}}) \mid f(x) = z_f^n \forall f \in \mathcal{M}\} \subset X \times \mathbb{C}^{\mathcal{M}}$$
and defining $\pi^\mathcal{M}: R_n(X, \mathcal{M}) \to X$ to be the natural projection. Let $A$ and $B$ be two subsets of $C(X)$ such that $A \subset B$. There is a natural projection $\pi^\mathcal{B}_A: R_n(X, B) \to R_n(X, A)$ defined by $\pi^\mathcal{B}_A[(x, (z_f)_{f \in B})] = (x, (z_f)_{f \in A})$. We let $R_n(X, \emptyset) = X$ and $\pi^\emptyset = \pi^B$.

We outline the ideas of our constructions in Sections 5 and 6. Suppose that we want to construct a space $X$ with n-th root closed $C(X)$. Take any space $X_1$ and resolve the n-th root problems for $X_1$ projectively using the space $X_2 = R_n(X_1, C(X_1))$. Then resolve all n-th root problems for $X_2$ projectively using $X_3$, and so on. This way we obtain an inverse spectrum of spaces $X_\lambda$ and define $X$ to be the inverse limit of this spectrum. To guarantee that the n-th root problems for $X$ can be solved, we need this spectrum to be factorizing in the following sense: for any map $f: X \to \mathbb{C}$ there exist a space $X_\lambda$ in the spectrum and a map $f_\lambda: X_\lambda \to \mathbb{C}$ such that $f = f_\lambda \circ p_\lambda$, where $p_\lambda: X_\lambda \to X$ is the limit projection. Then the projective resolution of the n-th root problem for $f$ gives us a solution of the n-th root problem for $f$. In order to obtain a factorizing spectrum we make its length uncountable. Namely, we construct the spectrum over $\omega_1$, the first uncountable ordinal.

The space described above is not metrizable for two reasons. First, the length of the spectrum used is not countable. Second, for a metric compactum $X_\lambda$ and a subset $\mathcal{M} \subset C(X_\lambda)$, the space $R_n(X_\lambda, \mathcal{M})$ is metrizable if and only if the set $\mathcal{M}$ is countable. If we want $C(X)$ to have just the approximate n-th root property, it is enough to construct a countable spectrum and for each space $X_\lambda$ to resolve the projective n-th root problem for a countable dense set of maps from $C(X_\lambda)$. Then the limit space $X$ is a metrizable compactum, if we start with a metrizable compactum $X_1$.

To guarantee that for the limit space $X = \lim_{\rightarrow \lambda} \{X_\lambda, p_\lambda^\nu, \Lambda\}$ we can (approximately) solve the n-th root problem for any function from $C(X)$ and for any $n > 1$, we represent the index set $\Lambda$ as the union of disjoint cofinal subsets $\{\Lambda_n\}_{n=2}^\infty$. Then we construct the spectrum by transfinite induction so that the space $X_{\lambda+1}$ and the projection $p_{\lambda+1}^\nu$ resolve projectively (almost) all n-th root problems on $X_\lambda$, where $\lambda \in \Lambda_n$. Since every set $\{\Lambda_n\}$ is cofinal, for any $n$ and any $\alpha$, (almost) every n-th root problem on $X_\alpha$ will be projectively resolved at some level $\lambda > \alpha$ where $\lambda \in \Lambda_n$.

To guarantee that the limit space $X$ has dimension $\dim X = m$, we start the construction with the space $X_1$ homeomorphic to the $m$-dimensional sphere $S^m$. Then we show that the homomorphism $(p_1)^*: H^m(S^m; \mathbb{Q}) \to H^m(X; \mathbb{Q})$ induced
by the limit projection is a monomorphism. Therefore the mapping \( p_1 : X \to S^m \)

is essential and hence \( \dim X \geq m \). To prove that the homomorphism above is a

monomorphism, we use a construction called transfer, that briefly can be described

as follows. Suppose \( G \) is a finite group acting on a compact Hausdorff space \( Y \).

Let \( Y/G \) be the quotient space and let \( \pi : Y \to Y/G \) be the natural projection.

Then there exists a homomorphism \( \mu^* : H^*(Y; \mathbb{Q}) \to H^*(Y/G; \mathbb{Q}) \) such that the

composition \( \mu^* \pi^* \) is the multiplication by the order of \( G \) in the group \( H^*(Y/G; \mathbb{Q}) \).

Therefore \( \pi^* \) is a monomorphism. See Chapter II, §19 of [1] for more information

on transfers.

3. Projective resolutions

In this section we establish some properties of projective resolutions needed for

our constructions in Sections 5 and 6. We begin with a summary of basic properties

of the space \( R_n(X, \mathcal{M}) \).

**Proposition 3.1.** Let \( X \) be a space, let \( \mathcal{M} \) be a subset of \( C(X) \), and let \( n > 1 \) be

an integer.

(a) \( R_n(X, \mathcal{M}) \) is the pull-back in the following diagram:

\[
\begin{array}{ccc}
R_n(X, \mathcal{M}) & \to & \mathbb{C}^\mathcal{M} \\
\downarrow \pi^\mathcal{M} & & \downarrow N \\
X & \to & \mathbb{C}^\mathcal{M}
\end{array}
\]

where \( F : X \to \mathbb{C} \) is defined by \( F(x) = (f(x))_{f \in \mathcal{M}} \) and \( N : \mathbb{C}^\mathcal{M} \to \mathbb{C}^\mathcal{M} \) is defined

by \( N((z_f)_{f \in \mathcal{M}}) = (z_f^n)_{f \in \mathcal{M}} \).

(b) For any \( f \in \mathcal{M} \) there exists \( g \in C(R_n(X, \mathcal{M})) \) such that \( f \circ \pi^\mathcal{M} = g^n \).

(c) If \( X \) is a compact Hausdorff space, then \( R_n(X, \mathcal{M}) \) is also a compact Hausdorff space and \( \dim R_n(X, \mathcal{M}) \leq \dim X \).

**Proof.** The statement (a) is obvious. To prove (b) we just let \( g[(x, (z_h)_{h \in A})] = z_f \).

To verify (c) we note first of all that \( R_n(X, \mathcal{M}) \) is a subset of the product \( X \times \prod_{f \in \mathcal{M}} \{z \mid z^n \in f(X)\} \) of compact Hausdorff spaces. Moreover, \( R_n(X, \mathcal{M}) \) is closed in this product due to (a), and the compactness follows. For the dimension part, observe that \( \pi^\mathcal{M} \) has zero-dimensional fibers and apply [7, Theorem 3.3.10]. \( \square \)

In what follows we shall omit the index \( n \) when this does not cause ambiguities.

**Proposition 3.2.** For any space \( X \) and any two subsets \( A \) and \( B \) of \( C(X) \) there

exists a natural homeomorphism \( h : R(R(X, A), B \circ \pi^A) \to R(X, A \cup B) \), where

\( B \circ \pi^A = \{f \circ \pi^A \mid f \in B\} \). This homeomorphism makes the following diagram

commutative:

\[
\begin{array}{ccc}
R(R(X, A), B \circ \pi^A) & \xrightarrow{h} & R(X, A \cup B) \\
\downarrow \pi_{B \circ \pi^A} & & \downarrow \pi^{A \cup B} \\
R(X, A) & \xrightarrow{\pi^A} & X
\end{array}
\]
Proposition 3.3. Let \( X \) be a compact Hausdorff space and let \( S \) be a subset of \( C(X) \). Let \( A \) be a family of subsets of \( S \), partially ordered by inclusion. Assume that \( A \) is a directed set with respect to this order and that \( \bigcup A = S \). Then \( R(X, S) \) is naturally homeomorphic to the limit of the inverse spectrum \( \{R(X, A), \pi^A, A\} \).

Proof. Put \( \mathcal{R} = \lim_{\longrightarrow} R(X, A), \pi^A, A \). Define \( h_A: R(X, S) \to R(X, A) \) for each \( A \in A \) letting \( h_A = \pi^A \). The family of maps \( \{h_A \mid A \in A\} \) induces the limit map \( h: R(X, S) \to \mathcal{R} \). By Proposition 1.2.13, we claim that \( h \) is a homeomorphism. Since both \( R(X, S) \) and \( \mathcal{R} \) are Hausdorff compacta, it is enough to check that \( h \) is bijective. Since all maps \( \pi^A \) are surjective, \( h \) is surjective by Theorem 3.2.14 in [6]. To verify the injectivity, it is enough, for any two distinct points from \( R(X, S) \), to find \( A \in A \) such that the images of these two points under \( h_A \) are distinct. Let \( y = (x, (z_f)_{f \in S}) \) and \( y' = (x', (z'_f)_{f \in S}) \) be two distinct points from \( R(X, S) \). If \( x \neq x' \), then any \( A \in A \) will do. Otherwise there exists \( f \in S \) such that \( z_f \neq z'_f \). Since \( \bigcup A = S \) there exists \( A \in A \) such that \( f \in A \), and one can easily see that \( h_A(y) \neq h_A(y') \). \( \square \)

Later we use the following special case of Corollary 14.6 from [1].

Proposition 3.4. Let \( S = \{X_\alpha, p^3_\alpha, A\} \) be an inverse spectrum consisting of Hausdorff compact spaces. Then there exists a natural isomorphism

\[
\lim H^*(X_\alpha; \mathbb{Q}) \cong H^*(\lim S; \mathbb{Q}).
\]

Proposition 3.5. Let \( X \) be a compact Hausdorff space and let \( S \) be any subset of \( C(X) \). Then for any integer \( n > 1 \)

\[
(\pi^S)^*: H^*(X; \mathbb{Q}) \to H^*(R_n(X, S); \mathbb{Q})
\]

is a monomorphism.

Proof. (i) First, we prove the proposition for any space \( X \) and a set \( S \) consisting of a single function \( f \). There is an action of \( \mathbb{Z}_n \) on \( R_n(X, f) \) whose orbit space is \( X \), with \( \pi^f \) being the quotient map. Namely, represent \( \mathbb{Z}_n \) as the group of \( n \)-th roots of 1 and put \( g \cdot (x, z_f) = (x, g \cdot z_f) \). The proposition now follows from Theorem 19.1 in [1]. Repeating the argument and applying Proposition 3.22 finitely many times, we see that the proposition holds for every finite set \( S \).

(ii) Finally, let \( S \) be any subset of \( C(X) \). Let \( S_{\text{fin}} \) denote the set of all finite subsets of \( S \), partially ordered by inclusion. Proposition 3.3 implies that \( R_n(X, S) \) is the limit of the inverse spectrum \( \{R_n(X, A), \pi^A_n, S_{\text{fin}}\} \). We apply step (i) of this proof to conclude that \( (\pi^S)^*: H^*(R_n(X, A); \mathbb{Q}) \to H^*(R_n(X, B); \mathbb{Q}) \) is a monomorphism for all \( A \subset B \) in \( S_{\text{fin}} \). An application of Proposition 3.4 completes the proof. \( \square \)
4. Characterizations

Lemma 4.1. If a map \( f : X \to C^* \) has an \( n \)-th root, then any map \( g : X \to C^* \) which is homotopic to \( f \) also has an \( n \)-th root.

Proof. Apply the homotopy lifting property to the \( n \)-th degree covering map \( C^* \to \mathbb{C}^* : z \mapsto z^n \).

Lemma 4.2. Let \( X \) be a normal space. The following conditions are equivalent:

(a) \( C(X) \) is approximately \( n \)-th root closed.

(b) \( C(A, C^*) \) is approximately \( n \)-th root closed for any closed subset \( A \) of \( X \).

(c) \( C(A, C^*) \) is \( n \)-th root closed for any closed subset \( A \) of \( X \).

Proof. For a positive number \( r \), let \( A(0, r) = \{ z \in \mathbb{C} : |z| \geq r \} \) and \( B(0, r) = \{ z \in \mathbb{C} : |z| \leq r \} \). Let \( \rho_z : C^* \to A(0, r) \) be the radial retraction. Note that \( \rho_z \) is homotopic to the identity map of \( C^* \).

(a) \( \Rightarrow \) (b) Take \( \varepsilon > 0 \) and consider a closed subset \( A \) of \( X \). Pick \( f \in C(A, C^*) \) and put \( h = \rho_z \circ f \). Extend \( h \) to a function \( F \) on \( X \), applying the hypothesis (a) to find an \( n \)-th root \( g \), and restricting \( g \) to \( A \), we obtain a function \( g : A \to C^* \) such that \( ||h - g^n|| < \varepsilon/2 \). This condition guarantees that \( g \in C(A, C^*) \). It is easy to see that \( ||f - g^n|| < \varepsilon + \varepsilon/2 < 2\varepsilon \).

(b) \( \Rightarrow \) (c) Again, consider \( f \in C(A, C^*) \), where \( A \) is a closed subset of \( X \), and put \( h = \rho_z \circ f \). Note that \( h \) is homotopic to \( f \). Find \( g : A \to C^* \) such that \( ||h - g^n|| < \varepsilon/2 \). This condition guarantees that \( g^n \) is homotopic to \( h \) and hence to \( f \). An application of Lemma 4.1 completes the proof.

(c) \( \Rightarrow \) (a) Take \( f \in C(X) \) and fix \( \varepsilon > 0 \). Consider \( A = f^{-1}(A(0, \varepsilon)) \) and \( B = f^{-1}(B(0, \varepsilon)) \). Find \( g \in C(A, C^*) \) such that \( f|_A = g^n \). Note that \( g(A \cap B) \subset B(0, \sqrt[n]{\varepsilon}) \), and we can extend \( g \) over \( X \) to \( \tilde{g} \) such that \( \tilde{g}(B) \subset B(0, \sqrt[n]{\varepsilon}) \). It is easy to check that \( ||f - \tilde{g}^n|| < 2\varepsilon \). \( \square \)

We let \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Suppose \( Y \) is a Hausdorff paracompact space. Huber’s Theorem \([9]\) implies the existence of a canonical isomorphism \( H^1(Y; \mathbb{Z}) \cong [Y, S^1] \). Here \([Y, S^1]\) denotes the group of all homotopy classes of maps from \( Y \) to \( S^1 \) with the group operation induced by the multiplication of maps in \( C(Y, S^1) \). We denote the homotopy class of a map \( f \in C(Y, S^1) \) by \([f]\).

Theorem 4.3. Let \( X \) be a Hausdorff paracompact space. Then \( C(X) \) is approximately \( n \)-th root closed iff \( H^1(A; \mathbb{Z}) \) is \( n \)-divisible for every closed subset \( A \) of \( X \).

Proof. Consider a closed subset \( A \) of \( X \). First, suppose that \( C(X) \) is approximately \( n \)-th root closed. Let \( f : A \to S^1 \) be a representative of an arbitrary element of \( H^1(A; \mathbb{Z}) \). By condition (c) of Lemma 4.2 there exist \( g : A \to S^1 \) such that \( g^n = f \) and hence \( n|g| = [f] \) in \( H^1(A; \mathbb{Z}) \).

In order to prove the converse part, we verify condition (c) of Lemma 4.1. Pick \( f \in C(A, C^*) \). Then \( f \) is homotopic to a map \( \tilde{f} : A \to S^1 \). Since \( \tilde{f} \) is divisible by \( n \) there exists \( h : A \to S^1 \) such that \( h^n \) is homotopic to \( f \) and hence to \( \tilde{f} \). Lemma 4.1 implies that \( f \) has an \( n \)-th root. \( \square \)

5. Compacta with Approximately Root Closed \( C(X) \)

Lemma 5.1. Let \( S = \{ X_i, p_i^{n+1} \} \) be an inverse sequence of compact metrizable spaces and let \( X = \lim S \). Consider the following two conditions.

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(a) $C(X)$ is approximately $n$-th root closed.

(b) For any $i$, any closed subset $A_i$ of $X_i$ and any map $h: A_i \to \mathbb{C}^*$, there exists $j > i$ such that the map $h \circ p_i^j: A_j \to \mathbb{C}^*$ has an $n$-th root, where $A_j = (p_i^j)^{-1}(A_i)$.

Condition (b) implies condition (a). Moreover if all projections of $S$ are surjective, then the converse implication (a)$\to$(b) also holds.

**Proof.** Put $X = \lim S$. First, we show that $C(X)$ is approximately $n$-th root closed by checking condition (b) of Lemma 1.2. Let $A$ be a closed subset of $X$ and let $f \in C(A, \mathbb{C}^*)$ be a function. Take any $\varepsilon > 0$. There exist $i$ and a mapping $f_i: p_i(A) \to \mathbb{C}^*$ such that $f_i \circ p_i|_A$ is $\varepsilon$-close to $f$. Let $A_i = p_i(A)$ and find $j > i$ such that the map $f_i \circ p_i^j: A_j \to \mathbb{C}^*$ has an $n$-th root. Then $g = h \circ f_j$ is an $n$-th root of $f_i \circ p_i|_A$. Obviously, $\|f - g^n\| < \varepsilon$.

Conversely, suppose $C(X)$ is approximately $n$-th root closed and all projections of $S$ are surjective. Pick $i$ and consider a closed subset $A_i$ of $X_i$ and a map $h: A_i \to \mathbb{C}^*$. Let $\varepsilon = \min\{|h(x)|: x \in A_i\}$. Put $A = (p_i)^{-1}(A_i)$. There exists $g: A \to \mathbb{C}^*$ such that $g^n$ is $\varepsilon/4$-close to $h \circ p_i|_A$. We can find $j > i$ and a map $g_j: p_j(A) \to \mathbb{C}^*$ such that $(g_j \circ p_j)^n$ is $\varepsilon/4$-close to $g^n$. Let $A_j = (p_i^j)^{-1}(A_i)$. Since all projections of $S$ are surjective, $p_j(A) = A_j$. Using this, it is not hard to verify that $(g_j)^n$ is $\varepsilon/2$-close, and hence homotopic, to $h \circ p_i^j$. Lemma 1.1 implies that $h \circ p_i^j$ has an $n$-th root. \hfill $\square$

**Theorem 5.2.** For every positive integer $m$ there exists an $m$-dimensional compact metrizable space $X$ such that $C(X)$ is approximately $n$-th root closed for all positive integers $n$.

**Proof.** We obtain $X$ as the inverse limit of a sequence $S = \{X_i, p_i^{i+1}\}$, consisting of $m$-dimensional metrizable compacta. The sequence is constructed by induction as follows. Represent the set of all positive integers as a union of disjoint infinite subsets $\{\Lambda_n\}_{n=2}^\infty$. Put $X_1 = S_{\infty}$, the $m$-dimensional sphere. Suppose the space $X_k$ has already been constructed. Fix a countable collection $B_k$ of closed subsets of $X_k$ such that for each closed subset $A$ of $X_k$ and for any open neighborhood $U$ of $A$ there exists $B \in B_k$ such that $A \subset B \subset U$. For each $B \in B_k$ we fix a family $F_B^i$ of maps from $B$ to $\mathbb{C}^*$ which is dense in the space $C(B, \mathbb{C}^*)$. For every map from the family $F^i_B$, we fix its extension to a map from $X_k$ to $\mathbb{C}$ and denote the family of these extensions by $F_B$. Let $\Phi_k = \bigsqcup_{B \in B_k} F_B$. Define $X_{k+1} = R_m(X_k, \Phi_k)$ where $n$ is such that $k \in \Lambda_n$, and let $p_{k+1}^j = \pi_{\Phi_k}$.

Put $X = \lim S$. To verify that $C(X)$ is approximately $n$-th root closed for each $n > 1$, it is enough to show that condition (b) of Lemma 5.1 is satisfied for the inverse sequence $S$. Fix $n > 1$. Pick $i$ and consider a closed subset $A_i$ of $X_i$ and a function $h: A_i \to \mathbb{C}^*$. Take a number $j > i$ such that $j - 1 \in \Lambda_n$. Let $A_j = (p_i^j)^{-1}(A_i)$. We show that the map $h \circ p_i^j: A_j \to \mathbb{C}^*$ has an $n$-th root. Put $A_{j-1} = (p_i^{j-1})^{-1}(A_i)$. Let $g$ be an extension of the map $h \circ p_i^{j-1}: A_{j-1} \to \mathbb{C}^*$ to some neighborhood $U$ of $A_{j-1}$. There exists $B \in B_k$ and a function $f: B \to \mathbb{C}^*$ such that $A_{j-1} \subset B \subset U$ and the restriction $g|_B$ is homotopic to $f$. Let $\tilde{f}: B \to \mathbb{C}$ be the extension of $f$ that belongs to the family $\Phi_k$. Since the map $p_{k-1}^j$ resolves the projective $n$-th root problem for $\tilde{f}$, the map $f \circ p_{k-1}^j|_{A_j}$ has an $n$-th root.
By Lemma \[\text{Lemma }1\text{] the map } g \circ p_{j-1}^j|_{A_j} \text{ has an } n\text{-th root. It remains to note that }
\[h \circ p_{j-1}^j|_{A_j} = g \circ p_{j-1}^j|_{A_j}.
\]

Note that dim\(X \leq m\) since all \(X_k\) are at most \(m\)-dimensional. Proposition 3.5 implies that \((p_k^{k+1})*: H^m(X_k; \mathbb{Q}) \to H^m(X_{k+1}; \mathbb{Q})\) is a monomorphism. Applying Proposition 3.3 we conclude that \((p_1)^*: H^m(S^m; \mathbb{Q}) \to H^m(X; \mathbb{Q})\) is a monomorphism. Thus the limit projection \(p_1: X \to S^m\) is essential and therefore \(\text{dim}\(X \geq m\).

By the condition of the lemma we can show that \(\mathcal{A}_r(n)\) contains a universal space for any \(\tau \geq \omega\). Also we may consider the subclass \(\mathcal{A}_{m,\tau}(n)\) consisting of all spaces in \(\mathcal{A}_r(n)\) of dimension at most \(m\). Theorem 1.2 of \[3\] also proves that the class \(\mathcal{A}_{m,\tau}(n)\) contains a universal space. A similar proof, based on the Mardešić factorization theorem \[\text{[11]}\], works to prove that the class \(\mathcal{A}_{m,\tau}(n)\) contains a universal space.

Corollary 5.3. Let \(Y\) be a universal space with respect to the class \(\mathcal{A}_\omega\) or \(\mathcal{A}_\omega(n)\). Then \(Y\) is infinite dimensional.

Hence, any universal space for the class \(\mathcal{A}_\omega(n)\) must be infinite dimensional for any \(\tau \geq \omega\).

Also we may consider the subclass \(\mathcal{A}_{m,\omega}(n)\) consisting of all spaces in \(\mathcal{A}_{m,\omega}(n)\) of dimension at most \(m\). Theorem 1.2 of \[3\] also proves that the class \(\mathcal{A}_{m,\omega}(n)\) contains a universal space. A similar proof, based on the Mardešić factorization theorem \[\text{[11]}\], works to prove that the class \(\mathcal{A}_{m,\omega}(n)\) contains a universal space.

6. Compacta with root closed \(C(X)\)

In this section, for any positive integer \(m\) we construct a compact Hausdorff space \(X\) with \(\text{dim}\(X = m\)\) such that \(C(X)\) is \(n\)-th root closed for all \(n\). Note that for a metrizable continuum \(Y\) the algebra \(C(Y)\) is square root closed if and only if \(Y\) is a dendrite, and therefore \(\text{dim}\(Y \leq 1\)\) \[10\], \[13\]. This forces the space \(X\) above to be non-metrizable.

Lemma 6.1. Let \(S = \{X_\alpha, p^\beta_\alpha, A\}\) be a factorizing spectrum. In order for \(C(\lim S)\) to be \(n\)-th root closed it is sufficient that for any \(\alpha \in A\) and any function \(h \in C(X_\alpha)\) there exists \(\beta > \alpha\) such that \(h \circ p^\beta_\alpha\) has an \(n\)-th root. If all limit projections of \(S\) are surjective, the above condition is also necessary.

Proof. Put \(X = \lim S\). Consider \(f \in C(X)\). Since \(S\) is factorizing there exists \(\alpha\) and \(f_\alpha \in C(X_\alpha)\) such that \(f = f_\alpha \circ p_\alpha\). By the condition of the lemma we can find \(\beta > \alpha\) and \(g_\beta: X_\beta \to \mathbb{C}\) such that \((g_\beta)^n = f_\alpha \circ p^\beta_\alpha\). It is easy to verify that \(g = g_\beta \circ p_\beta\) is an \(n\)-th root of \(f\).

Now suppose that all limit projections of \(S\) are surjective and \(C(X)\) is \(n\)-th root closed. Consider \(\alpha \in A\) and \(h \in C(X_\alpha)\). There exists \(g \in C(X)\) such that \(g^n = h \circ p_\alpha\). Since \(S\) is factorizing, there exists \(\beta > \alpha\) and \(g_\beta: X_\beta \to \mathbb{C}\) such that \(g = g_\beta \circ p_\beta\). Since the projection \(p_\beta\) is surjective, \((g_\beta)^n = h \circ p^\beta_\alpha\).

Theorem 6.2. For each positive integer \(m\), there exists a compact Hausdorff space \(X\) with \(\text{dim}\(X = m\)\) and such that \(C(X)\) is \(n\)-th root closed for any \(n\).
Proof. Represent the ordinal \( \omega_1 \) as the union of countably many disjoint uncountable subsets \( \{ \Lambda_n \}_{n=2}^{\infty} \). Starting with \( X_0 = S^m \), where \( S^m \) denotes an \( m \)-dimensional sphere, by transfinite induction we define an inverse spectrum \( \mathcal{S} = \{ X_\alpha, p_\alpha, \omega_1 \} \) as follows. If \( \beta = \alpha + 1 \), then define \( X_\beta = R_n(X_\alpha, C(X_\alpha)) \), where \( n \) is such that \( \alpha \in \Lambda_n \), and let \( p_\alpha^\beta = \pi C(X_\alpha) \). If \( \beta \) is a limit ordinal, then define \( X_\beta = \lim\{ X_\alpha, p_\alpha^\beta, \alpha < \beta \} \) and, for \( \alpha < \beta \), let \( p_\alpha^\beta \) be the limit projection.

Put \( X = \operatorname{lim}\mathcal{S} \). To verify that \( C(X) \) is \( n \)-th root closed for each \( n > 1 \), it is enough to check the condition of Lemma 6.1 for the spectrum \( \mathcal{S} \). Consider \( n > 1 \). Since the spectrum \( \mathcal{S} \) has length \( \omega_1 \), it is factorizing \[ \text{[2 Corollary 1.3.2].} \] Consider a function \( h: X_\alpha \rightarrow \mathbb{C} \) and take an ordinal \( \gamma > \alpha \) such that \( \gamma \in \Lambda_n \). Since the map \( p_\alpha^{\gamma+1} \) resolves the projective \( n \)-th root problem for \( h \circ p_\alpha^\gamma \), the map \( h \circ p_\alpha^{\gamma+1} \) has an \( n \)-th root.

Note that \( \dim X_\alpha \leq m \) for each \( \alpha \) and hence \( \dim X \leq m \). We claim that \( (p_\alpha^\beta)^*: H^*(X_\alpha; \mathbb{Q}) \rightarrow H^*(X_\alpha; \mathbb{Q}) \) is a monomorphism for all \( \alpha < \beta < \omega_1 \). Indeed, in the case \( \beta = \alpha + 1 \) it follows from Proposition 3.1 and then in a general case it is due to Proposition 3.4. Finally, again with the help of Proposition 3.4 we conclude that \( p_\alpha^*: H^m(S^m; \mathbb{Q}) \rightarrow H^m(X; \mathbb{Q}) \) is a monomorphism and hence the map \( p_\alpha : X \rightarrow S^m \) is essential. This implies \( \dim X \geq m \).

It is not hard to verify that if \( C(Y) \) is \( n \)-th root closed for some (completely regular) space \( Y \), then \( C(\beta Y) \) is also \( n \)-th root closed. Here by \( \beta Y \) we denote the Stone-C\'ech compactification of \( Y \).

**Corollary 6.3.** There exists an infinite-dimensional compact Hausdorff space \( X \) such that \( C(X) \) is \( n \)-th root closed for all \( n \).

**Proof.** For each \( m \), let \( X_m \) denote compactum provided by Theorem 6.2. We put \( X = \beta(\bigoplus\{ X_m \mid m \in \omega \}) \). \[ \square \]

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