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ROOT CLOSED FUNCTION ALGEBRAS ON COMPACTA OF LARGE DIMENSION

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ABSTRACT. Let X be a Hausdorff compact space and let $C(X)$ be the algebra of all continuous complex-valued functions on X, endowed with the supremum norm. We say that $C(X)$ is (approximately) n-th root closed if any function from $C(X)$ is (approximately) equal to the n-th power of another function. We characterize the approximate n-th root closedness of $C(X)$ in terms of n-divisibility of the first Čech cohomology groups of closed subsets of X. Next, for each positive integer m we construct an m-dimensional metrizable compactum X such that $C(X)$ is approximately n-th root closed for any n. Also, for each positive integer m we construct an m-dimensional compact Hausdorff space X such that $C(X)$ is n-th root closed for any n.

1. INTRODUCTION

Relations between algebraic closedness of the algebra of continuous bounded complex-valued functions $C(X)$ on a space X and topological properties of X have been studied since the 1960s [5]. Recall that the algebra $C(X)$ is called algebraically closed if each monic polynomial with coefficients in $C(X)$ has a root in $C(X)$. For a locally connected compact Hausdorff space, the algebra $C(X)$ is algebraically closed if and only if $\dim X \leq 1$ and $H^1(X;\mathbb{Z}) = 0$ [8], [13], where $H^1(X;\mathbb{Z})$ denotes the first Čech cohomology group of X with the integer coefficient (see section 2). It is proved in [13] that for a first-countable compact Hausdorff space X, algebraic closedness of $C(X)$ is equivalent to a weaker property of square root closedness. The latter means that every function from $C(X)$ is a square of another function. It should be noted that this property appears in the study of subalgebras of $C(X)$ [4].

An even weaker property of approximate square root closedness was introduced by Miura [12] and was proved to be equivalent to the square root closedness when the underlying compact Hausdorff space X is locally connected.

There is a nice characterization of algebraic closedness of $C(X)$ when X is a metrizable continuum. Namely, in this case $C(X)$ is algebraically closed if and only if X is a dendrite (i.e. a Peano continuum containing no simple closed curves) [10], [13].

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The approximate n-th root closedness of $C(X)$ was studied by Kawamura and Miura and was proved to be equivalent to n-divisibility of $H^1(X;\mathbb{Z})$ under the additional assumption $\dim X \leq 1$. The universal space for metrizable compacta with the approximately n-th root closed $C(X)$ is constructed in [3].

In this paper we characterize the approximate n-th root closedness of $C(X)$ for any Hausdorff paracompact space X. Namely, $C(X)$ is approximately n-th root closed if and only if the group $H^1(A;\mathbb{Z})$ is n-divisible for every closed subset A of X. If $\dim X \leq 1$, then the n-divisibility of $H^1(X;\mathbb{Z})$ implies the n-divisibility of $H^1(A;\mathbb{Z})$, so this generalizes Theorem 1.3 of [10]. Further, for each positive integer m we construct an m-dimensional metrizable compactum X such that $C(X)$ is approximately n-th root closed for any n. Note that such examples were known in dimension 1 only. Also, for each positive integer m we construct an m-dimensional compact Hausdorff space X such that $C(X)$ is n-th root closed for any n. This example solves the problem posed in [10]: for a compact Hausdorff space X, does square root closedness of $C(X)$ imply $\dim X \leq 1$?

2. Notations, definitions, and ideas of constructions

All maps considered in this paper are continuous. For spaces X and Y, we denote the set of all maps from X to Y by $C(X,Y)$. As usual, by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ we denote the integers, the rational numbers, and the complex numbers, respectively. We let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the multiplicative subgroup of $\mathbb{C}$. An inverse spectrum over a directed partially ordered set $(\mathcal{A}, <)$ consisting of spaces X_α, $\alpha \in \mathcal{A}$, and projections $p^\beta_\alpha \colon X_\beta \to X_\alpha$, $\alpha, \beta \in \mathcal{A}$, $\beta > \alpha$, is denoted by $\{X_\alpha, p^\beta_\alpha, \mathcal{A}\}$. Throughout this section $n > 1$ denotes an integer.

By $H^k(X;G)$ we denote the k-th Čech cohomology group of the space X with an abelian coefficient group G. Note that in the case when X is a Hausdorff paracompact space the Čech cohomologies are naturally isomorphic to the Alexander-Spanier cohomologies [14, p. 334]. Note also that due to the Huber's theorem [9] for a Hausdorff paracompact space X there exists a natural isomorphism between the group of all homotopy classes of maps from X to $K(G,k)$ and the group $H^k(X;G)$ if G is countable. Here $K(G,k)$ denotes the Eilenberg-Mac Lane complex.

For a space X, by $C(X)$ we denote the algebra of all bounded complex-valued functions on X, endowed with the supremum norm. We say that $C(X)$ is *approximately n-th root closed* if for every $f \in C(X)$ and every $\varepsilon > 0$ there exists $g \in C(X)$ such that $||f - g^n|| < \varepsilon$. The algebra $C(X)$ is said to be *n-th root closed* if any $f \in C(X)$ has an n-th root, which means that there exists $g \in C(X)$ such that $f = g^n$. Note that if $C(X)$ is (approximately) n-th root closed, then $C(A)$ is also (approximately) n-th root closed for any closed subset A of X.

We consider $C(X,\mathbb{C}^*)$ as a multiplicative subgroup of $C(X)$ with a metric inherited from $C(X)$. We say that $C(X,\mathbb{C}^*)$ is (approximately) n-th root closed if any $f \in C(X,\mathbb{C}^*)$ has an (approximate) n-th root in $C(X,\mathbb{C}^*)$.

The basic idea explored in this paper — the construction of a projective n-th root resolution — is outlined as follows. The simplest case has been known in the theory of uniform algebra, and it is called the Cole construction (cf. [15, Chapter 3, §19, pp. 194-197]).

Given a space X and a function $f \colon X \to \mathbb{C}$ it is not always possible to solve, even approximately, the problem of finding an n-th root of f (consider for instance any homotopically non-trivial map from a circle S^1 to $\mathbb{C}^*$). Nevertheless, it is always possible to solve the n-th root problem *projectively* in the following sense. There exists a space denoted $R_n(X, f)$ and a map $\pi^f\colon R_n(X, f) \to X$ such that the composition $f \circ \pi^f$ has an n-th root. The space $R_n(X, f)$ is simply the graph of the (multivalued) n-th root of f,

$$R_n(X, f) = \{(x, z) \mid f(x) = z^n\} \subset X \times \mathbb{C},$$

and the map π^f is the natural projection on X. Obviously, the projection of $R_n(X, f)$ to $\mathbb{C}$ is an n-th root of the composition $f \circ \pi^f$. We say that the space $R_n(X, f)$ together with the map π^f resolve the n-th root problem for f projectively.

Given any family of maps $\mathcal{M} \subset C(X)$ we can projectively resolve the n-th root problem for all maps from $\mathcal{M}$ simultaneously by using the space

$$R_n(X, \mathcal{M}) = \{(x, (z_f)_{f\in\mathcal{M}}) \mid f(x) = z_f^n \quad \forall f \in \mathcal{M}\} \subset X \times \mathbb{C}^{\mathcal{M}}$$

and defining $\pi^{\mathcal{M}}\colon R_n(X, \mathcal{M}) \to X$ to be the natural projection. Let A and B be two subsets of $C(X)$ such that $A \subset B$. There is a natural projection $\pi_A^B\colon R_n(X, B) \to R_n(X, A)$ defined by $\pi_A^B[(x, (z_f)_{f\in B})] = (x, (z_f)_{f\in A})$. We let $R_n(X, \emptyset) = X$ and $\pi_\emptyset^B = \pi^B$.

We outline the ideas of our constructions in Sections 5 and 6. Suppose that we want to construct a space X with n-th root closed $C(X)$. Take any space X_1 and resolve the n-th root problems for X_1 projectively using the space $X_2 = R_n(X_1, C(X_1))$. Then resolve all n-th root problems for X_2 projectively using X_3, and so on. This way we obtain an inverse spectrum of spaces X_λ and define X to be the inverse limit of this spectrum. To guarantee that the n-th root problems for X can be solved, we need this spectrum to be *factorizing* in the following sense: for any map $f\colon X \to \mathbb{C}$ there exist a space X_λ in the spectrum and a map $f_\lambda\colon X_\lambda \to \mathbb{C}$ such that $f = f_\lambda \circ p_\lambda$, where $p_\lambda\colon X \to X_\lambda$ is the limit projection. Then the projective resolution of the n-th root problem for f_λ gives us a solution of the n-th root problem for f. In order to obtain a factorizing spectrum we make its length uncountable. Namely, we construct the spectrum over ω_1, the first uncountable ordinal.

The space described above is not metrizable for two reasons. First, the length of the spectrum used is not countable. Second, for a metric compactum X_λ and a subset $\mathcal{M} \subset C(X_\lambda)$, the space $R_n(X_\lambda, \mathcal{M})$ is metrizable if and only if the set $\mathcal{M}$ is countable. If we want $C(X)$ to have just the *approximate* n-th root property, it is enough to construct a countable spectrum and for each space X_λ to resolve the projective n-th root problem for a countable *dense* set of maps from $C(X_\lambda)$. Then the limit space X is a metrizable compactum, if we start with a metrizable compactum X_1.

To guarantee that for the limit space $X = \varprojlim\{X_\lambda, p_\lambda^\mu, \Lambda\}$ we can (approximately) solve the n-th root problem for any function from $C(X)$ and for any $n > 1$, we represent the index set Λ as the union of disjoint cofinal subsets $\{\Lambda_n\}_{n=2}^\infty$. Then we construct the spectrum by transfinite induction so that the space $X_{\lambda+1}$ and the projection $p_\lambda^{\lambda+1}$ resolve projectively (almost) all n-th root problems on X_λ, where $\lambda \in \Lambda_n$. Since every set $\{\Lambda_n\}$ is cofinal, for any n and any α, (almost) every n-th root problem on X_α will be projectively resolved at some level $\lambda > \alpha$ where $\lambda \in \Lambda_n$.

To guarantee that the limit space X has dimension $\dim X \geq m$, we start the construction with the space X_1 homeomorphic to the m-dimensional sphere S^m. Then we show that the homomorphism $(p_1)^*\colon H^m(S^m; \mathbb{Q}) \to H^m(X; \mathbb{Q})$ induced by the limit projection is a monomorphism. Therefore the mapping $p_1\colon X \to S^m$ is essential and hence $\dim X \geq m$. To prove that the homomorphism above is a monomorphism, we use a construction called *transfer*, that briefly can be described as follows. Suppose G is a finite group acting on a compact Hausdorff space Y. Let Y/G be the quotient space and let $\pi\colon Y \to Y/G$ be the natural projection. Then there exists a homomorphism $\mu^*\colon H^*(Y;\mathbb{Q}) \to H^*(Y/G;\mathbb{Q})$ such that the composition $\mu^*\pi^*$ is the multiplication by the order of G in the group $H^*(Y/G;\mathbb{Q})$. Therefore π^* is a monomorphism. See Chapter II, §19 of [1] for more information on transfers.

3. Projective resolutions

In this section we establish some properties of projective resolutions needed for our constructions in Sections 5 and 6. We begin with a summary of basic properties of the space $R_n(X,\mathcal{M})$.

Proposition 3.1. *Let X be a space, let $\mathcal{M}$ be a subset of $C(X)$, and let $n > 1$ be an integer.*

(a) *$R_n(X,\mathcal{M})$ is the pull-back in the following diagram:*

$$\begin{tikzcd} R_n(X,\mathcal{M}) \arrow[r] \arrow[d, "\pi^{\mathcal{M}}"] & \mathbb{C}^{\mathcal{M}} \arrow[d, "N"] \\ X \arrow[r, "F"] & \mathbb{C}^{\mathcal{M}} \end{tikzcd}$$

where $F\colon X \to \mathbb{C}$ is defined by $F(x) = (f(x))_{f\in\mathcal{M}}$ and $N\colon \mathbb{C}^{\mathcal{M}} \to \mathbb{C}^{\mathcal{M}}$ is defined by $N((z_f)_{f\in\mathcal{M}}) = (z_f^n)_{f\in\mathcal{M}}$.

(b) *For any $f \in \mathcal{M}$ there exists $g \in C(R_n(X,\mathcal{M}))$ such that $f\circ\pi^{\mathcal{M}} = g^n$.*

(c) *If X is a compact Hausdorff space, then $R_n(X,\mathcal{M})$ is also a compact Hausdorff space and $\dim R_n(X,\mathcal{M}) \leq \dim X$.*

Proof. The statement (a) is obvious. To prove (b) we just let $g[(x,(z_h)_{h\in A})] = z_f$. To verify (c) we note first of all that $R_n(X,\mathcal{M})$ is a subset of the product $X \times \prod_{f\in\mathcal{M}}\{z \mid z^n \in f(X)\}$ of compact Hausdorff spaces. Moreover, $R_n(X,\mathcal{M})$ is closed in this product due to (a), and the compactness follows. For the dimension part, observe that $\pi^{\mathcal{M}}$ has zero-dimensional fibers and apply [7, Theorem 3.3.10]. □

In what follows we shall omit the index n when this does not cause ambiguities.

Proposition 3.2. *For any space X and any two subsets A and B of $C(X)$ there exists a natural homeomorphism $h\colon R(R(X,A), B\circ\pi^A) \to R(X, A\cup B)$, where $B\circ\pi^A = \{f\circ\pi^A \mid f\in B\}$. This homeomorphism makes the following diagram commutative:*

$$\begin{tikzcd} R(R(X,A),B\circ\pi^A) \arrow[r, "h"] \arrow[d, "\pi^{B\circ\pi^A}"] & R(X,A\cup B) \arrow[d, "\pi^{A\cup B}"] \\ R(X,A) \arrow[r, "\pi^A"] & X \end{tikzcd}$$

Proof. Note that both $R(X, A\cup B)$ and $R(R(X,A), B\circ\pi^A)$ can be viewed as subsets of $X\times\mathbb{C}^A\times\mathbb{C}^B$. Namely,

$$R(X, A\cup B) = \{(x, (z_f)_{f\in A}, (z_g)_{g\in B}) \mid z_f^n = f(x), z_g^n = g(x)\},$$

$$R(R(X,A), B\circ\pi^A) = \{(x, (z_f)_{f\in A}, (z_{g\circ\pi^A})_{g\in B}) \mid z_f^n = f(x), z_{g\circ\pi^A}^n = (g\circ\pi^A)[(x,(z_f)_{f\in A})]\}.$$

It remains to note that these subsets coincide since

$$(g\circ\pi^A)[(x,(z_f)_{f\in A})] = g(\pi^A[(x,(z_f)_{f\in A})]) = g(x)$$

by the definition of π^A. □

Proposition 3.3. *Let X be a compact Hausdorff space and let S be a subset of $C(X)$. Let $\mathcal{A}$ be a family of subsets of S, partially ordered by inclusion. Assume that $\mathcal{A}$ is a directed set with respect to this order and that $\bigcup\mathcal{A} = S$. Then $R(X,S)$ is naturally homeomorphic to the limit of the inverse spectrum $\{R(X,A), \pi_A^B, \mathcal{A}\}$.*

Proof. Put $\Re = \varprojlim\{R(X,A), \pi_A^B, \mathcal{A}\}$. Define $h_A\colon R(X,S)\to R(X,A)$ for each $A\in\mathcal{A}$ letting $h_A = \pi_A^S$. The family of maps $\{h_A \mid A\in\mathcal{A}\}$ induces the limit map $h\colon R(X,S)\to\Re$ [2, Proposition 1.2.13]. We claim that h is a homeomorphism. Since both $R(X,S)$ and $\Re$ are Hausdorff compacta, it is enough to check that h is bijective. Since all maps π_A^S are surjective, h is surjective by Theorem 3.2.14 in [6]. To verify the injectivity, it is enough, for any two distinct points from $R(X,S)$, to find $A\in\mathcal{A}$ such that the images of these two points under h_A are distinct. Let $y = (x, (z_f)_{f\in S})$ and $y' = (x', (z'_f)_{f\in S})$ be two distinct points from $R(X,S)$. If $x\neq x'$, then any $A\in\mathcal{A}$ will do. Otherwise there exists $f\in S$ such that $z_f\neq z'_f$. Since $\bigcup\mathcal{A} = S$ there exists $A\in\mathcal{A}$ such that $f\in A$, and one can easily see that $h_A(y)\neq h_A(y')$. □

Later we use the following special case of Corollary 14.6 from [1].

Proposition 3.4. *Let $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \mathcal{A}\}$ be an inverse spectrum consisting of Hausdorff compact spaces. Then there exists a natural isomorphism*

$$\varinjlim H^*(X_\alpha;\mathbb{Q}) \cong H^*(\varprojlim\mathcal{S};\mathbb{Q}).$$

Proposition 3.5. *Let X be a compact Hausdorff space and let S be any subset of $C(X)$. Then for any integer $n>1$*

$$(\pi^S)^*\colon H^*(X;\mathbb{Q})\to H^*(R_n(X,S);\mathbb{Q})$$

is a monomorphism.

Proof. (i) First, we prove the proposition for any space X and a set S consisting of a single function f. There is an action of $\mathbb{Z}_n$ on $R_n(X,f)$ whose orbit space is X, with π^f being the quotient map. Namely, represent $\mathbb{Z}_n$ as the group of n-th roots of 1 and put $g\cdot(x,z_f) = (x, g\cdot z_f)$. The proposition now follows from Theorem 19.1 in [1]. Repeating the argument and applying Proposition 3.2 finitely many times, we see that the proposition holds for every finite set S.

(ii) Finally, let S be any subset of $C(X)$. Let S_{fin} denote the set of all finite subsets of S, partially ordered by inclusion. Proposition 3.3 implies that $R_n(X,S)$ is the limit of the inverse spectrum $\{R_n(X,A), \pi_A^B, S_{\mathrm{fin}}\}$. We apply step (i) of this proof to conclude that $\left(\pi_A^B\right)^*: H^*(R_n(X,A);\mathbb{Q})\to H^*(R_n(X,B);\mathbb{Q})$ is a monomorphism for all $A\subset B$ in S_{fin}. An application of Proposition 3.4 completes the proof. □

4. Characterizations

Lemma 4.1. *If a map $f\colon X \to \mathbb{C}^*$ has an n-th root, then any map $g\colon X \to \mathbb{C}^*$ which is homotopic to f also has an n-th root.*

Proof. Apply the homotopy lifting property to the n-th degree covering map $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$. □

Lemma 4.2. *Let X be a normal space. The following conditions are equivalent:*

(a) *$C(X)$ is approximately n-th root closed.*
(b) *$C(A, \mathbb{C}^*)$ is approximately n-th root closed for any closed subset A of X.*
(c) *$C(A, \mathbb{C}^*)$ is n-th root closed for any closed subset A of X.*

Proof. For a positive number r, let $A(0, r) = \{z \in \mathbb{C}\colon |z| \geq r\}$ and $B(0, r) = \{z \in \mathbb{C}\colon |z| \leq r\}$. Let $\rho_\varepsilon\colon \mathbb{C}^* \to A(0, \epsilon)$ be the radial retraction. Note that ρ_ε is homotopic to the identity map of $\mathbb{C}^*$.

(a) ⇒ (b) Take $\varepsilon > 0$ and consider a closed subset A of X. Pick $f \in C(A, \mathbb{C}^*)$ and put $h = \rho_\varepsilon \circ f$. Extend h to a function F on X, applying the hypothesis (a) to find an n-th root g, and restricting g to A, we obtain a function $g\colon A \to \mathbb{C}$ such that $||h - g^n|| < \varepsilon/2$. This condition guarantees that $g \in C(A, \mathbb{C}^*)$. It is easy to see that $||f - g^n|| < \varepsilon + \varepsilon/2 < 2\varepsilon$.

(b) ⇒ (c) Again, consider $f \in C(A, \mathbb{C}^*)$, where A is a closed subset of X, and put $h = \rho_\varepsilon \circ f$. Note that h is homotopic to f. Find $g\colon A \to \mathbb{C}^*$ such that $||h - g^n|| < \varepsilon/2$. This condition guarantees that g^n is homotopic to h and hence to f. An application of Lemma 4.1 completes the proof.

(c) ⇒ (a) Take $f \in C(X)$ and fix $\varepsilon > 0$. Consider $A = f^{-1}(A(0, \varepsilon))$ and $B = f^{-1}(B(0, \varepsilon))$. Find $g \in C(A, \mathbb{C}^*)$ such that $f|_A = g^n$. Note that $g(A \cap B) \subset B(0, \sqrt[n]{\varepsilon})$, and we can extend g over X to $\overline{g}$ such that $\overline{g}(B) \subset B(0, \sqrt[n]{\varepsilon})$. It is easy to check that $||f - \overline{g}^n|| < 2\varepsilon$. □

We let $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. Suppose Y is a Hausdorff paracompact space. Huber's Theorem [9] implies the existence of a canonical isomorphism $H^1(Y; \mathbb{Z}) \cong [Y, S^1]$. Here $[Y, S^1]$ denotes the group of all homotopy classes of maps from Y to S^1 with the group operation induced by the multiplication of maps in $C(Y, S^1)$. We denote the homotopy class of a map $f \in C(Y, S^1)$ by $[f]$.

Theorem 4.3. *Let X be a Hausdorff paracompact space. Then $C(X)$ is approximately n-th root closed iff $H^1(A; \mathbb{Z})$ is n-divisible for every closed subset A of X.*

Proof. Consider a closed subset A of X. First, suppose that $C(X)$ is approximately n-th root closed. Let $f\colon A \to S^1$ be a representative of an arbitrary element of $H^1(A; \mathbb{Z})$. By condition (c) of Lemma 4.2 there exist $g\colon A \to S^1$ such that $g^n = f$ and hence $n[g] = [f]$ in $H^1(A; \mathbb{Z})$.

In order to prove the converse part, we verify condition (c) of Lemma 4.2. Pick $f \in C(A, \mathbb{C}^*)$. Then f is homotopic to a map $\widetilde{f}\colon A \to S^1$. Since $[\widetilde{f}\,] \in H^1(A; \mathbb{Z})$ is divisible by n there exists $h\colon A \to S^1$ such that h^n is homotopic to $\widetilde{f}$ and hence to f. Lemma 4.1 implies that f has an n-th root. □

5. Compacta with approximately root closed $C(X)$

Lemma 5.1. *Let $\mathcal{S} = \{X_i, p_i^{i+1}\}$ be an inverse sequence of compact metrizable spaces and let $X = \varprojlim \mathcal{S}$. Consider the following two conditions.*

(a) $C(X)$ *is approximately* n*-th root closed.*
(b) *For any* i*, any closed subset* A_i *of* X_i *and any map* $h\colon A_i \to \mathbb{C}^*$*, there exists* $j > i$ *such that the map* $h \circ p_i^j\colon A_j \to \mathbb{C}^*$ *has an* n*-th root, where* $A_j = (p_i^j)^{-1}(A_i)$*.*

Condition (b) *implies condition* (a)*. Moreover if all projections of* $\mathcal{S}$ *are surjective, then the converse implication* (a)$\to$(b) *also holds.*

Proof. Put $X = \varprojlim \mathcal{S}$. First, we show that $C(X)$ is approximately n-th root closed by checking condition (b) of Lemma 4.2. Let A be a closed subset of X and let $f \in C(A, \mathbb{C}^*)$ be a function. Take any $\varepsilon > 0$. There exist i and a mapping $f_i\colon p_i(A) \to \mathbb{C}^*$ such that $f_i \circ p_i|_A$ is ε-close to f. Let $A_i = p_i(A)$ and find $j > i$ such that the map $f_i \circ p_i^j\colon A_j \to \mathbb{C}^*$ has an n-th root. Then $g = h \circ f_j$ is an n-th root of $f_i \circ p_i|_A$. Obviously, $||f - g^n|| < \varepsilon$.

Conversely, suppose $C(X)$ is approximately n-th root closed and all projections of $\mathcal{S}$ are surjective. Pick i and consider a closed subset A_i of X_i and a map $h\colon A_i \to \mathbb{C}^*$. Let $\varepsilon = \min\{|h(x)|\colon x \in A_i\}$. Put $A = (p_i)^{-1}(A_i)$. There exists $g\colon A \to \mathbb{C}^*$ such that g^n is $\varepsilon/4$-close to $h \circ p_i|_A$. We can find $j > i$ and a map $g_j\colon p_j(A) \to \mathbb{C}^*$ such that $(g_j \circ p_j)^n$ is $\varepsilon/4$-close to g^n. Let $A_j = (p_i^j)^{-1}(A_i)$. Since all projections of $\mathcal{S}$ are surjective, $p_j(A) = A_j$. Using this, it is not hard to verify that $(g_j)^n$ is $\varepsilon/2$-close, and hence homotopic, to $h \circ p_i^j$. Lemma 4.1 implies that $h \circ p_i^j$ has an n-th root. □

Theorem 5.2. *For every positive integer* m *there exists an* m*-dimensional compact metrizable space* X *such that* $C(X)$ *is approximately* n*-th root closed for all positive integers* n*.*

Proof. We obtain X as the inverse limit of a sequence $\mathcal{S} = \{X_i, p_i^{i+1}\}$, consisting of m-dimensional metrizable compacta. The sequence is constructed by induction as follows. Represent the set of all positive integers as a union of disjoint infinite subsets $\{\Lambda_n\}_{n=2}^{\infty}$. Put $X_1 = S^m$, the m-dimensional sphere. Suppose the space X_k has already been constructed. Fix a countable collection $\mathcal{B}_k$ of closed subsets of X_k such that for each closed subset A of X_k and for any open neighborhood U of A there exists $B \in \mathcal{B}_k$ such that $A \subset B \subset U$. For each $B \in \mathcal{B}_k$ fix a family $\mathcal{F}_B^*$ of maps from B to $\mathbb{C}^*$ which is dense in the space $C(B, \mathbb{C}^*)$. For every map from the family $\mathcal{F}_B^*$ we fix its extension to a map from X_k to $\mathbb{C}$ and denote the family of these extensions by $\mathcal{F}_B$. Let $\Phi_k = \bigcup\{\mathcal{F}_B \mid B \in \mathcal{B}_k\}$. Define $X_{k+1} = R_n(X_k, \Phi_k)$ where n is such that $k \in \Lambda_n$, and let $p_k^{k+1} = \pi^{\Phi_k}$.

Put $X = \varprojlim \mathcal{S}$. To verify that $C(X)$ is approximately n-th root closed for each $n > 1$, it is enough to show that condition (b) of Lemma 5.1 is satisfied for the inverse sequence $\mathcal{S}$. Fix $n > 1$. Pick i and consider a closed subset A_i of X_i and a function $h\colon A_i \to \mathbb{C}^*$. Take a number $j > i$ such that $j - 1 \in \Lambda_n$. Let $A_j = \left(p_i^j\right)^{-1}(A_i)$. We show that the map $h \circ p_i^j\colon A_j \to \mathbb{C}^*$ has an n-th root. Put $A_{j-1} = \left(p_i^{j-1}\right)^{-1}(A_i)$. Let g be an extension of the map $h \circ p_i^{j-1}\colon A_{j-1} \to \mathbb{C}^*$ to some neighborhood U of A_{j-1}. There exists $B \in \mathcal{B}_k$ and a function $f\colon B \to \mathbb{C}^*$ such that $A_{j-1} \subset B \subset U$ and the restriction $g|_B$ is homotopic to f. Let $\widetilde{f}\colon B \to \mathbb{C}$ be the extension of f that belongs to the family Φ_k. Since the map p_{j-1}^j resolves the projective n-th root problem for $\widetilde{f}$, the map $f \circ p_{j-1}^j|_{A_j}$ has an n-th root.

By Lemma 4.1 the map $g \circ p^j_{j-1}|_{A_j}$ has an n-th root. It remains to note that $h \circ p^j_i|_{A_j} = g \circ p^j_{j-1}|_{A_j}$.

Note that $\dim X \leq m$ since all X_k are at most m-dimensional. Proposition 3.5 implies that $(p^{k+1}_k)^*\colon H^m(X_k;\mathbb{Q}) \to H^m(X_{k+1};\mathbb{Q})$ is a monomorphism. Applying Proposition 3.4 we conclude that $(p_1)^*\colon H^m(S^m;\mathbb{Q}) \to H^m(X;\mathbb{Q})$ is a monomorphism. Thus the limit projection $p_1\colon X \to S^m$ is essential and therefore $\dim X \geq m$. □

Let $\mathcal{K}$ be a class of spaces. A space $Z \in \mathcal{K}$ is called a universal space for the class $\mathcal{K}$, if every space in $\mathcal{K}$ is topologically embedded in Z. For a positive integer n and $\tau \geq \omega$, let $\mathcal{A}_\tau(n)$ ($\mathcal{A}_\tau$ resp.) be the class of all compact Hausdorff spaces X such that $w(X) \leq \tau$ and $C(X)$ is approximate n-th root closed ($C(X)$ is approximate n-th root closed for each $n > 1$ resp.). It was shown in [3, Corollary 1.3], that $\mathcal{A}_\tau(n)$ contains a universal space for any $\tau \geq \omega$ and any $n > 1$. Using the idea of the proof of Theorem 1.2 from [3] one can show that $\mathcal{A}_\tau$ also contains a universal space.

Corollary 5.3. *Let Y be a universal space with respect to the class $\mathcal{A}_\omega$ or $\mathcal{A}_\omega(n)$. Then Y is infinite dimensional.*

Hence, any universal space for the class $\mathcal{A}_\tau(n)$ ($\mathcal{A}_\tau$ resp.) must be infinite dimensional for any $\tau \geq \omega$.

Also we may consider the subclass $\mathcal{A}_{m,\tau}(n)$ ($\mathcal{A}_{m,\tau}$ resp.) consisting of all spaces in $\mathcal{A}_\tau(n)$ ($\mathcal{A}_\tau$ resp.) of dimension at most m. Theorem 1.2 of [3] also proves that the class $\mathcal{A}_{1,\tau}(n)$ contains a universal space. A similar proof, based on the Mardešič factorization theorem [11], works to prove that the class $\mathcal{A}_{m,\tau}(n)$ ($\mathcal{A}_{m,\tau}$ resp.) contains a universal space.

6. Compacta with root closed $C(X)$

In this section, for any positive integer m we construct a compact Hausdorff space X with $\dim X = m$ such that $C(X)$ is n-th root closed for all n. Note that for a metrizable continuum Y the algebra $C(Y)$ is square root closed if and only if Y is a dendrite, and therefore $\dim Y \leq 1$ [10], [13]. This forces the space X above to be non-metrizable.

Lemma 6.1. *Let $\mathcal{S} = \{X_\alpha, p^\beta_\alpha, \mathcal{A}\}$ be a factorizing spectrum. In order for $C(\varprojlim \mathcal{S})$ to be n-th root closed it is sufficient that for any $\alpha \in \mathcal{A}$ and any function $h \in C(X_\alpha)$ there exists $\beta > \alpha$ such that $h \circ p^\beta_\alpha$ has an n-th root. If all limit projections of $\mathcal{S}$ are surjective, the above condition is also necessary.*

Proof. Put $X = \varprojlim \mathcal{S}$. Consider $f \in C(X)$. Since $\mathcal{S}$ is factorizing there exists α and $f_\alpha \in C(X_\alpha)$ such that $f = f_\alpha \circ p_\alpha$. By the condition of the lemma we can find $\beta > \alpha$ and $g_\beta\colon X_\beta \to \mathbb{C}$ such that $(g_\beta)^n = f_\alpha \circ p^\beta_\alpha$. It is easy to verify that $g = g_\beta \circ p_\beta$ is an n-th root of f.

Now suppose that all limit projections of $\mathcal{S}$ are surjective and $C(X)$ is n-th root closed. Consider $\alpha \in \mathcal{A}$ and $h \in C(X_\alpha)$. There exists $g \in C(X)$ such that $g^n = h \circ p_\alpha$. Since $\mathcal{S}$ is factorizing, there exists $\beta > \alpha$ and $g_\beta\colon X_\beta \to \mathbb{C}$ such that $g = g_\beta \circ p_\beta$. Since the projection p_β is surjective, $(g_\beta)^n = h \circ p^\beta_\alpha$. □

Theorem 6.2. *For each positive integer m, there exists a compact Hausdorff space X with $\dim X = m$ and such that $C(X)$ is n-th root closed for any n.*

Proof. Represent the ordinal ω_1 as the union of countably many disjoint uncountable subsets $\{\Lambda_n\}_{n=2}^{\infty}$. Starting with $X_0 = S^m$, where S^m denotes an m-dimensional sphere, by transfinite induction we define an inverse spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \omega_1\}$ as follows. If $\beta = \alpha + 1$, then define $X_\beta = R_n(X_\alpha, C(X_\alpha))$, where n is such that $\alpha \in \Lambda_n$, and let $p_\alpha^\beta = \pi^{C(X_\alpha)}$. If β is a limit ordinal, then define $X_\beta = \varprojlim\{X_\alpha, p_\alpha^\gamma, \alpha < \beta\}$ and, for $\alpha < \beta$, let p_α^β be the limit projection.

Put $X = \varprojlim \mathcal{S}$. To verify that $C(X)$ is n-th root closed for each $n > 1$, it is enough to check the condition of Lemma 6.1 for the spectrum $\mathcal{S}$. Consider $n > 1$. Since the spectrum $\mathcal{S}$ has length ω_1, it is factorizing [2, Corllary 1.3.2]. Consider a function $h\colon X_\alpha \to \mathbb{C}$ and take an ordinal $\gamma > \alpha$ such that $\gamma \in \Lambda_n$. Since the map $p_\gamma^{\gamma+1}$ resolves the projective n-th root problem for $h \circ p_\alpha^\gamma$, the map $h \circ p_\alpha^{\gamma+1}$ has an n-th root.

Note that $\dim X_\alpha \leq m$ for each α and hence $\dim X \leq m$. We claim that $(p_\alpha^\beta)^*\colon H^*(X_\alpha;\mathbb{Q}) \to H^*(X_\alpha;\mathbb{Q})$ is a monomorphism for all $\alpha < \beta < \omega_1$. Indeed, in the case $\beta = \alpha + 1$ it follows from Proposition 3.5, and then in a general case it is due to Proposition 3.4. Finally, again with the help of Proposition 3.4, we conclude that $p_0^*\colon H^m(S^m;\mathbb{Q}) \to H^m(X;\mathbb{Q})$ is a monomorphism and hence the map $p_0\colon X \to S^m$ is essential. This implies $\dim X \geq m$. □

It is not hard to verify that if $C(Y)$ is n-th root closed for some (completely regular) space Y, then $C(\beta Y)$ is also n-th root closed. Here by βY we denote the Stone-Čech compactification of Y.

Corollary 6.3. *There exists an infinite-dimensional compact Hausdorff space X such that $C(X)$ is n-th root closed for all n.*

Proof. For each m, let X_m denote compactum provided by Theorem 6.2. We put $X = \beta(\bigoplus\{X_m \mid m \in \omega\})$. □

References

[1] G. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967. MR0221500 (36:4552)

[2] A. Chigogidze, *Inverse Spectra*, North Holland, Amsterdam, 1996. MR1406565 (97g:54001)

[3] A. Chigogidze, A. Karasev, K. Kawamura, and V. Valov, *On commutative and non-commutative C^*-algebras with the approximate n-th root property*, preprint.

[4] E.M. Čirka, *Approximation of continuous functions by functions holomorphic on Jordan arcs in $\mathbb{C}^n$*, Soviet Math. **7** (1966), 336–338.

[5] R.S. Countryman, *On the characterization of compact Hausdorff X for which $C(X)$ is algebraically closed*, Pacific J. Math. **20** (1967), 433–438. MR0208410 (34:8220)

[6] R. Engelking, *General Topology, Revised and completed edition*, Sigma Ser. in Pure Math. 6, Heldermann Verlag, 1989. MR1039321 (91c:54001)

[7] R. Engelking, *Theory of Dimesions, Finite and Infinite*, Sigma Ser. in Pure Math. 10, Heldermann Verlag, 1995. MR1363947 (97j:54033)

[8] O. Hatori and T. Miura, *On a characterization of the maximal ideal spaces of commutative C^*-algebras in which every element is the square of another*, Proc. Amer. Math. Soc. **128** (1999), 239–242. MR1690991 (2000k:46072a)

[9] P.J. Huber, *Homotopical cohomology and Čech cohomology*, Math. Ann. **144** (1961), 73–76. MR0133821 (24:A3646)

[10] K. Kawamura and T. Miura, On the existence of continuous (approximate) roots of algebraic equations, preprint.

[11] S. Mardešič, *On covering dimension and inverse limit of compact spaces*, Illinois J. of Math. **4** (1960), 278-291. MR0116306 (22:7101)

[12] T. Miura, *On commutative C^*-algebras in which every element is almost the square of another*, Contemporary Mathematics **232** (1999), 239–242. MR1678337 (2000k:46072b)

[13] T. Miura and K. Niijima, *On a characterization of the maximal ideal spaces of algebraically closed commutative C^*-algebras*, Proc. Amer. Math. Soc. **131** (2003), 2869–2876. MR1974344 (2004c:46094)

[14] E.H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR0210112 (35:1007)

[15] E.L. Stout, *The theory of uniform algebras*, Bogden-Quigley, 1971. MR0423083 (54:11066)

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