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FIXED POINT THEORY FOR WEAKLY INWARD KAKUTANI MAPS: THE PROJECTIVE LIMIT APPROACH

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Abstract. New fixed point results are presented for weakly inward Kakutani condensing maps defined on a Fréchet space E. The proofs rely on the notion of an essential map and viewing E as the projective limit of a sequence of Banach spaces.

1. Introduction

In Section 2 we first present a new fixed point result for weakly inward mappings defined between Fréchet spaces (see [2, 7]). Also new applicable Leray–Schauder results will be presented in Section 2 and our theory will be based on a new notion of an essential map. Our theory is based on results in Banach spaces and on viewing a Fréchet space as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (here $N = \{1, 2, ...\}$). The usual Leray–Schauder alternatives in the nonnormable situation are rarely of interest from an application viewpoint since the set constructed is usually open and bounded and so has empty interior.

For the remainder of this section we present some definitions and known facts. Let Q be a subset of a Hausdorff topological space X and $x \in X$. The <u>inward set</u> $I_Q(x)$ is defined by

$$I_Q(x) = \{x + r\,(y - x) : \ y \in Q, \ r \geq 0\}.$$

If Q is convex and $x \in Q$, then

$$I_Q(x) = x + \{r\,(y - x) : \ y \in Q, \ r \geq 1\}.$$

A mapping $F : Q \to 2^X$ (here 2^X denotes the family of all nonempty subsets of X) is said to be weakly inward with respect to Q if $F(x) \cap \overline{I_Q(x)} \neq \emptyset$ for $x \in Q$.

Let (X, d) be a metric space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty]$ defined by (here $A \in \Omega_X$)

$$\alpha(A) = \inf\{r > 0 : \ A \subseteq \bigcup_{i=1}^{n} A_i \ \text{ and } \ diam\,(A_i) \leq r\}.$$

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Let S be a nonempty subset of X. For each $x \in X$, define $d(x,S) = \inf_{y\in S} d(x,y)$. We say a set is countably bounded if it is countable and bounded. Now suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then $G : S \to 2^X$ is

(i) countably k–set contractive (here $k \geq 0$) if $G(S)$ is bounded and $\alpha(G(W)) \leq k\,\alpha(W)$ for all countably bounded sets W of S,
(ii) countably condensing if $G(S)$ is bounded, G is countably 1–set contractive and $\alpha(G(W)) < \alpha(W)$ for all countably bounded sets W of S with $\alpha(W) \neq 0$,
(iii) hemicompact if each sequence $\{x_n\}_{n\in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature [1].

Theorem 1.1. *Let (Y,d) be a metric space, D a nonempty, complete subset of Y, and $G : D \to 2^Y$ a countably condensing map. Then G is hemicompact.*

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : \ x_\alpha = \pi_{\alpha,\beta}(x_\beta)\ \forall\, \alpha,\, \beta \in I,\, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [7, p. 439] $\bigcap_{\alpha\in I} E_\alpha$).

Existence in Section 2 is based on the following results in the literature [2, 5].

Theorem 1.2. *Let E be a Banach space and C a closed bounded convex subset of E. Suppose $F : C \to CK(E)$ is an upper semicontinuous condensing map with $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in C$; here $CK(E)$ denotes the family of nonempty convex compact subsets of E. Then F has a fixed point in E.*

In our next definitions E is a Banach space, C a closed convex subset of E and U_0 a bounded open subset of E. We will let $U = U_0 \cap C$. In our definitions $\overline{U}$ and ∂U denote the closure and the boundary of U in C respectively.

Definition 1.1. We say $F \in K(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is an upper semicontinuous condensing map with $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in \overline{U}$.

Definition 1.2. A map $F \in K_{\partial U}(\overline{U}, E)$ if $F \in K(\overline{U}, E)$ with $x \notin F\,x$ for $x \in \partial U$.

Definition 1.3. A map $F \in K_{\partial U}(\overline{U}, E)$ is essential in $K_{\partial U}(\overline{U}, E)$ if for every $G \in K_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G\,x$.

Definition 1.4. Two maps $F, G \in K_{\partial U}(\overline{U}, E)$ are homotopic in $K_{\partial U}(\overline{U}, E)$, written $F \cong G$ in $K_{\partial U}(\overline{U}, E)$, if there exists an upper semicontinuous condensing map $N : \overline{U} \times [0,1] \to CK(E)$ such that $N_t(u) = N(t,u) : \overline{U} \to CK(E)$ belongs to $K_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$ and $N_0 = F$, $N_1 = G$.

The topological transversality theorem for weakly inward Kakutani maps was established in [5].

Theorem 1.3. *Let E, C, U_0 and U be as above. Suppose F and G are maps in $K_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $K_{\partial U}(\overline{U}, E)$. Then F is essential in $K_{\partial U}(\overline{U}, E)$ iff G is essential in $K_{\partial U}(\overline{U}, E)$.*

Remark 1.1. If $0 \in U$, then the zero map is essential in $K_{\partial U}(\overline{U}, E)$; see [5] for details (the proof uses Theorem 1.2).

Remark 1.2. If the map F in Definition 1.1 (and throughout) was countably condensing instead of condensing, then we would have to assume $F(x) \cap I_C(x) \neq \emptyset$ for $x \in \overline{U}$ instead of $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in \overline{U}$ in Definition 1.1 (and throughout); see [6] for details.

The following Krasnoselskii type result was established in [5] (there is also an obvious analogue for countably condensing maps if we note Remark 1.2).

Theorem 1.4. *Let E be a Banach space, C a closed convex subset of E, and W and V open bounded subsets of E with $U_1 = W \cap C$ and $U_2 = V \cap C$. Suppose $0 \in U_1 \subseteq \overline{U_1} \subseteq U_2$ and $F : \overline{U_2} \to CK(E)$ is an upper semicontinuous, condensing, weakly inward with respect to C (i.e. $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in \overline{U_2}$) map. In addition assume the following conditions are satisfied:*

$$x \notin \lambda F x \quad \text{for} \quad x \in \partial U_2 \quad \text{and} \quad \lambda \in [0,1], \tag{1.1}$$

$$\exists\, v \in C \setminus \{0\} \quad \text{with} \quad x \notin F x + \delta v \quad \text{for} \quad \delta \geq 0 \quad \text{and} \quad x \in \partial U_1, \tag{1.2}$$

$$\begin{cases} F(\,.\,) + \mu v : \overline{U_1} \to CK(E) \ \text{ is a weakly inward with respect} \\ \text{to } C \ \text{ (i.e. } [F(x) + \mu v] \cap \overline{I_C(x)} \neq \emptyset \ \text{ for } x \in \overline{U_1}) \\ \text{map for all } \ \mu \geq 0. \end{cases} \tag{1.3}$$

Then F has a fixed point in $\overline{U_2} \setminus U_1$.

2. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \ldots \quad \text{for every} \quad x \in E. \tag{2.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E / \sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m / \sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \ \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } \ j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (i) For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii) Usually in applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii) Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$), then $x \in E$. However if $x \in E_n$, then x is not necessarily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$E_1 \supseteq E_2 \supseteq ... \text{ and for each } n \in N,\ |x|_n \le |x|_{n+1}\ \forall\ x \in E_{n+1}. \tag{2.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\bigcap_1^\infty E_n$ where $\bigcap_1^\infty$ is the generalized intersection [4]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\, \mu_{n,m}\, j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\, \mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote respectively the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by [3]

$$pseudo - int\,(X) = \{x \in X :\ j_n\, \mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$.

We begin by extending Theorem 1.2 to the Fréchet space setting. Our first result is for Volterra type operators.

Theorem 2.1. *Let E and E_n be as described above, C a closed bounded convex subset of E and $F : C \to 2^E$. Suppose the following conditions are satisfied:*

$$\begin{cases} \textit{for each } \ n \in N,\ F : \overline{C_n} \to CK(E_n) \ \textit{ is an} \\ \textit{upper semicontinuous condensing map,} \end{cases} \tag{2.5}$$

$$\textit{for each } \ n \in N,\ F(x) \cap \overline{I_{\overline{C_n}}(x)} \ne \emptyset \ \textit{ for } \ x \in \overline{C_n}, \tag{2.6}$$

and

$$\begin{cases} \textit{for each } \ n \in \{2,3,...\} \ \textit{ if } \ y \in \overline{C_n} \ \textit{ solves } \ y \in F\,y \ \textit{ in } \ E_n, \\ \textit{then } \ y \in \overline{C_k} \ \textit{ for } \ k \in \{1,...,n-1\}. \end{cases} \tag{2.7}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. We would like to apply Theorem 1.2. To do so we need to show

$$\overline{C_n} \ \text{ is convex and bounded.} \tag{2.8}$$

We need only check convexity. To see this let $\hat{x}$, $\hat{y} \in \mu_n(C)$ and $\lambda \in [0,1]$. Then for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$ we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda\hat{x} + (1-\lambda)\hat{y} = \lambda\mu_n(x) + (1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$, so as a result

$$\lambda\hat{x} + (1-\lambda)\hat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C),$$

and so $\mu_n(C)$ is convex. Now since j_n is linear we have $C_n = j_n(\mu_n(C))$ is convex and as a result $\overline{C_n}$ is convex.

Theorem 1.2 guarantees that there exists $y_n \in \overline{C_n}$ with $y_n \in F\,y_n$. Let's look at $\{y_n\}_{n\in N}$. Notice $y_1 \in \overline{C_1}$ and $y_k \in \overline{C_1}$ for $k \in N\backslash\{1\}$ from (2.7). As a result $y_n \in \overline{C_1}$ for $n \in N$, $y_n \in F\,y_n$ in E_n together with (2.5) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{C_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in \overline{C_2}$ for $n \in N_1$ together with (2.5) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{C_2}$ with $y_n \to z_2$ in E_2 as $n \to \infty$

in $N_2^\star$. Note from (2.4) that $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ..., \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \overline{C_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F\,y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : \overline{C_k} \to CK(E_k)$ is upper semicontinuous (note $y_n \in \overline{C_k}$ for $n \in N_k$) implies $y \in F\,y$ in E_k. We can do this for each $k \in N$, so as a result we have $y \in F\,y$ in E. □

Our next result was motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (2.13).

Theorem 2.2. *Let E and E_n be as described in the beginning of Section 2, C a closed bounded convex subset of E and $F : C \to 2^E$. Suppose the following conditions are satisfied:*

$$\overline{C_1} \supseteq \overline{C_2} \supseteq ..., \tag{2.9}$$

$$\text{for each } n \in N,\ F_n : \overline{C_n} \to CK(E_n) \text{ is upper semicontinuous,} \tag{2.10}$$

$$\text{for each } n \in N,\ F_n(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \text{ for } x \in \overline{C_n}, \tag{2.11}$$

$$\begin{cases} \text{for each } n \in N, \text{ the map } \mathcal{K}_n : \overline{C_n} \to 2^{E_n}, \text{ given by} \\ \mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y) \text{ (see Remark 2.2), is condensing,} \end{cases} \tag{2.12}$$

and

$$\begin{cases} \text{if there exists a } w \in E \text{ and a sequence } \{y_n\}_{n\in N} \\ \text{with } y_n \in \overline{C_n} \text{ and } y_n \in F_n\,y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \text{ of } N \text{ with } y_n \to w \text{ in } E_k \\ \text{as } n \to \infty \text{ in } S, \text{ then } w \in F\,w \text{ in } E. \end{cases} \tag{2.13}$$

Then F has a fixed point in E.

Remark 2.2. The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{C_n}$ and $y \notin \overline{C_{n+1}}$, then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{C_{n+1}}$ and $y \notin \overline{C_{n+2}}$, then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in N$. Theorem 1.2 guarantees that there exists $y_n \in \overline{C_n}$ with $y_n \in F_n\,y_n$ in E_n. Let's look at $\{y_n\}_{n\in N}$. Now Theorem 1.1 (with $Y = E_1$, $G = \mathcal{K}_1$, $D = \overline{C_1}$ and note $d_1(y_n, \mathcal{K}_1(y_n)) = 0$ for each $n \in N$ since $|x|_1 \leq |x|_n$ for all $x \in E_n$ and $y_n \in F_n\,y_n$ in E_n; here $d_1(x, Z) = \inf_{y\in Z} |x - y|_1$ for $Z \subseteq Y$) guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n\in N_1}$. Now Theorem 1.1 (with $Y = E_2$, $G = \mathcal{K}_2$ and $D = \overline{C_2}$) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1^\star$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ..., \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (2.13) implies $y \in F\,y$ in E. □

For our next definitions E and E_n are as described in the beginning of Section 2, C is a closed convex subset of E and V a bounded pseudo–open subset of E. We let $U = V \cap C$ and $F : \overline{U} \to 2^E$.

Definition 2.1. $F \in K(\overline{U}, E)$ if for each $n \in N$ we have $F \in K(\overline{U_n}, E_n)$ (i.e. for each $n \in N$, $F : \overline{U_n} \to CK(E_n)$ is an upper semicontinuous condensing map with $F(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset$ for $x \in \overline{U_n}$); here $U_n = V_n \cap \overline{C_n}$ and $\overline{U_n}$ denotes the closure of U_n in $\overline{C_n}$.

Definition 2.2. $F \in K_\partial(\overline{U}, E)$ if $F \in K(\overline{U}, E)$ and for each $n \in N$ we have $x \notin F(x)$ for $x \in \partial U_n$; here ∂U_n denotes the boundary of U_n in $\overline{C_n}$.

Definition 2.3. A map $F \in K_\partial(\overline{U}, E)$ is essential in $K_\partial(\overline{U}, E)$ if for each $n \in N$ we have that $F \in K_{\partial U_n}(\overline{U_n}, E_n)$ is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$ (i.e. for each $n \in N$, every map $G \in K_{\partial U_n}(\overline{U_n}, E_n)$ with $G|_{\partial U_n} = F|_{\partial U_n}$ has a fixed point in $\overline{U_n} \setminus \partial U_n$).

Definition 2.4. $F, G \in K_\partial(\overline{U}, E)$ are homotopic in $K_\partial(\overline{U}, E)$, written $F \cong G$ in $K_\partial(\overline{U}, E)$, if for each $n \in N$ we have $F \cong G$ in $K_{\partial U_n}(\overline{U_n}, E_n)$.

Theorem 2.3. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E and V a bounded pseudo-open subset of E. Now suppose $F \in K_\partial(\overline{U}, E)$ with $U = V \cap C$. Also assume the following conditions are satisfied:*

$$G \in K_\partial(\overline{U}, E) \ \textit{ is essential in } \ K_\partial(\overline{U}, E), \tag{2.14}$$

$$F \cong G \ \textit{ in } \ K_\partial(\overline{U}, E), \tag{2.15}$$

and

$$\begin{cases} \textit{for each } \ n \in \{2, 3, ...\} \ \textit{ if } \ y \in \overline{U_n} \ \textit{ solves } \ y \in F\,y \ \textit{ in } \ E_n, \\ \textit{then } \ y \in \overline{U_k} \ \textit{ for } \ k \in \{1, ..., n-1\}. \end{cases} \tag{2.16}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. We wish to apply Theorem 1.3. To do this we need to show

$$V_n \ \text{ is a bounded open subset of } \ E_n. \tag{2.17}$$

Clearly V_n is bounded since V is bounded (note if $y \in V_n$, then there exists $x \in V$ with $y = j_n \mu_n(x)$). It remains to show V_n is open. First notice $V_n \subseteq \overline{V_n} \setminus \partial V_n$ since if $y \in V_n$, then there exists $x \in V$ with $y = j_n \mu_n(x)$ and this together with $V = pseudo - int\, V$ yields $j_n \mu_n(x) \in \overline{V_n} \setminus \partial V_n$ i.e. $y \in \overline{V_n} \setminus \partial V_n$. In addition notice

$$\overline{V_n} \setminus \partial V_n = (int\, V_n \cup \partial V_n) \setminus \partial V_n = int\, V_n \setminus \partial V_n = int\, V_n$$

since $int\, V_n \cap \partial V_n = \emptyset$. Consequently

$$V_n \subseteq \overline{V_n} \setminus \partial V_n = int\, V_n, \quad \text{so} \quad V_n = int\, V_n.$$

As a result V_n is open in E_n. Thus (2.17) holds.

Now (2.14) and (2.15) and Theorem 1.3 guarantee that F is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$. In particular there exists $y_n \in \overline{U_n}$ with $y_n \in F\,y_n$. Let's look at $\{y_n\}_{n \in N}$. Now $y_n \in \overline{U_1}$ for $n \in N$, $y_n \in F\,y_n$ in E_n together with the fact that $F \in K(\overline{U_1}, E_1)$ guarantees that there exists a subsequence $N_1^\star$ of N and a

$z_1 \in \overline{U_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Proceed inductively (as in Theorem 2.1) to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ..., \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \overline{U_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Essentially the same argument as in Theorem 2.1 guarantees that $y \in F\,y$ in E. □

Remark 2.3. If for each $n \in N$ the map $F : \overline{U_n} \to CK(E_n)$ is countably condensing instead of condensing in Definition 2.1 (and throughout), then we assume $F(x) \cap I_{\overline{C_n}}(x) \neq \emptyset$ for $x \in \overline{U_n}$ instead of $F(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset$ for $x \in \overline{U_n}$ in Definition 2.1 (and throughout).

Corollary 2.1. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E and V a bounded pseudo-open subset of E. Now suppose $F \in K_\partial(\overline{U}, E)$ with $U = V \cap C$ and assume $0 \in U$. Also suppose* (2.16) *and the following condition hold:*

$$\begin{cases} \text{for each } n \in N,\ y \notin \lambda\, F\, y \quad \text{in } E_n \text{ for all} \\ \lambda \in (0,1) \quad \text{and} \quad y \in \partial\, U_n. \end{cases} \tag{2.18}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. We first show

$$0 \in U_n. \tag{2.19}$$

Now since V is pseudo-open and $0 \in V$, then $0 \in pseudo{-}int\, V$, so $0 = j_n\, \mu_n(0) \in \overline{V_n} \setminus \partial V_n$ (here $\overline{V_n}$ and ∂V_n denote the closure and boundary of V_n in E_n respectively). Of course

$$\overline{V_n} \setminus \partial V_n = (V_n \cup \partial V_n) \setminus \partial V_n = V_n \setminus \partial V_n,$$

so $0 \in V_n \setminus \partial V_n$, and in particular $0 \in V_n$. Thus $0 \in V_n \cap \overline{C_n}$, so (2.19) holds.

Now Remark 1.1 guarantees that the zero map is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$, so (2.14) holds with $G = 0$. Also (2.15) is immediate if we take for each $n \in N$, $H_n(x, \lambda) = \lambda\, F(x)$ for $(x, \lambda) \in \overline{U_n} \times [0, 1]$. Our result now follows from Theorem 2.3. □

We now describe an essential map approach motivated by the Urysohn operator. In this case the map F_n will be related to F by the closure property (2.23). Here E and E_n are as described in the beginning of Section 2, C is a closed convex subset of E and V a bounded pseudo–open subset of E. We let $U = V \cap C$ and $F : \overline{U} \to 2^E$ and $F : \overline{U} \to 2^E$. We also assume $0 \in U$ for Definition 2.8.

Definition 2.5. $F \in A(\overline{U}, E)$ if for each $n \in N$ the map $F_n \in K(\overline{U_n}, E_n)$.

Definition 2.6. $F \in A_\partial(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ and for each $n \in N$ we have $x \notin F_n\,(x)$ for $x \in \partial U_n$.

Definition 2.7. A map $F \in A_\partial(\overline{U}, E)$ is essential in $A_\partial(\overline{U}, E)$ if for each $n \in N$ we have that $F_n \in K_{\partial U_n}(\overline{U_n}, E_n)$ is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$.

Remark 2.4. Note if $0 \in U$, then $0 \in A_\partial(\overline{U}, E)$ is essential in $A_\partial(\overline{U}, E)$ by Remark 1.1 (here $F = 0$ and $F_n = 0$ in Definition 2.7).

Definition 2.8. (We assume $0 \in U$ here.) $F, 0 \in A_\partial(\overline{U}, E)$ are homotopic in $A_\partial(\overline{U}, E)$, written $F \cong 0$ in $A_\partial(\overline{U}, E)$, if for each $n \in N$ we have $F_n \cong j_n \, \mu_n(0) = 0$ in $K_{\partial U_n}(\overline{U_n}, E_n)$.

Theorem 2.4. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E and V a bounded pseudo–open subset of E. Now suppose $F \in A_\partial(\overline{U}, E)$ with $U = V \cap C$ and assume $0 \in U$. Also suppose the following conditions are satisfied:*

$$\overline{U_1} \supseteq \overline{U_2} \supseteq ..., \tag{2.20}$$

$$F \cong 0 \quad in \quad A_\partial(\overline{U}, E), \tag{2.21}$$

$$\begin{cases} \textit{for each } n \in N, \textit{ the map } \mathcal{K}_n : \overline{U_n} \to 2^{E_n}, \textit{ given by} \\ \mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y) \textit{ is condensing,} \end{cases} \tag{2.22}$$

and

$$\begin{cases} \textit{if there exists a } w \in E \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in \overline{U_n} \textit{ and } y_n \in F_n \, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \textit{ of } N \textit{ with } y_n \to w \textit{ in } E_k \\ \textit{as } n \to \infty \textit{ in } S, \textit{ then } w \in F \, w \textit{ in } E. \end{cases} \tag{2.23}$$

Then F has a fixed point in E.

Remark 2.5. One could also have a remark in this situation similar to Remark 2.3.

Remark 2.6. Notice $0 \in U$ and (2.21) could be replaced by $F \cong G$ in $A_\partial(\overline{U}, E)$ (of course we assume $G \in A_\partial(\overline{U}, E)$ and we must specify G_n for $n \in N$ here).

Proof. Fix $n \in N$. As in Corollary 2.1 we have $0 \in U_n$, so the zero map is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$. Now Theorem 1.3 guarantees that F_n is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$, so in particular there exists $y_n \in \overline{U_n}$ with $y_n \in F_n \, y_n$ in E_n. Let's look at $\{y_n\}_{n \in N}$. Now Theorem 1.1 (applied to $\mathcal{K}_1$) guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ..., \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Essentially the same argument as in Theorem 2.2 guarantees that $y \in F \, y$ in E. □

Corollary 2.2. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E and V a bounded pseudo-open subset of E. Now suppose $F \in K_\partial(\overline{U}, E)$ with $U = V \cap C$ and assume $0 \in U$. Also suppose* (2.20), (2.22) *and* (2.23) *hold and in addition assume*

$$\begin{cases} \textit{for each } n \in N, \; y \notin \lambda \, F_n \, y \quad \textit{in } E_n \textit{ for all} \\ \lambda \in (0,1) \textit{ and } y \in \partial \, U_n. \end{cases} \tag{2.24}$$

Then F has a fixed point in E.

Proof. Notice (2.24) guarantees (2.21), so the result follows from Theorem 2.4. □

We now extend Theorem 1.4 to the Fréchet space setting.

Theorem 2.5. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E, and U and V are bounded pseudo-open subsets of E with $0 \in U \subseteq \overline{U} \subseteq V$ and $F : C \cap \overline{V} \to 2^E$. Suppose the following conditions are satisfied:*

$$\left\{\begin{array}{l} \text{for each } n \in N,\ F : \overline{W_n} \to CK(E_n) \text{ is an upper} \\ \text{semicontinuous condensing map with } F(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \\ \text{for } x \in \overline{W_n};\ \text{ here } W_n = V_n \cap \overline{C_n} \text{ and } \overline{W_n} \\ \text{denotes the closure of } W_n \text{ in } \overline{C_n}; \end{array}\right. \tag{2.25}$$

$$\left\{\begin{array}{l} \text{for each } n \in N,\ y \notin \lambda\, F\, y \text{ in } E_n \text{ for all} \\ \lambda \in [0,1] \text{ and } y \in \partial\, W_n; \end{array}\right. \tag{2.26}$$

$$\left\{\begin{array}{l} \text{for each } n \in N,\ \exists\, v_n \in \overline{C_n} \setminus \{0\} \text{ with } x \notin F\, x + \delta\, v_n \\ \text{for } \delta \geq 0 \text{ and } x \in \partial\Omega_n;\ \text{ here } \Omega_n = U_n \cap \overline{C_n}; \end{array}\right. \tag{2.27}$$

$$\left\{\begin{array}{l} \text{for each } n \in N,\ F(\,.\,) + \mu\, v_n : \overline{\Omega_n} \to CK(E_n) \text{ is} \\ \text{weakly inward with respect to } \overline{C_n} \text{ for all } \mu \geq 0 \\ \text{(i.e. } [F(x) + \mu\, v_n] \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \text{ for } x \in \overline{\Omega_n}); \end{array}\right. \tag{2.28}$$

$$\left\{\begin{array}{l} \text{for each } n \in \{2,3,...\} \text{ if } y \in \overline{W_n} \text{ solves } y \in F\, y \\ \text{in } E_n,\ \text{ then } y \in \overline{W_k} \text{ for } k \in \{1,...,n-1\}; \end{array}\right. \tag{2.29}$$

and

$$\left\{\begin{array}{l} \text{for every } k \in N \text{ and any subsequence } A \subseteq \{k, k+1, ...\} \\ \text{if } x \in \overline{C_n} \text{ is such that } x \in \overline{W_n} \setminus \Omega_n \text{ for some } n \in A, \\ \text{then there exists a } \gamma > 0 \text{ with } |x|_k \geq \gamma. \end{array}\right. \tag{2.30}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. Now $\overline{C_n}$ is a cone (see Theorem 2.2) and U_n, V_n are open bounded subsets of E_n (see Theorem 2.3) with $0 \in U_n \subseteq V_n$. Also since $j_n \mu_n$ is continuous we have $U_n \subseteq j_n \mu_n(\overline{U}) \subseteq \overline{j_n\, \mu_n(U)} = \overline{U_n}$. It is easy to see that $\overline{\mu_n(U)} \subseteq \mu_n(V)$ (note $\overline{U} \subseteq V$) so since j_n is an isometry

$$\overline{U_n} = \overline{j_n\, \mu_n(U)} = j_n\, \overline{\mu_n(U)} \subseteq j_n\, \mu_n(V) = V_n.$$

Theorem 1.3 guarantees that there exists $y_n \in \overline{W_n} \setminus \Omega_n$ with $y_n \in F\, y_n$ in E_n. Let's look at $\{y_n\}_{n \in N}$. Notice $y_n \in \overline{W_1}$ for $n \in N$ from (2.29). Then there exists a subsequence $N_1^\star$ of N and a $z_1 \in \overline{W_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Also $y_n \in \overline{W_n} \setminus \Omega_n$ for $n \in N$ together with (2.30) yields $|y_n|_1 \geq \gamma$ for $n \in N$, and so $|z_1|_1 \geq \gamma$. Let $N_1 = N_1^\star \setminus \{1\}$. Proceed inductively (as in Theorem 2.1) to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ..., \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \overline{U_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$ and $|z_k|_k \geq \gamma$ for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Essentially the same argument as in Theorem 2.1 guarantees that $y \in F\, y$ in E. □

Remark 2.7. Notice (2.30) is only needed to guarantee that the fixed point $y \in E$ satisfies $|z_k|_k \geq \gamma$ for $k \in N$; here $y = z_k$ in E_k. If we assume (2.25)-(2.29), then once again F has a fixed point in E but the above property is not guaranteed.

Theorem 2.6. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E, U and V are bounded pseudo-open subsets of E with $0 \in U \subseteq \overline{U} \subseteq V$ and $F : C \cap \overline{V} \to 2^E$. Suppose the following conditions are satisfied:*

$$\overline{W_1} \supseteq \overline{W_2} \supseteq ...; \quad \text{here } W_n = \overline{C_n} \cap V_n; \tag{2.31}$$

$$\begin{cases} \text{for each } n \in N,\ F_n : \overline{W_n} \to CK(E_n) \text{ is an upper} \\ \text{semicontinuous condensing map with } F(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \\ \text{for } x \in \overline{W_n}; \end{cases} \tag{2.32}$$

$$\begin{cases} \text{for each } n \in N,\ y \notin \lambda F_n y \text{ in } E_n \text{ for all} \\ \lambda \in [0,1] \text{ and } y \in \partial W_n; \end{cases} \tag{2.33}$$

$$\begin{cases} \text{for each } n \in N,\ \exists\, v_n \in \overline{C_n} \setminus \{0\} \text{ with } x \notin F_n x + \delta v_n \\ \text{for } \delta \geq 0 \text{ and } x \in \partial\Omega_n; \text{ here } \Omega_n = U_n \cap \overline{C_n}; \end{cases} \tag{2.34}$$

$$\begin{cases} \text{for each } n \in N,\ F_n(\,.\,) + \mu v_n : \overline{\Omega_n} \to CK(E_n) \text{ is} \\ \text{weakly inward with respect to } \overline{C_n} \text{ for all } \mu \geq 0 \\ (\text{i.e. } [F_n(x) + \mu v_n] \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \text{ for } x \in \overline{\Omega_n}); \end{cases} \tag{2.35}$$

$$\begin{cases} \text{for each } n \in N, \text{ the map } \mathcal{K}_n : \overline{W_n} \to 2^{E_n}, \text{ given by} \\ \mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y), \text{ is condensing}; \end{cases} \tag{2.36}$$

and

$$\begin{cases} \text{if there exists a } w \in E \text{ and a sequence } \{y_n\}_{n \in N} \\ \text{with } y_n \in \overline{W_n} \setminus \Omega_n \text{ and } y_n \in F_n y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \text{ of } N \text{ with } y_n \to w \text{ in } E_k \\ \text{as } n \to \infty \text{ in } S, \text{ then } w \in F w \text{ in } E. \end{cases} \tag{2.37}$$

Finally suppose (2.30) *holds. Then F has a fixed point in E.*

Remark 2.8. A similar remark to Remark 2.7 applies here.

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