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STRONG UNIQUE CONTINUATION FOR m-TH POWERS OF A LAPLACIAN OPERATOR WITH SINGULAR COEFFICIENTS

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Abstract. In this paper we prove strong unique continuation for u satisfying an inequality of the form $|\triangle^m u| \leq f(x, u, Du, \cdots, D^k u)$, where k is up to $[3m/2]$. This result gives an improvement of a work by Colombini and Grammatico (1999) in some sense. The proof of the main theorem is based on Carleman estimates with three-parameter weights $|x|^{2\sigma_1}(\log|x|)^{2\sigma_2}\exp(\frac{\beta}{2}(\log|x|)^2)$.

1. Introduction

Let Ω be a connected open subset of $\mathbb{R}^n$ $(n \geq 2)$ containing 0. In [6], Colombini and Grammatico prove some results of strong unique continuation for u satisfying an inequality of the form

$$|\triangle^m u| \leq f(x, u, Du, \cdots, D^k u) \quad \text{for } k = m. \tag{1.1}$$

In this paper we are interested in u satisfying the following forms:

$$|\triangle^m u| \leq C_0 \textstyle\sum_{|\alpha|\leq m-1} |x|^{-2m+|\alpha|}|D^\alpha u| + C_0 \sum_{|\alpha|=m}^{[3m/2]} |x|^{-2m+|\alpha|+\epsilon}|D^\alpha u|, \tag{1.2}$$

where $0 < \epsilon < 1/2$ and the orders of lower order terms are up to $[3m/2]$.

Theorem 1.1. *Let $u \in H^{2m}_{loc}(\Omega)$ be a solution of (1.2), and for all $N > 0$ let*

$$\int_{|x|\leq R} |u|^2 dx = O(R^N), \quad R \to 0.$$

Then u is identically zero in Ω.

This problem for $m = 1$ has drawn a lot of attention in partial differential equations and mathematical physics. For the development of this problem, the author refers the readers to [15]. In particular, their work was focused on second order equations in which the coefficients of the lower order terms are allowed to be singular (see [2], [3], [11] and [5]). Here we mention two articles which are closely related to our work in this paper.

Hörmander [8] proved that if $u \in H^1_{loc}(\Omega)$ satisfies

$$|\Delta u| \leq C_1|x|^{-2+\epsilon}|u| + C_2|x|^{-1+\epsilon}|\nabla u|, \quad \epsilon > 0,$$

and u vanishes of infinite order at 0, then u is identically zero in Ω.

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Regbaoui [13] extended Hörmander's result to the sharp case $\epsilon = 0$ with small C_2. His proof was also based on suitable Carleman estimates, but he worked with the strictly convex weights $\varphi_\beta(x) = \exp(\frac{\beta}{2}(\log|x|)^2)$ rather than the usual polynomial weights. As for negative results, some counterexamples were given when C_2 is not small (such as [1] and [16]).

For the case $m = 2$, Borgne [4] got some results about strong unique continuation when $k = 3$ in (1.1). On the other hand, Watanabe [14] established some uniqueness results for $m = 3$ with $k = 5$. For $m = 4$, Ōkaji [10] solved unique continuation when $k = 7$. For the higher orders, Protter [12] got the unique continuation for inequalities of the form (1.1) in which k is up to $[3m/2]$. According to the counterexamples for $n = 2$ in [7], Colombini and Grammatico [6] proved the sharp results about the coefficients of the m-th lower order terms for strong unique continuation with some cases when $k = m$. In this paper, we consider the cases (1.2).

We will follow Regbaoui's approach in [13] and organize this paper as follows. In Section 2, we shall study the asymptotic behavior of the function u near 0 which guarantees the use of the singular weights $|x|^{2\sigma_1}(\log|x|)^{2\sigma_2}\varphi_\beta^2$. Some of the key Carleman estimates with the weights $|x|^{2\sigma_1}(\log|x|)^{2\sigma_2}\varphi_\beta^2$ will be derived in Section 3. Using these Carleman estimates, we prove Theorem 1.1 in Section 4.

2. Carleman estimates with polynomial weights

Lemma 2.1. *There exists a positive constant C_m such that for any $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ and for any $\tau \in \{k + 1/2, k \in \aleph\}$, we have the estimate*

$$\sum_{|\alpha|\le 2m} \int \tau^{2m-2|\alpha|}|x|^{-2\tau+2|\alpha|-n}|D^\alpha u|^2 dx \le C_m \int |x|^{-2\tau+4m-n}|\triangle^m u|^2 dx. \tag{2.1}$$

Proof. In [13], Regbaoui proved in Lemma 2.1 for any $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ and for any $\tau \in \{k + 1/2, k \in \aleph\}$ that

$$\sum_{|\alpha|\le 2} \int \tau^{2-2|\alpha|}|x|^{-2\tau+2|\alpha|-n}|D^\alpha u|^2 dx \le C \int |x|^{-2\tau+4-n}|\triangle u|^2 dx. \tag{2.2}$$

By the repeated use of (2.2), we can get (2.1).

Remark 2.2. The estimate (2.1) in Lemma 2.1 remains valid if we assume $u \in H^{2m}_{loc}(\Omega)$ with compact support and satisfying for all $|\alpha| \le 2m$ and all $N > 0$, $\int_{|x|\le R} |D^\alpha u|^2 dx = O(R^N)$ as $R \to 0$. This can be easily obtained by cutting u off for small $|x|$ and regularizing.

Motivated by [8] and [13], we get the following theorem. The proof of this theorem is based on Lemma 2.1. From now on, c stands for a generic constant and its value may vary from line to line.

Theorem 2.3. *Let $u \in H^{2m}_{loc}(\Omega)$ be a solution to* (1.2). *If $|\alpha| \le 2m$, then*

$$\int_{|x|\le R} |D^\alpha u|^2 dx = O(e^{-BR^{-\epsilon/m}}), \quad R \to 0, \tag{2.3}$$

for some positive constant B.

Proof. Following Hörmander's argument in [9] (Corollary 17.1.4., p. 8), we show that if $u \in H^{2m}_{loc}(\Omega)$ is a solution of (1.2), then for all $|\alpha| \le 2m$ and for all $N > 0$

$$\int_{|x|\le R} |D^\alpha u|^2 dx = O(R^N), \quad R \to 0. \tag{2.4}$$

In view of Remark 2.2, we can apply (2.1) to the function ξu, where $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \le R$ and $\xi(x) = 0$ for $|x| \ge 2R$ ($R > 0$ sufficiently small). Here the number R is not yet fixed and is given by $R = (\gamma\tau)^{-m/\epsilon}$, where $\gamma > 0$ is a large constant which will be chosen later. Using the estimate (2.1) and the equation (1.2), we can derive that

$$\begin{aligned}
&\sum_{|\alpha|\le m-1} \tau^{2m-2|\alpha|} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx \\
&\quad + \sum_{|\alpha|=m}^{2m} \gamma^{2m} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n+2\epsilon} |D^\alpha u|^2 dx \\
&= \sum_{|\alpha|\le m-1} \tau^{2m-2|\alpha|} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx \\
&\quad + \sum_{|\alpha|=m}^{2m} R^{-2\epsilon}\tau^{-2m} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n+2\epsilon} |D^\alpha u|^2 dx \\
&\le \sum_{|\alpha|\le m-1} \tau^{2m-2|\alpha|} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx \\
&\quad + \sum_{|\alpha|=m}^{2m} \tau^{2m-2|\alpha|} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx \\
&\le \sum_{|\alpha|\le 2m} \tau^{2m-2|\alpha|} \int |x|^{-2\tau+2|\alpha|-n} |D^\alpha(\chi u)|^2 dx \\
&\le C_m \int |x|^{-2\tau+4m-n} |\triangle^m(\chi u)|^2 dx \\
&\le cC_m \int_{|x|\le R} |x|^{-2\tau+4m-n} |\triangle^m u|^2 dx \\
&\quad + cC_m \int_{|x|>R} |x|^{-2\tau+4m-n} |[\chi, \triangle^m] u|^2 dx \\
&\le cC_m C_0 \sum_{|\alpha|\le m-1} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx \\
&\quad + cC_m C_0 \sum_{|\alpha|=m}^{[3m/2]} \int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n+2\epsilon} |D^\alpha u|^2 dx \\
&\quad + cC_m \int_{|x|>R} |x|^{-2\tau+4m-n} |[\chi, \triangle^m] u|^2 dx,
\end{aligned} \tag{2.5}$$

where $[\cdot,\cdot]$ denotes the commutator.

Therefore, carefully checking terms on both sides of (2.5), we can choose γ and let τ be large enough such that all terms with $\int_{|x|\le R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx$ and

$\int_{|x|\le R}|x|^{-2\tau+2|\alpha|-n+2\epsilon}|D^\alpha u|^2dx$ on the right side of (2.5) are absorbed by the left-hand side. We now fix such γ. By construction of χ we have $|D^\alpha\chi|\le c_1R^{-|\alpha|}$, where c_1 is a positive constant. Consequently, it follows for $R<1/2$ and (2.5) that

$$\begin{aligned}
&\tau^{-2m}(R/2)^{-2\tau+4m-n+2\epsilon}\sum_{|\alpha|\le 2m}\int_{|x|\le R/2}|D^\alpha u|^2dx\\
&\le \sum_{|\alpha|\le 2m}\tau^{2m-2|\alpha|}(R/2)^{-2\tau+2|\alpha|-n+2\epsilon}\int_{|x|\le R/2}|D^\alpha u|^2dx\\
&\le \sum_{|\alpha|\le 2m}\tau^{2m-2|\alpha|}\int_{|x|\le R/2}|x|^{-2\tau+2|\alpha|-n+2\epsilon}|D^\alpha u|^2dx\\
&\le \sum_{|\alpha|\le m-1}\tau^{2m-2|\alpha|}\int_{|x|\le R}|x|^{-2\tau+2|\alpha|-n}|D^\alpha u|^2dx\\
&\quad+\sum_{|\alpha|=m}^{2m}\gamma^{2m}\int_{|x|\le R}|x|^{-2\tau+2|\alpha|-n+2\epsilon}|D^\alpha u|^2dx\\
&\le c\int_{|x|>R}|x|^{-2\tau+4m-n}|[\chi,\triangle^m]u|^2dx\\
&\le cR^{-4m}R^{-2\tau+4m-n}\|u\|^2_{H^{2m}}\\
&= c\|u\|^2_{H^{2m}}R^{-2\tau-n},
\end{aligned}\tag{2.6}$$

where $\|u\|^2_{H^{2m}}$ is the H^{2m} norm of u in the ball $B(0,2R)$.

Recall that for $R=(\gamma\tau)^{-m/\epsilon}$, we have

$$\begin{aligned}
\sum_{|\alpha|\le 2m}\int_{|x|\le R/2}|D^\alpha u|^2dx &\le cR^{-4m-4\epsilon}2^{-2\gamma^{-1}R^{-\epsilon/m}}\|u\|^2_{H^{2m}}\\
&\le ce^{-\tilde{B}R^{-\epsilon/m}}.
\end{aligned}\tag{2.7}$$

It should be noted that (2.7) is valid for $\tau\in\mathbb{N}+\frac{1}{2}$ and $R=(\gamma\tau)^{-m/\epsilon}$. Therefore, if we choose $\tau\in\{j+\frac{1}{2}:j\in\mathbb{N}\}$, then (2.7) only holds for $R_j=(\gamma(j+\frac{1}{2}))^{-m/\epsilon}$. Nevertheless, we can see that

$$R_{j+1}<R_j<2R_{j+1}\quad\text{for } j\quad\text{large enough}\quad\text{and}\quad R_j\to 0\text{ as } j\to\infty.$$

Thus, we can conclude that

$$\sum_{|\alpha|\le 2m}\int_{|x|\le R}|D^\alpha u|^2dx\le ce^{-BR^{-\epsilon/m}},$$

for all sufficiently small $R>0$ with $B=2^{-2\epsilon/m}\tilde{B}$.

3. Carleman estimates with more singular weights

To prove the following Carleman estimates, we introduce polar coordinates in $\mathbb{R}^n\backslash\{0\}$ by setting $x=r\omega$, with $r=|x|$, $\omega=(\omega_1,\cdots,\omega_n)\in S^{n-1}$ when $x\ne 0$. Furthermore, setting $t=\log r$, we can substitute a new coordinate t for r such that

$$\frac{\partial}{\partial x_j}=e^{-t}(\omega_j\partial_t+\Omega_j),$$

where Ω_j is a vector field in S^{n-1}. Then the Laplacian becomes

$$e^{2t}\Delta = \partial_t^2 + (n-2)\partial_t + \Delta_\omega, \tag{3.1}$$

where $\Delta_\omega = \sum_{j=1}^n \Omega_j^2$ is the Laplace-Beltrami operator in S^{n-1}. The vector field Ω_j has the properties

$$\sum_{j=1}^n \omega_j \Omega_j = 0, \qquad \sum_{j=1}^n \Omega_j \omega_j = n-1.$$

The adjoint of Ω_j as an operator in $L^2(S^2)$ is

$$\Omega_j^* = (n-1)\omega_j - \Omega_j \tag{3.2}$$

and

$$\sum_{j=1}^n \Omega_j^* \Omega_j = -\Delta_\omega. \tag{3.3}$$

Hereafter we shall use the following notation:

$$\begin{aligned} &D_0 = (1/i)\partial_t; \\ &D_k = (1/i)\Omega_k, \quad k = 1, \cdots, n; \\ &D^\alpha = D_0^{\alpha_1} \cdots D_n^{\alpha_n}, \qquad \alpha = (\alpha_0, \cdots, \alpha_n) \in \mathbb{N}^{n+1}. \end{aligned} \tag{3.4}$$

When $r \to 0$, $t \to -\infty$, we will be interested in values of t in a neighborhood of $-\infty$. Motivated by [13], we derive the following Carleman estimate with weights $\varphi_\beta = \varphi_\beta(x) = \exp(\frac{\beta}{2}(\log|x|)^2)$.

Theorem 3.1. *Given $\sigma_1 \in \mathbb{Z}$ and $\sigma_2 \in \mathbb{Z}$, there exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$ depending on n, σ_1 and σ_2 such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $\beta \geq \beta_0$, we have that*

$$\begin{aligned} &C \sum_{|\alpha| \leq 2} \beta^{3-2|\alpha|} \int \varphi_\beta^2 |x|^{2\sigma_1 + 2|\alpha| - n} (\log|x|)^{2\sigma_2 + 2 - 2|\alpha|} |D^\alpha u|^2 dx \\ &\leq \int \varphi_\beta^2 |x|^{2\sigma_1 + 4 - n} (\log|x|)^{2\sigma_2} |\Delta u|^2 dx, \end{aligned} \tag{3.5}$$

where $U_{r_0} = \{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \mathrm{supp}(u) \subset B_{r_0}\}$ and C is a positive constant.

Proof. By the polar coordinate system, we have

$$\begin{aligned} &\int \varphi_\beta^2 |x|^{2\sigma_1 + 4 - n} (\log|x|)^{2\sigma_2} |\Delta u|^2 dx \\ &= \int\int e^{\beta t^2} e^{2\sigma_1 t + 4t} t^{2\sigma_2} |\Delta u|^2 dt d\omega \\ &= \int\int |e^{\beta t^2/2} e^{\sigma_1 t} t^{\sigma_2} e^{2t} \Delta u|^2 dt d\omega. \end{aligned} \tag{3.6}$$

If we set $u = e^{-\beta t^2/2} e^{-\sigma_1 t} t^{-\sigma_2} v$ and use (3.1), then

$$e^{\beta t^2/2} e^{\sigma_1 t} t^{\sigma_2} e^{2t} \Delta u = (\partial_t - \beta t)^2 v + (n-2)(\partial_t - \beta t) v + \Delta_\omega v + a \partial_t v + bv, \tag{3.7}$$

where $b = 2\beta\sigma_1 t + 2\beta\sigma_2 + \sigma_1^2 + 2\sigma_1\sigma_2 t^{-1} - \sigma_2 t^{-2} + \sigma_2^2 t^{-2} - (n-2)(\sigma_1 + \sigma_2 t^{-1})$ and $a = -2\sigma_1 - 2\sigma_2 t^{-1}$.

By (3.6), (3.7) and β large enough, (3.5) holds if we have

$$\sum_{|\alpha|\le 2} \beta^{3-2|\alpha|} \int\int t^{2-2|\alpha|}|D^\alpha v|^2 dtd\omega \le C \int\int |\Delta_\beta v|^2 dtd\omega, \tag{3.8}$$

where $\Delta_\beta v = (\partial_t - \beta t)^2 v + (n-2)(\partial_t - \beta t)v + \Delta_\omega v$ and C is a positive constant.

Note that

$$\Delta_\beta v = \partial_t^2 v + (n-2)\partial_t v - 2\beta t\partial_t v - (n-2)\beta t v - \beta v + \beta^2 t^2 v + \Delta_\omega v.$$

Denote

$$\Delta_\beta^- v = \partial_t^2 v - (n-2)\partial_t v + 2\beta t\partial_t v - (n-2)\beta t v - \beta v + \beta^2 t^2 v + \Delta_\omega v.$$

Using integration by parts, we get

$$\begin{aligned} I :=& \int\int |\Delta_\beta v|^2 dtd\omega - \int\int |\Delta_\beta^- v|^2 dtd\omega \\ =& \int\int (12\beta^3 t^2 + \beta^2 O(t))|v|^2 dtd\omega + 4\beta \int\int |\partial_t v|^2 dtd\omega \\ &- 4\beta \int\int \sum_j |\Omega_j v|^2 dtd\omega. \end{aligned} \tag{3.9}$$

Similarly, we can get

$$\begin{aligned} J :=& \int\int t^{-2}|\Delta_\beta v|^2 dtd\omega + \int\int t^{-2}|\Delta_\beta^- v|^2 dtd\omega \\ =& \int\int (2\beta^4 t^2 + \beta^3 O(t))|v|^2 dtd\omega + \int (8\beta^2 + \beta O(t^{-1}))|\partial_t v|^2 dtd\omega \\ &+ 2\int\int t^{-2}|\partial_t^2 v|^2 dtd\omega - \int\int \sum_j (4\beta^2 + \beta O(t^{-1}))|\Omega_j v|^2 dtd\omega \\ &+ 2\int\int t^{-2}|\Delta_\omega v|^2 dtd\omega + 4\int\int t^{-2}\sum_j |\partial_t \Omega_j v|^2 dtd\omega. \end{aligned} \tag{3.10}$$

If $v \in C_0^\infty(B(0, e^{T_0}))$ and $\beta >> |T_0|$ is large enough, then it follows from (3.9) and (3.10) that

$$\begin{aligned} \beta I + J \ge& 13\beta^4 \int\int t^2|v|^2 dtd\omega + 2\int\int t^{-2}|\Delta_\omega v|^2 dtd\omega \\ &- 9\beta^2 \int\int \sum_j |\Omega_j v|^2 dtd\omega + 11\beta^2 \int\int |\partial_t v|^2 dtd\omega \\ &+ 2\int\int t^{-2}|\partial_t^2 v|^2 dtd\omega + 4\int\int t^{-2}\sum_j |\partial_t \Omega_j v|^2 dtd\omega \\ \ge& K + \frac{13}{39}\beta^4 \int\int t^2|v|^2 dtd\omega + \frac{1}{20}\int\int t^{-2}|\Delta_\omega v|^2 dtd\omega \\ &+ \beta^2 \int\int \sum_j |\Omega_j v|^2 dtd\omega + 11\beta^2 \int\int |\partial_t v|^2 dtd\omega \\ &+ 2\int\int t^{-2}|\partial_t^2 v|^2 dtd\omega + 4\int\int t^{-2}\sum_j |\partial_t \Omega_j v|^2 dtd\omega, \end{aligned} \tag{3.11}$$

where

$$K = \frac{500}{39}\beta^4 \int\int t^2|v|^2 dtd\omega - 10\beta^2 \int\int \sum_j |\Omega_j v|^2 dtd\omega + \frac{39}{20}\int\int t^{-2}|\Delta_\omega v|^2 dtd\omega.$$

Now we shall get a lower bound of K. Then, by (3.3), we have

$$-\beta^2 \int\int \sum_j |\Omega_j v|^2 dtd\omega = \beta^2 \int\int \bar{v}\Delta_\omega v dtd\omega. \tag{3.12}$$

Since $10|ab| \le 500a^2/39 + 39b^2/20$, it follows that

$$10|\beta^2 v\Delta_\omega v| \le \frac{500}{39}\beta^4 t^2|v|^2 + \frac{39}{20}t^{-2}|\Delta_\omega v|^2. \tag{3.13}$$

Combining (3.12) and (3.13), we get a lower bound of K:

$$\begin{aligned} K &= \frac{500}{39}\beta^4 \int\int t^2|v|^2 dtd\omega - 10\beta^2 \int\int \sum_j |\Omega_j v|^2 dtd\omega + \frac{39}{20}\int\int t^{-2}|\Delta_\omega v|^2 dtd\omega \\ &\ge 0. \end{aligned} \tag{3.14}$$

In addition, the ellipticity of Δ_ω implies that there exists a new positive constant C such that

$$\sum_{|\alpha|=2} \int\int t^{-2}|\Omega^\alpha v|^2 dtd\omega \le C\int\int t^{-2}|\Delta_\omega v|^2 dtd\omega. \tag{3.15}$$

Thus, it follows from (3.11), (3.14) and (3.15) that

$$\begin{aligned} C(\beta I + J) \ge{}& \beta^4 \int\int t^2|v|^2 dtd\omega + \int\int t^{-2}|\Delta_\omega v|^2 dtd\omega \\ &+ \beta^2 \int\int \sum_j |\Omega_j v|^2 dtd\omega + \beta^2 \int\int |\partial_t v|^2 dtd\omega \\ &+ \int\int t^{-2}|\partial_t^2 v|^2 dtd\omega + \int\int t^{-2}\sum_j |\partial_t \Omega_j v|^2 dtd\omega. \end{aligned}$$

Together with

$$\beta I + J \le (\beta+1)\int\int |\Delta_\beta v|^2 dtd\omega,$$

(3.8) holds. So we have the result.

Corollary 3.1. *Given $\sigma_1 \in \mathbb{Z}$ and $\sigma_2 \in \mathbb{Z}$, there exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$ depending on n, m, σ_1 and σ_2 such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $\beta \ge \beta_0$, we have that*

$$\begin{aligned} &C\sum_{|\alpha|\le 2m} \beta^{3m-2|\alpha|} \int \varphi_\beta^2 |x|^{2\sigma_1+2|\alpha|-n}(\log|x|)^{2\sigma_2+2m-2|\alpha|}|D^\alpha u|^2 dx \\ &\le \int \varphi_\beta^2 |x|^{2\sigma_1+4m-n}(\log|x|)^{2\sigma_2}|\Delta^m u|^2 dx, \end{aligned}$$

where C is a positive constant.

Corollary 3.2. *There exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$ depending on n and m such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $\beta \geq \beta_0$, we have that*

$$\begin{aligned} &C \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx \\ &\leq \int \varphi_\beta^2 |x|^{4m-n} |\Delta^m u|^2 dx, \end{aligned} \tag{3.16}$$

where C is a positive constant.

Remark 3.2. The estimate (3.16) in Corollary 3.2 remains valid if we assume $u \in H^{2m}_{loc}(\Omega)$ with compact support and satisfies for all $|\alpha| \leq 2m$, $\int_{|x| \leq R} |D^\alpha u|^2 dx = O(e^{-BR^{-\epsilon/m}})$ as $R \to 0$, $B > 0$.

4. Proof of Theorem 1.1

Let $u \in H^{2m}_{loc}(\Omega)$ be a solution of (1.2). By Lemma 2.1 and Theorem 2.3, u is in $H^{2m}_{loc}(\Omega)$ and satisfies (2.3). Thus by Remark 3.2 we can apply (3.16) with the function ξu, where $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \leq R$ and $\xi(x) = 0$ for $|x| \geq 2R$ ($R > 0$ small enough). Then

$$\begin{aligned} &\sum_{|\alpha| \leq [\frac{3m}{2}]} \beta^{3m-2|\alpha|} \int_{|x|<R} \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx \\ &\leq \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int_{|x|<R} \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx \\ &\leq \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha (\xi u)|^2 dx \\ &\leq c \int \varphi_\beta^2 |x|^{4m-n} |\Delta^m (\xi u)|^2 dx \\ &\leq c \int_{|x| \leq R} \varphi_\beta^2 |x|^{4m-n} |\triangle^m u|^2 dx + c \int_{|x|>R} \varphi_\beta^2 |x|^{4m-n} |[\xi, \triangle^m] u|^2 dx \\ &\leq c \sum_{|\alpha| \leq m-1} \int_{|x| \leq R} \varphi_\beta^2 |x|^{2|\alpha|-n} |D^\alpha u|^2 dx \\ &\quad + c \sum_{|\alpha|=m}^{[3m/2]} \int_{|x| \leq R} \varphi_\beta^2 |x|^{2|\alpha|-n+2\epsilon} |D^\alpha u|^2 dx \\ &\quad + c \int_{|x|>R} \varphi_\beta^2 |x|^{4m-n} |[\xi, \triangle^m] u|^2 dx, \end{aligned} \tag{4.1}$$

because $\lim_{|x|=r \to 0} r^{2\epsilon} (\log r)^{-2m} = 0$ which implies $c|x|^{2\epsilon} \leq (\log |x|)^{2m}$ as $R \leq R_0$ for some small $R_0 > 0$. Let $\beta > \beta_0$ (β_0 large enough); the first two terms on the right-hand side will be swallowed by the left-hand side. Recall that $\varphi_\beta = \varphi_\beta(x) =$

$\exp(\frac{\beta}{2}(\log|x|)^2)$. Therefore, we obtain that

$$
\begin{aligned}
&\beta^{3m}\exp(\beta(\log R_0)^2)\int_{|x|<R_0}|x|^{-n}(\log|x|)^{2m}|u|^2dx\\
&\le \sum_{|\alpha|\le[\frac{3m}{2}]}\beta^{3m-2|\alpha|}\int_{|x|<R_0}\varphi_\beta^2|x|^{2|\alpha|-n}(\log|x|)^{2m-2|\alpha|}|D^\alpha u|^2dx\\
&\le c\int_{|x|>R_0}\varphi_\beta^2|x|^{4m-n}|[\xi,\triangle^m]u|^2dx\\
&\le c\exp(\beta(\log R_0)^2)\int_{|x|>R_0}|x|^{4m-n}|[\xi,\triangle^m]u|^2dx.
\end{aligned}
\tag{4.2}
$$

Divide $\exp(\beta(\log R_0)^2)$ into both sides of (4.2) and then let $\beta\to\infty$; this implies that $u=0$ in $B(0,R_0)$. By standard arguments, we can get $u=0$ in Ω. This completes the proof.

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