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WREATH PRODUCTS AND KALUZHNIN-KRASNER EMBEDDING FOR LIE ALGEBRAS

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Abstract. The wreath product of groups $A \wr B$ is one of basic constructions in group theory. We construct its analogue, a wreath product of Lie algebras.

Consider Lie algebras H and G over a field K. Let $U(G)$ be the universal enveloping algebra. Then $\bar{H} = \mathrm{Hom}_K(U(G), H)$ has the natural structure of a Lie algebra, where the multiplication is defined via the comultiplication in $U(G)$. Also, G acts by derivations on $\bar{H}$ via the (left) coregular action. The semidirect sum $\bar{H} \leftthreetimes G$ we call the wreath product and denote by $H \wr G$. As a main result, we prove that an arbitrary extension of Lie algebras $0 \to H \to L \to G \to 0$ can be embedded into the wreath product $L \hookrightarrow H \wr G$.

1. Introduction. Wreath products of groups and Lie algebras

We recall the notion of the wreath product of groups. This is a basic construction in group theory. Let A and B be arbitrary groups. Consider the set of functions $\bar{A} = \{f : B \to A\}$; this is a group under the pointwise multiplication $(f * h)(b) = f(b)\,h(b)$ for $b \in B$ and $f, h \in \bar{A}$. In other words, we consider the Cartesian product $\bar{A} \cong A^B$. For any $z \in B$ define the mapping $\bar{A} \ni f \mapsto {}^z\!f \in \bar{A}$ by setting $({}^z\!f)(b) = f(b \cdot z)$, $b \in B$. We obtain the embedding $B \hookrightarrow \mathrm{Aut}\,\bar{A}$. The *Cartesian wreath product* $A \,\bar{\wr}\, B$ is the semidirect product $\bar{A} \leftthreetimes B = \{(a, b) \mid a \in \bar{A}, b \in B\}$ with the multiplication

$$(a, b) * (a_1, b_1) = (a({}^b a_1), bb_1), \quad a, a_1 \in \bar{A},\ b, b_1 \in B.$$

Assume that $0 \to A \to G \xrightarrow{\theta} B \to 0$ is an extension of groups. Let $\tau : B \to G$ be a mapping such that $\theta(\tau(b)) = b$ for all $b \in B$. The Kaluzhnin-Krasner theorem [6, 4] claims that there exists an embedding $\rho : G \hookrightarrow A \,\bar{\wr}\, B$, and it is given by the following formula:

$$\rho(g) = (f_g, \theta(g)), \quad \text{where } f_g \in \bar{A},\ f_g(b) = \tau(b)g \cdot (\tau(b\theta(g)))^{-1},\ b \in B;\ g \in G.$$

A formal check shows that this is indeed an embedding.

A. Shmelkin introduced a *verbal wreath product* of groups [13] and a *verbal wreath product* of Lie algebras [14]. These constructions are widely used to study relatively free groups (Lie algebras) of products of varieties of groups (Lie algebras) [1, 5, 9].

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But what is an analogue of the ordinary wreath product for Lie algebras? Our goal is to define this notion, namely the *wreath product of Lie algebras.* Suppose that H and G are Lie algebras over a field K. Consider the vector space $\bar{H} = \mathrm{Hom}_K(U(G), H)$, where $U(G)$ is the universal enveloping algebra. Let $\langle -, - \rangle$ denote the natural pairing $\bar{H} \times U(G) \to H$. Letting $f, f_1 \in \bar{H}$, we define the product $[f, f_1]$ as follows:

$$\langle [f, f_1], a \rangle = \sum \Big[\langle f, a_{(1)} \rangle, \langle f_1, a_{(2)} \rangle \Big], \quad a \in U(G),$$

where $\Delta a = \sum a_{(1)} \otimes a_{(2)}$ denotes the comultiplication Δ for $a \in U(G)$ in Sweedler's notations [3, 8, 15]. We also define the action of G on $\bar{H}$ as follows:

$$(z \circ f)(u) = -f(zu), \qquad z \in G,\ f \in \bar{H},\ u \in U(G).$$

A formal check shows that $\bar{H}$ is a Lie algebra and G acts by derivations on it. In Section 2, we interpret $\bar{H}$ as a formal divided power series ring $H^O[[X_I]]$ with coefficients in H, and G acts by special derivations of this ring. Now we can define the *(standard) wreath product* of G and H

$$H \wr G = \bar{H} \rightthreetimes G = \mathrm{Hom}_K(U(G), H) \rightthreetimes G,$$

where $[(f, g), (f_1, g_1)] = ([f, f_1] + g \circ f_1 - g_1 \circ f, [g, g_1])$, for all $f, f_1 \in \bar{H}$, $g, g_1 \in G$.

The main result of the paper is an analogue of the Kaluzhnin-Krasner embedding theorem (Theorem 2). Let L be an extension of Lie algebras $0 \to H \to L \to G \to 0$ over an arbitrary field K. Then there exists an embedding into the wreath product $L \hookrightarrow H \wr G$. Thus, the wreath product $H \wr G$ is a universal object that contains all such extensions.

The pioneering work on the application of coalgebras and so-called produced modules was done by Blattner [2]. These methods were used by Radford to establish different embedding theorems for Lie algebras [10]. Initially, our embedding theorem into the wreath product was established in terms of produced modules in the case of characteristic zero [7, 12]. Now we essentially use embeddings into special derivations of formal divided power series rings. This approach simplifies arguments and allows us to consider Lie algebras of arbitrary dimension over arbitrary fields; the methods are more close to the works of Blattner [2] and Radford [10]. Finally, remark that similar notions of the wreath product and the embedding theorem can be established for restricted Lie algebras and Lie superalgebras.

2. Dual algebra to enveloping algebra

In this section we observe basic facts on the algebra dual to a universal enveloping algebra. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let L be a Lie algebra with a linearly ordered basis $E = \{v_i \mid i \in I\}$. By the Poincaré-Birkhoff-Witt theorem, the universal enveloping algebra $U(L)$ has a canonical basis

$$U(L) = \left\langle v_{i_1}^{\alpha_{i_1}} \cdots v_{i_n}^{\alpha_{i_n}} \mid i_1 < \cdots < i_n;\ \alpha_i \geq 0,\ n \geq 0 \right\rangle_K. \tag{1}$$

Consider functions $\alpha : I \to \mathbb{N}_0$, whose values we denote by α_i, $i \in I$. Denote by $\mathbb{N}_0^I$ the set of functions with finitely many nonzero values α_i. We write (1) as v^α, where $\alpha \in \mathbb{N}_0^I$. Denote $|\alpha| = \sum_{i \in I} \alpha_i$, $\alpha! = \prod_{i \in I} \alpha_i!$ and $\binom{\alpha}{\beta} = \prod_{i \in I} \binom{\alpha_i}{\beta_i}$ for $\alpha_i, \beta_i \in \mathbb{N}_0^I$. Elements of $\mathbb{N}_0^I$ are added componentwise. Let $\alpha, \beta \in \mathbb{N}_0^I$; then we write $\alpha \preceq \beta$ if and only if $\alpha_i \leq \beta_i$ for all $i \in I$. Let $\alpha \in \mathbb{N}_0^I$; we write $\alpha = 0$ if $\alpha_i = 0$ for all $i \in I$.

The space $U(L)$ is a *Hopf algebra* [15], [8]. The structure of a cocommutative coassociative coalgebra is given by the comultiplication Δ and the counit ε:

$$\Delta : U(L) \to U(L) \otimes U(L), \qquad \Delta(v^\alpha) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} v^\beta \otimes v^\gamma, \tag{2}$$

$$\varepsilon : U(L) \to K, \qquad \varepsilon(v^\alpha) = \begin{cases} 1, & |\alpha| = 0, \\ 0, & |\alpha| > 0. \end{cases} \tag{3}$$

We use Sweedler's notations [15] and write $\Delta a = \sum a_{(1)} \otimes a_{(2)}$, for all $a \in U(L)$.

The structure of a coalgebra on $U(L)$ gives rise to the associated dual algebra $U(L)^* = \mathrm{Hom}_K(U(L), K)$. By $\langle -, - \rangle$ denote the natural pairing $U(L)^* \times U(L) \to K$. Let $f, g \in U(L)^*$; then the product is defined as follows:

$$\langle f \cdot g, a\rangle = \langle f \otimes g, \Delta a\rangle = \sum \langle f, a_{(1)}\rangle \langle g, a_{(2)}\rangle, \quad a \in U(L). \tag{4}$$

Let us consider a partial case; suppose that $I = \{1, 2, \ldots, m\}$ and $\operatorname{char} K = 0$. It is well known that the dual algebra to the coalgebra $U(L)$ is isomorphic to the formal power series ring $U(L)^* \cong K[[X_1, \ldots, X_m]]$. Denote $X^\alpha = X_1^{\alpha_1} \cdots X_m^{\alpha_m}$ for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^I$. Then the isomorphism is given by [3, 2.7.5]:

$$\phi : \; U(L)^* \ni f \longmapsto \sum_{\alpha \in \mathbb{N}_0^I} \langle f, v^\alpha \rangle \frac{X^\alpha}{\alpha!} \in K[[X_1, \ldots, X_m]]. \tag{5}$$

Now we describe the structure of the algebra $U(L)^*$ in case that I and K are arbitrary. Moreover, we consider a more general situation. Suppose that A is a linear algebra over a field K and $*$ is a binary operation in A (say A is a Lie, Jordan, or associative algebra). Consider the vector space $\bar{A} = \mathrm{Hom}_K(U(L), A)$. Let $\langle -, - \rangle$ denote the natural mapping $\bar{A} \times U(L) \to A$. Define the multiplication on $\bar{A}$ as follows. Let $f, g \in \bar{A}$; then

$$\langle f * g, a\rangle = \langle f \otimes g, \Delta a\rangle = \sum \langle f, a_{(1)}\rangle * \langle g, a_{(2)}\rangle, \quad a \in U(L). \tag{6}$$

The structure of the algebra $\bar{A} = \mathrm{Hom}_K(U(L), A)$ can be described in other terms. We introduce *formal divided power series with coefficients* in A in variables X_I as follows. Let $A^O[[X_I]]$ be the set of all formal sums $\sum_{\alpha \in \mathbb{N}_0^I} \lambda_\alpha X^{(\alpha)}$, where $\lambda_\alpha \in A$. Here the elements $X^{(\alpha)}$, $\alpha \in \mathbb{N}_0^I$, are just formal symbols. The letter A indicates that the coefficients belong to A and ^{O} indicates that the series are "divided". By definition, we put

$$\Big(\sum_{\alpha \in \mathbb{N}_0^I} \lambda_\alpha X^{(\alpha)}\Big) * \Big(\sum_{\beta \in \mathbb{N}_0^I} \mu_\beta X^{(\beta)}\Big) = \sum_{\gamma \in \mathbb{N}_0^I} \Big(\sum_{\alpha+\beta=\gamma} \binom{\gamma}{\alpha} \lambda_\alpha * \mu_\beta\Big) X^{(\gamma)}. \tag{7}$$

Since $\gamma \in \mathbb{N}_0^I$ have finitely many nonzero entrees, the multiplication is well-defined. Let A be a Lie algebra; then $A^O[[X_I]]$ is a Lie algebra as well. Indeed, a check of the Jacobi identity is reduced to the following:

$$\begin{aligned}(\lambda_\alpha X^{(\alpha)} * \lambda_\beta X^{(\beta)}) * \lambda_\gamma X^{(\gamma)} + (\lambda_\beta X^{(\beta)} * \lambda_\gamma X^{(\gamma)}) * \lambda_\alpha X^{(\alpha)}& \\ + (\lambda_\gamma X^{(\gamma)} * \lambda_\alpha X^{(\alpha)}) * \lambda_\beta X^{(\beta)}& \\ = \frac{(\alpha+\beta+\gamma)!}{\alpha!\beta!\gamma!}\Big((\lambda_\alpha * \lambda_\beta) * \lambda_\gamma + (\lambda_\beta * \lambda_\gamma) * \lambda_\alpha + (\lambda_\gamma * \lambda_\alpha) * \lambda_\beta\Big) X^{(\alpha+\beta+\gamma)} &= 0.\end{aligned}$$

Similarly, one checks that $A^O[[X_I]]$ satisfies all multilinear identical relations of the algebra A of arbitrary signature. From (6), (7) and (2) it follows that we have the isomorphism

$$\phi:\ \mathrm{Hom}_K(U(L),A)\cong A^O[[X_I]], \tag{8}$$

$$\phi:\ \mathrm{Hom}_K(U(L),A)\ni f\longmapsto \sum_{\alpha\in\mathbb{N}_0^I}\langle f,v^\alpha\rangle X^{(\alpha)}\in A^O[[X_I]]. \tag{9}$$

Suppose that $I=\{1,\dots,m\}$ and $\mathrm{char}K=0$. Let $K[[X_I]]=K[[X_1,\dots,X_m]]$ be the ordinary *power series ring*; it consists of elements $\sum_{\alpha\in\mathbb{N}_0^I}\lambda_\alpha X^\alpha$, where $\lambda_\alpha\in K$ and $X^\alpha=X_1^{\alpha_1}\cdots X_m^{\alpha_m}$ for $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}_0^I$. We have the isomorphism $\psi: K^O[[X_I]]\cong K[[X_I]]$, given by $\psi(\sum_{\alpha\in\mathbb{N}_0^I}\lambda_\alpha X^{(\alpha)})=\sum_{\alpha\in\mathbb{N}_0^I}\lambda_\alpha/\alpha!X^\alpha$. Thus, (8) yields the isomorphism

$$\mathrm{Hom}_K(U(L),A)\cong A\otimes_K K[[X_1,\dots,X_m]] \qquad \text{in case } \mathrm{char}\,K=0,\ \dim_K L=m,\ \dim A<\infty. \tag{10}$$

Let $i\in I$, and consider the tuple $\epsilon(i)\in\mathbb{N}_0^I$ such that $\epsilon(i)_j=\delta_{i,j}$ for $j\in I$. Denote $X_i=X^{(\epsilon(i))}$ for all $i\in I$. Over a field of characteristic zero X_i behaves like a variable in a polynomial ring, because $(X_i)^n=(X^{(\epsilon(i))})^n=n!X^{(n\epsilon(i))}$ for all $n\ge 1$. In particular, if I is finite, then X_i can be identified with the respective variable of the ring $K[[X_1,\dots,X_m]]$. But in case $\mathrm{char}K=p$, we have $(X_i)^p=0$, while $X^{(p^k\epsilon(i))}$, $k\ge 0$, are different nonzero elements; these are just the symbols $X^{(\alpha)}$ for $\alpha=p^k\epsilon(i)\in\mathbb{N}_0^I$.

Consider an arbitrary Lie algebra L; we have $U(L)^*\cong K^O[[X_I]]$. One has the *left regular action* of L on $U(L)$ by left multiplications. The conjugate action $\rho(x)$ on $U(L)^*$ is called the *left coregular action* and is given by

$$\langle\rho(x)f,w\rangle=-\langle f,x\cdot w\rangle,\qquad f\in U(L)^*,\ w\in U(L),\ x\in L.$$

Let A be an arbitrary linear K-algebra. We extend the coregular action onto $\bar A=\mathrm{Hom}_K(U(L),A)$ as follows:

$$\langle z\circ h,u\rangle=-\langle h,zu\rangle,\qquad z\in L,\ h\in\bar A,\ u\in U(L).$$

Let $h\in\mathrm{Hom}_K(U(L),A)$. By (9), we have $h=\sum_{\alpha\in\mathbb{N}_0^I}h_\alpha X^{(\alpha)}$, where $h_\alpha=\langle h,v^\alpha\rangle\in A$. In this record, let us formally identify $X^{(\alpha)}$ with elements of $U(L)^*$ such that $\langle X^{(\alpha)},v^\beta\rangle=\delta_{\alpha,\beta}$ for all $\alpha,\beta\in\mathbb{N}_0^I$. Fix $\beta\in\mathbb{N}_0^I$ and $z\in L$. Denote by T the finite subset of $\mathbb{N}_0^I$ consisting of the tuples that appear in the expansion of zv^β via (1). Then we have

$$\begin{aligned}\langle z\circ h,v^\beta\rangle=-\langle h,zv^\beta\rangle=-\langle\sum_{\alpha\in\mathbb{N}_0^I}h_\alpha X^{(\alpha)},zv^\beta\rangle=-\sum_{\alpha\in T}h_\alpha\langle X^{(\alpha)},zv^\beta\rangle\\ =\sum_{\alpha\in T}h_\alpha\langle\rho(z)X^{(\alpha)},v^\beta\rangle=\sum_{\alpha\in\mathbb{N}_0^I}h_\alpha\langle\rho(z)X^{(\alpha)},v^\beta\rangle,\end{aligned}$$

where $\rho(x)$ denotes the left coregular action on $U(L)^*$. Therefore, we can act on $A^O[[X_I]]$ formally via the left coregular action

$$z\circ\left(\sum_{\alpha\in\mathbb{N}_0^I}h_\alpha X^{(\alpha)}\right)=\sum_{\alpha\in\mathbb{N}_0^I}h_\alpha(\rho(z)X^{(\alpha)}),\quad z\in L. \tag{11}$$

Let $x \in L$. Then $\rho(x)$ is a derivation of $U(L)^*$; see e.g. [3, 2.7.7]. The same arguments prove that the extension of the action onto $\mathrm{Hom}_K(U(L), A)$ consists of derivations. Indeed, let $z \in L$, $f, h \in \mathrm{Hom}_K(U(L), A)$, and $w \in U(L)$. Then

$$\begin{aligned}\langle z \circ (f * h), w\rangle &= \langle f * h, -zw\rangle = \langle f \otimes h, -\Delta(zw)\rangle \\ &= \langle f \otimes h, -(z \otimes 1 + 1 \otimes z) \sum w_{(1)} \otimes w_{(2)}\rangle \\ &= -\sum f(zw_{(1)}) * h(w_{(2)}) - \sum f(w_{(1)}) * h(zw_{(2)}) \\ &= \sum (z \circ f)(w_{(1)}) * h(w_{(2)}) + \sum f(w_{(1)}) * (z \circ h)(w_{(2)}) \\ &= \langle (z \circ f) * h + f * (z \circ h), w\rangle.\end{aligned}$$

Recall that all derivations of the polynomial algebra $K[X_1, \dots, X_n]$ over a field of characteristic zero form an infinite-dimensional simple Lie algebra, called the *Witt* algebra [11]:

$$\mathbf{W}_n(K) = \mathrm{Der}\, K[X_1, \dots, X_n] = \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial X_i} \;\middle|\; f_i \in K[X_1, \dots, X_n] \right\}.$$

In a general situation, we need to study derivations of the formal divided power series ring $K^O[[X_I]]$, where I and K are arbitrary. To study the coregular action, it is sufficient to consider so-called *special derivations* [10]. For any $i \in I$ define the mapping $\partial_{X_i} : K^O[[X_I]] \to K^O[[X_I]]$ by

$$\partial_{X_i}(X^{(\alpha)}) = \begin{cases} X^{(\alpha - \epsilon(i))}, & \alpha_i > 0, \\ 0, & \alpha_i = 0, \end{cases} \qquad \alpha \in \mathbb{N}_0^I.$$

An easy check shows that $\partial_{X_i} \in \mathrm{Der}\, K^O[[X_I]]$ for all $i \in I$. In case (10), we have that $\partial_{X_i} = \frac{\partial}{\partial X_i}$, $i \in I$, are ordinary partial derivations. Consider the set of all formal sums

$$\mathbf{W}(X_I, K) = \left\{ \sum_{\alpha \in \mathbb{N}_0^I} X^{(\alpha)} \sum_{j=1}^{m(\alpha)} \lambda_{\alpha, i_j}\, \partial_{X_{i_j}} \;\middle|\; \lambda_{\alpha, i_j} \in K,\ i_j \in I \right\}.$$

It is essential that the sum is finite at each $X^{(\alpha)}$, $\alpha \in \mathbb{N}_0^I$. We define the Lie product formally:

$$\begin{aligned}&\Big[\sum_{\alpha \in \mathbb{N}_0^I} X^{(\alpha)} \sum_{j=1}^{n(\alpha)} \lambda_{\alpha, i_j}\, \partial_{X_{i_j}}, \sum_{\beta \in \mathbb{N}_0^I} X^{(\beta)} \sum_{k=1}^{m(\beta)} \mu_{\beta, i_k}\, \partial_{X_{i_k}} \Big] \\ &\quad = \sum_{\gamma \in \mathbb{N}_0^I} X^{(\gamma)} \sum_{\alpha + \beta + \epsilon(q) = \gamma} \binom{\alpha + \beta}{\alpha} \\ &\qquad \cdot \Bigg(\sum_{k=1}^{m(\beta + \epsilon(q))} \lambda_{\alpha, q}\, \mu_{\beta + \epsilon(q), i_k}\, \partial_{X_{i_k}} - \sum_{j=1}^{n(\alpha + \epsilon(q))} \lambda_{\alpha + \epsilon(q), i_j}\, \mu_{\beta, q}\, \partial_{X_{i_j}} \Bigg).\end{aligned}$$

The product is well defined because the sums are finite. So, $\mathbf{W}(X_I, K)$ is a Lie algebra. Let A be a linear algebra over K. We define the action of $\mathbf{W}(X_I, K)$ on

$A^O[[X_I]]$ by

$$\Big(\sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)}\sum_{j=1}^{n(\alpha)}\lambda_{\alpha,i_j}\,\partial_{X_{i_j}}\Big)\Big(\sum_{\beta\in\mathbb{N}_0^I} a_\beta X^{(\beta)}\Big)$$
$$=\sum_{\gamma\in\mathbb{N}_0^I}\Big(\sum_{\alpha+\beta+\epsilon(q)=\gamma}\binom{\alpha+\beta}{\alpha}\lambda_{\alpha,q}\,a_{\beta+\epsilon(q)}\Big)X^{(\gamma)},\qquad \lambda_{\alpha,i_j}\in K,\ a_\beta\in A.$$

The action is well defined and yields derivations. The elements of $\mathbf{W}(X_I,K)$ were called *special derivations* of $K^O[[X_I]]$ in [10]; see also the monograph [11]. Thus, $\mathbf{W}(X_I,K)\subseteq \operatorname{Der} K^O[[X_I]]$. Following [11], $\mathbf{W}(X_I,K)$ is called the *complete Cartan Lie algebra of general type* of rank $|I|$. We shall denote it also as $\mathbf{W}(X_I)$.

The fact that the coregular action $\rho: L\to \operatorname{Der} U(L)^*$ yields the embedding into the special derivations $\rho: L\hookrightarrow \mathbf{W}(X_I)$ was proved in [10]. We shall use a more precise formula from [11].

Theorem 1 ([10], [11, Theorem 44.1]). *Let L be a Lie algebra over a field K with a basis $\{v_i \mid i\in I\}$ and $U(L)^*\cong K^O[[X_I]]$. Then the left coregular action $\rho: L\hookrightarrow \operatorname{Der} U(L)^*$ yields the embedding $\rho: L\hookrightarrow \mathbf{W}(X_I)$ such that*

$$\rho(z)=\sum_{i\in I}\big(\rho(z)(X_i)\big)\,\partial_{X_i},\qquad z\in L.$$

3. Main result: Embedding theorem

We recall our definition of the wreath product of Lie algebras. Let H and G be Lie algebras over an arbitrary field K. We have $U(G)^*\cong K^O[[X_I]]$. We define the *base* of the wreath product as $\bar H=\operatorname{Hom}_K(U(G),H)\cong H^O[[X_I]]$; this is a Lie algebra. Also, we have the action by derivations $G\hookrightarrow \operatorname{Der}(H^O[[X_I]])$. We define the *(standard) wreath product* of G and H as follows:

$$H\wr G=\bar H\leftthreetimes G=\operatorname{Hom}_K(U(G),H)\leftthreetimes G=H^O[[X_I]]\leftthreetimes G.$$

Let us also define a *general wreath product*. Consider Lie algebras H and G, and assume that $U(G)^*\cong K^O[[X_I]]$. Suppose that we are given a homomorphism $\phi: G\to \mathbf{W}(X_I)$. It is extended to the homomorphism $\tilde\rho: G\to \operatorname{Der}(H^O[[X_I]])$. Then we define

$$H\wr_\phi G=H^O[[X_I]]\leftthreetimes G.$$

Now consider an extension of Lie algebras $0\to H\to L\to G\to 0$ over K. Let $\{b_j\mid j\in J\}$ be a basis of H and $\{a_i\mid i\in I\}$ a basis that complements H in L. We fix a linear order on these bases so that $a_i<b_j$. By the PBW theorem, we have the basis

$$U(L)=\langle a_{i_1}^{\alpha_1}\cdots a_{i_n}^{\alpha_n}b_{j_1}^{\beta_1}\cdots b_{j_m}^{\beta_m}\mid i_1<\cdots<i_n;\ j_1<\cdots<j_m;\ \alpha_i,\beta_j\geq 0,\ i\in I, j\in J\rangle_K. \tag{12}$$

For short write these monomials as $a^\alpha b^\beta$, where $\alpha\in\mathbb{N}_0^I$, $\beta\in\mathbb{N}_0^J$.

Now we define an important notation. By $\Psi: U(L)\to H=\langle b_j\mid j\in J\rangle_K$ denote the projection such that $\Psi(b_j)=b_j$, $j\in J$, where the other basis monomials (12) are mapped into zero. For each $\alpha\in\mathbb{N}_0^I$, we define the linear mapping

$$\chi_\alpha: L\to H;\qquad \chi_\alpha(z)=\Psi(z\cdot a^\alpha),\quad z\in L.$$

Theorem 2. *Suppose that we are given an extension of Lie algebras over a field K*

$$0 \to H \to L \xrightarrow{\theta} G \to 0.$$

Suppose that $U(G)^* \cong K^O[[X_I]]$. Let $H \wr G = \bar{H} \leftthreetimes G$ be the standard wreath product, where $\bar{H} = H^O[[X_I]]$ is the formal divided power series ring with coefficients in H. Then

(1) *there exists an embedding $\sigma : L \hookrightarrow H \wr G$;*
(2) *it is given by*

$$\sigma(z) = \Big(\sum_{\alpha \in \mathbb{N}_0^I} \chi_\alpha(z)\, X^{(\alpha)},\ \theta(z) \Big), \quad z \in L;$$

(3) *$(\sigma(L) + \bar{H})/\bar{H} \cong G$;*
(4) *for any $\alpha \in \mathbb{N}_0^I$ let $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ denote all nonzero values, where $i_1 < \cdots < i_k$. Then $\sigma(H) \subset \bar{H}$ and*

$$\sigma(h) = \Big(\sum_{\alpha \in \mathbb{N}_0^I} [\ldots[h, a_{i_1}^{\alpha_{i_1}}], \ldots, a_{i_k}^{\alpha_{i_k}}]\, X^{(\alpha)},\ 0 \Big), \quad h \in H.$$

We need the following technical but crucial lemma.

Lemma 1. *Let $z \in L$. Then for any $\alpha \in \mathbb{N}_0^I$*

$$z \cdot a^\alpha = \chi_\alpha(z) + \sum_{\gamma \preceq \alpha, |\gamma| > 0} a^\gamma h(z, \alpha, \gamma) + \sum_{|\delta| > 0} \lambda(z, \delta) a^\delta,$$

where $h(z, \alpha, \gamma) \in H$ and $\lambda(z, \delta) \in K$.

Proof. Consider an arbitrary Lie algebra L with a linearly ordered basis $E = \{v_i | i \in I\}$. The basis monomials (1) are called *straightened*. Suppose that $u = v_{i_1} \cdots v_{i_n}$ is straightened. Then we define a *monomial with singularity* $w \in E$ on k-th place by putting w inside u:

$$u \dashv_k w = v_{i_1} \cdots v_{i_{k-1}} \cdot w \cdot v_{i_k} \cdots v_{i_n}. \tag{13}$$

The monomial u is called the *straightened part* of $u \dashv_k w$. Suppose that $u = v^\alpha$, $\alpha \in \mathbb{N}_0^I$. Then α is called the *tuple* of u. Also, α is referred to as the *tuple* of $u \dashv_k w$.

Let us take a monomial with singularity (13) and apply the standard algorithm that expresses all monomials in E via the straightened monomials. Suppose that (13) is not straightened. We have either 1) $v_{i_{k-1}} > w$, or 2) $w > v_{i_k}$. Let us consider the second case; the first case is treated in the same way. Suppose that $[w, v_{i_k}] = \sum_{j \in I} \lambda_j v_j$, where $v_j \in E$, $\lambda_j \in K$. We present (13) as

$$\begin{aligned} u \dashv_k w &= v_{i_1} \cdots v_{i_k} w v_{i_{k+1}} \cdots v_{i_n} + \sum_{j \in I} \lambda_j v_{i_1} \cdots v_{i_{k-1}} v_j v_{i_{k+1}} \cdots v_{i_n} \\ &= u \dashv_{k+1} w + \sum_{j \in I} \lambda_j (v_{i_1} \cdots v_{i_{k-1}} \cdot v_{i_{k+1}} \cdots v_{i_n}) \dashv_k v_j. \end{aligned}$$

We obtained a sum of monomials with singularity. The first term has the same tuple. The other monomials have one letter erased, i.e. one of α_i is decreased by one in comparison with the previous tuple. In other words, we obtain tuples that are less with respect to $\prec$. There are two possibilities for each of these monomials with singularity. a) the monomial is straightened, and we stop the process by "forgetting" about the singularity. Remark that at this moment the erased singularity increases one of α_i in the tuple by one. b) the monomial is not straightened, and we continue the process.

Now we return to our basis $E = \{a_i | i \in I\} \cup \{b_j | j \in J\}$. The basis of $U(L)$ consists of straightened monomials $\{a^\alpha b^\beta \mid \alpha \in \mathbb{N}_0^I,\ \beta \in \mathbb{N}_0^J\}$. Without loss of generality we suppose that $z \in E$. We consider $z \cdot a^\alpha = a^\alpha \dashv_1 z$ as the monomial with singularity on the first place. Remark that $a^\alpha \dashv_1 z$ has the tuple $(\alpha_i, \beta_j \mid i \in I;\ \beta_j = 0,\ j \in J)$. Except for the last step, we obtain monomials with singularity and tuples $(\tilde\alpha_i, \tilde\beta_j \mid \tilde\alpha_i \le \alpha_i,\ i \in I;\ \tilde\beta_j = 0,\ j \in J)$.

At the last step, while we erase a singularity, one of these components increases by one. There are two cases. 1) The component for some $i \in I$ is increased. Then the resulting straightened monomial is of type $a^{\tilde\alpha}$, where $\tilde\alpha \ne 0$. Such monomials yield the last sum in the claimed formula. 2) The component for some $j \in J$ is increased. Then we obtain the monomial $a^{\tilde\alpha} b_j$, where $\tilde\alpha \preceq \alpha$ and $b_j \in E$. Such monomials with $\tilde\alpha = 0$ yield $\chi_\alpha(z)$, while the monomials with $|\tilde\alpha| > 0$ are collected into the first sum. The lemma is proved. □

Proof of Theorem 2. The idea is as follows. We consider the left coregular action $\rho : L \hookrightarrow \operatorname{Der} U(L)^*$ and construct a subalgebra $H_0 \oplus \tilde G \subset \operatorname{Der} U(L)^*$ isomorphic to the wreath product $H \wr G$ and such that $\rho(L) \subset H_0 \oplus \tilde G$.

We use basis (12) for $U(L)$. Let $\alpha \in \mathbb{N}_0^I$, $\beta \in \mathbb{N}_0^J$. By $X^{(\alpha)} Y^{(\beta)} \in U(L)^*$ denote the linear function such that $\langle X^{(\alpha)} Y^{(\beta)}, a^{\alpha_0} b^{\beta_0}\rangle = \delta_{\alpha,\alpha_0}\delta_{\beta,\beta_0}$ for all $\alpha_0 \in \mathbb{N}_0^I$, $\beta_0 \in \mathbb{N}_0^J$. Let $\mathbb{N}_0^J \ni \beta = 0$. Then we get elements denoted by $X^{(\alpha)} \in U(L)^*$, where $\alpha \in \mathbb{N}_0^I$, similarly we get $Y^{(\beta)} \in U(L)^*$. We present all elements of $U(L)^*$ as infinite sums

$$f = \sum_{\alpha \in \mathbb{N}_0^I, \beta \in \mathbb{N}_0^J} \langle f, a^\alpha b^\beta \rangle X^{(\alpha)} Y^{(\beta)}, \quad f \in U(L)^*; \tag{14}$$

these elements are multiplied similar to (7). Thus, we identify $U(L)^*$ with the formal divided power series ring. Let $K^O[[X_I, Y_J]]$ denote this ring and $\mathbf{W}(X_I \cup Y_J)$ denote its Lie algebra of special derivations. By Theorem 1, we have the embedding $\rho : L \hookrightarrow \mathbf{W}(X_I \cup Y_J)$.

The mapping $\theta : L \to G$ yields the natural epimorphism $\pi : U(L) \to U(G)$ of coalgebras and L-modules, where L acts via the left regular action. Namely, we have $\pi(z \cdot v) = \theta(z)\pi(v)$ for all $z \in L$, $v \in U(L)$. Similar to (14), we identify $U(G)^* = K^O[[\bar X_I]]$ so that $\langle \bar X^{(\alpha)}, \pi(a^{\alpha_0})\rangle = \delta_{\alpha,\alpha_0}$ for $\alpha, \alpha_0 \in \mathbb{N}_0^I$. We have the respective embedding $\rho_1 : G \hookrightarrow \mathbf{W}(\bar X_I)$. The dual to π is a monomorphism π^* of commutative rings and L-modules

$$\begin{gathered} \pi^* : U(G)^* = K^O[[\bar X_I]] \ \hookrightarrow\ U(L)^* = K^O[[X_I, Y_J]], \\ \rho(z)(\pi^* g) = \pi^*(\rho_1(\theta(z))g), \qquad g \in K^O[[\bar X_I]],\ z \in L. \end{gathered} \tag{15}$$

We compare values of linear functions at bases of monomials and conclude that $\pi^*(\bar X^{(\alpha)}) = X^{(\alpha)}$ for $\alpha \in \mathbb{N}_0^I$. In particular, $\pi^*(\bar X_i) = X_i$, $i \in I$. We use (15)

$$\rho(z) X_i = \pi^*(\rho_1(\theta(z))\bar X_i), \qquad i \in I,\ z \in L. \tag{16}$$

Also we identify $U(H)^* = K[[\bar{Y}_J]]$ using the basis $U(H) = \langle b^\beta | \beta \in \mathbb{N}_0^J \rangle$. Consider the respective embedding $\rho_2 : H \hookrightarrow \mathbf{W}(\bar{Y}_J)$. We have the embedding $\iota : U(H) \hookrightarrow U(L)$ of coalgebras and left H-modules. The conjugate mapping ι^* is an epimorphism of H-modules:

$$\iota^* : U(L)^* = K^O[[X_I, Y_J]] \to U(H)^* = K^O[[\bar{Y}_J]],$$
$$\iota^*(\rho(h)f) = \rho_2(h)(\iota^* f), \qquad f \in K^O[[X_I, Y_J]],\ h \in H.$$

It is described as follows. Let $f \in U(L)^*$; then $\iota^* f = f|_{U(H)} \in U(H)^*$. Moreover, $\iota^*(X^{(\alpha)} Y^{(\beta)}) = \delta_{\alpha,0} \bar{Y}^{(\beta)}$ for $\alpha \in \mathbb{N}_0^I$, $\beta \in \mathbb{N}_0^J$. In particular, $\iota^*(Y_j) = \bar{Y}_j$, $j \in J$. We also have the natural embedding of algebras $\nu : K^O[[\bar{Y}_J]] \hookrightarrow K^O[[X_I, Y_J]]$ (but this is not a homomorphism of H-modules!).

Let us extend the linear functions in H^* to the whole of $U(L)$ by zero values at the basis monomials (12) distinct from $\{b_j | j \in J\}$. Denote the set of all such functions by H'. We have $U(L)^* = K^O[[X_I, Y_J]] \supset H' = \{\sum_{j \in J} \mu_j Y_j \mid \mu_j \in K\}$. Fix $f \in H' \subset U(L)^*$. Let us compute the action of $z \in L$ on it. We apply Lemma 1

$$\begin{aligned}(17)\quad \langle \rho(z) f, a^\alpha b^\beta \rangle &= -\langle f, z \cdot a^\alpha b^\beta \rangle \\ &= -\left\langle f, \left(\chi_\alpha(z) + \sum_{\gamma \preceq \alpha, |\gamma| > 0} a^\gamma h(z, \alpha, \gamma) + \sum_{|\delta| > 0} \lambda(z, \delta) a^\delta \right) b^\beta \right\rangle \\ &= -\langle f, \chi_\alpha(z) \cdot b^\beta \rangle,\end{aligned}$$

because the elements in both sums yield the basis monomials of type $a^\epsilon b^\delta$, where $|\epsilon| > 0$. By construction, such monomials belong to the kernel of f. Since $\chi_\alpha(z) \in H$ and $b^\beta \in U(H)$, we continue with

$$(18)\qquad -\langle f, \chi_\alpha(z) \cdot b^\beta \rangle = -\langle \iota^* f, \chi_\alpha(z) \cdot b^\beta \rangle = \langle \rho_2(\chi_\alpha(z))(\iota^* f), b^\beta \rangle.$$

Combining (17) and (18), we obtain

$$\langle \rho(z) f, a^\alpha b^\beta \rangle = \langle \rho_2(\chi_\alpha(z))(\iota^* f), b^\beta \rangle, \quad \alpha \in \mathbb{N}_0^I, \beta \in \mathbb{N}_0^J.$$

We apply this formula and (14)

$$\begin{aligned}\rho(z) f &= \sum_{\alpha \in \mathbb{N}_0^I, \beta \in \mathbb{N}_0^J} \langle \rho(z) f, a^\alpha b^\beta \rangle X^{(\alpha)} Y^{(\beta)} \\ &= \sum_{\alpha \in \mathbb{N}_0^I} X^{(\alpha)} \sum_{\beta \in \mathbb{N}_0^J} \langle \rho_2(\chi_\alpha(z))(\iota^* f), b^\beta \rangle Y^{(\beta)} \\ &= \sum_{\alpha \in \mathbb{N}_0^I} X^{(\alpha)} \sum_{\beta \in \mathbb{N}_0^J} \langle \rho_2(\chi_\alpha(z))(\iota^* f), b^\beta \rangle \nu(\bar{Y}^{(\beta)}) \\ &= \sum_{\alpha \in \mathbb{N}_0^I} X^{(\alpha)} \nu\Big(\sum_{\beta \in \mathbb{N}_0^J} \langle \rho_2(\chi_\alpha(z))(\iota^* f), b^\beta \rangle \bar{Y}^{(\beta)} \Big) \\ &= \sum_{\alpha \in \mathbb{N}_0^I} X^{(\alpha)} \nu\Big(\rho_2(\chi_\alpha(z))(\iota^* f) \Big).\end{aligned}$$

We apply this formula to $Y_j \in H'$, $j \in J$:

$$\rho(z)Y_j = \sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)}\nu\Big(\rho_2(\chi_\alpha(z))\bar{Y}_j\Big), \qquad j\in J,\ z\in L. \tag{19}$$

The embedding $\pi^* : K^O[[\bar{X}_I]] \hookrightarrow K^O[[X_I, Y_J]]$ yields the embedding which we shall denote by the same letter $\pi^* : \mathbf{W}(\bar{X}_I) \hookrightarrow \mathbf{W}(X_I \cup Y_J)$ by setting

$$\pi^*\bigg(\sum_{i\in I} f_i \partial_{\bar{X}_i}\bigg) = \sum_{i\in I} \pi^*(f_i)\partial_{X_i}, \quad f_i \in K^O[[\bar{X}_I]].$$

Similarly, we get $\nu : \mathbf{W}(\bar{Y}_J) \hookrightarrow \mathbf{W}(X_I \cup Y_J)$. Consider an arbitrary $z \in L$; we apply Theorem 1, (16), and (19). The derivation $\rho(z) \in \mathbf{W}(X_I \cup Y_J)$ is presented as

$$\begin{aligned}\rho(z) &= \sum_{i\in I}(\rho(z)X_i)\,\partial_{X_i} + \sum_{j\in J}(\rho(z)Y_j)\,\partial_{Y_j}\\ &= \sum_{i\in I}(\pi^*(\rho_1(\theta(z))\bar{X}_i))\,\partial_{X_i} + \sum_{j\in J}\sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)}\nu(\rho_2(\chi_\alpha(z))\bar{Y}_j)\,\partial_{Y_j}\\ &= \pi^*\bigg(\sum_{i\in I}(\rho_1(\theta(z))\bar{X}_i)\,\partial_{\bar{X}_i}\bigg) + \sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)}\nu\bigg(\sum_{j\in J}(\rho_2(\chi_\alpha(z))\bar{Y}_j)\,\partial_{\bar{Y}_j}\bigg)\\ &= \pi^*(\rho_1(\theta(z))) + \sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)}\nu(\rho_2(\chi_\alpha(z))), \quad z\in L.\end{aligned} \tag{20}$$

Let $\mathbf{W}(X_I)$ and $\mathbf{W}(Y_J)$ be the subalgebras of $\mathbf{W}(X_I \cup Y_J)$ written in symbols X_I and Y_J, respectively. Denote by $\tilde{G}$ and $\tilde{H}$ the images of the embeddings $\pi^*\rho_1 : G \hookrightarrow \mathbf{W}(X_I)$ and $\nu\rho_2 : H \hookrightarrow \mathbf{W}(Y_J)$.

Recall the structure of the base of the wreath product

$$\bar{H} = \mathrm{Hom}_K(U(G), H) = H^O[[\bar{X}_I]] = \{\sum_{\alpha\in\mathbb{N}_0^I} h_\alpha \bar{X}^{(\alpha)} \mid h_\alpha \in H\}.$$

Consider $H_0 = \{\sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)} h_\alpha \mid h_\alpha \in \tilde{H}\} \subset \mathbf{W}(X_I \cup Y_J)$ and define $\phi : \bar{H} \to H_0$ by

$$\phi(\sum_{\alpha\in\mathbb{N}_0^I} h_\alpha \bar{X}^{(\alpha)}) = \sum_{\alpha\in\mathbb{N}_0^I} \pi^*(\bar{X}^{(\alpha)})\,\nu\rho_2(h_\alpha) = \sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)}\,\nu\rho_2(h_\alpha), \quad h_\alpha \in H. \tag{21}$$

Since $\nu\rho_2(h_\alpha) \in \tilde{H} \subset \mathbf{W}(Y_J)$, these elements commute with $X^{(\alpha)}$. Hence, we have an isomorphism of Lie algebras $\phi : \bar{H} \cong H_0$. Consider the wreath product $H \wr G = \bar{H} \rightthreetimes G$ and extend ϕ to the mapping $\phi : H \wr G \to \mathbf{W}(X_I \cup Y_J)$ by $\phi(g) = \pi^*\rho_1(g)$ for $g \in G$. We obtain the isomorphism of vector spaces $\phi : H \wr G \cong H_0 \oplus \tilde{G}$. The next computation proves that this is an isomorphism of Lie algebras. Let $g \in G$

and $h = \sum_{\alpha\in\mathbb{N}_0^I} h_\alpha \bar{X}^{(\alpha)} \in \bar{H}$, where $h_\alpha \in H$. We use (11) and (21) to obtain

$$\begin{aligned}\phi([g,h]) &= \phi\Bigg(g \circ \sum_{\alpha\in\mathbb{N}_0^I} h_\alpha \bar{X}^{(\alpha)}\Bigg) = \phi\Bigg(\sum_{\alpha\in\mathbb{N}_0^I} h_\alpha \left(\rho_1(g)\bar{X}^{(\alpha)}\right)\Bigg)\\ &= \sum_{\alpha\in\mathbb{N}_0^I} (\pi^*(\rho_1(g)\bar{X}^{(\alpha)}))\,\nu\rho_2(h_\alpha) = \sum_{\alpha\in\mathbb{N}_0^I} (\pi^*(\rho_1(g))X^{(\alpha)})\,\nu\rho_2(h_\alpha)\\ &= [\pi^*\rho_1(g), \sum_{\alpha\in\mathbb{N}_0^I} X^{(\alpha)}\,\nu\rho_2(h_\alpha)] = [\phi(g),\phi(h)],\end{aligned}$$

because $\pi^*\rho_1(g) \in \mathbf{W}(X_I)$ and $\nu\rho_2(h_\alpha) \in \mathbf{W}(Y_J)$. Now, (20) yields $\rho : L \hookrightarrow H_0 \oplus \tilde{G}$. We obtain the claimed embedding $\sigma = \phi^{-1}\rho : L \hookrightarrow H \wr G$; it is given by the formula

$$\sigma(z) = \Bigg(\sum_{\alpha\in\mathbb{N}_0^I} \chi_\alpha(z)\,\bar{X}^{(\alpha)},\ \theta(z)\Bigg), \quad z \in L.$$

Now let us compute σ on arbitrary $h \in H$. Let $\alpha \in \mathbb{N}_0^I$ have nonzero entrees $(\alpha_{i_1},\dots,\alpha_{i_k})$, where $i_1 < \cdots < i_k$. We use basis (12). One proves the following commutation relation by induction on $|\alpha|$ (see e.g. [3, 2.2.22]):

$$h \cdot a^\alpha = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} a^\beta \cdot [\cdots[[h, a_{i_1}^{\gamma_{i_1}}], a_{i_2}^{\gamma_{i_2}}],\dots, a_{i_k}^{\gamma_{i_k}}], \qquad \alpha \in \mathbb{N}_0^I.$$

The term for $\gamma = \alpha$ yields $\chi_\alpha(h) = [\cdots[h, a_{i_1}^{\alpha_{i_1}}],\dots, a_{i_k}^{\alpha_{i_k}}]$, and the last claim follows. □

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