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# HARMONIC HOMEOMORPHISMS OF THE CLOSED DISC TO ITSELF NEED BE IN $W^{1,p}$, $p < 2$, BUT NOT $W^{1,2}$

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Abstract. Harmonic maps $u$ from the closed disc onto bounded convex sets of the plane obey $u \in W^{1,p}, p < 2$.

The (complex-valued) Poisson extension $\mathcal{P}F$ of a homeomorphism $F$ of the circle $\mathbb{T}$ onto the boundary of a bounded convex domain $\Omega \subset \mathbb{C}$ extends $F$ to a homeomorphism of the closed disc $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. This theorem was first proved by Hellmuth Kneser [Kne26]. See Chapter 3 of [Dur04].

**Theorem 0.1** (H. Kneser)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded convex domain. Let $F : \mathbb{T} \to \partial\Omega$ be a homeomorphism. Then*

$$u(z) = \mathcal{P}F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} F(e^{it})dt\ ,\ z \in \mathbb{D},$$

*is a homeomorphism of $\mathbb{D}$ onto $\Omega$.*

The question of Sobolev regularity in $\mathbb{D}$ for such harmonic homeomorphisms has recently arisen [AIMO]. There the authors study homeomorphisms of $\Omega$ onto $\mathbb{D}$ with integrable distortion function. The problem of determining when a given homeomorphism between boundaries has an extension to a homeomorphism with integrable distortion is shown to be equivalent to asking a question about the inverse mapping: when does a homeomorphism of the boundaries have an extension with finite Dirichlet integral? Here we prove

**Theorem 0.2.** *With $\Omega, F$ and $u$ as in Kneser's Theorem*

(i) *$u \in W^{1,p}(\mathbb{D})$ when $p < 2$*

*and*

(ii) *there exist $F : \mathbb{T} \to \mathbb{T}$ so that $u \notin W^{1,2}(\mathbb{D})$.*

Here $W^{1,p}(\mathbb{D})$ is the Sobolev space of $L^p(\mathbb{D})$ functions $u$ that have distributional derivatives $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \in L^p(\mathbb{D})$ and norm $\|u\|_{1,p} = \left(\int_{\mathbb{D}} |u|^p dxdy\right)^{\frac{1}{p}} + \left(\int_{\mathbb{D}} |\nabla u|^p dxdy\right)^{\frac{1}{p}}$, $1 \le p < \infty$. So (ii) implies that there exists a homeomorphism of $\partial\Omega$ onto $\mathbb{T}$ which does not admit an extension with integrable distortion [AIMO].

For $p = 2$ Friedrich Prym [Pry71] showed that the Poisson extension of a continuous function on $\mathbb{T}$ could fail to have finite Dirichlet integral in $\mathbb{D}$, i.e. fail to be

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in $W^{1,2}(\mathbb{D})$. By a recent account [MP97], p. 259 (see also [Die81], p. 38), Prym's result was ignored. Hadamard [Had06] later exhibited a lacunary series example which can be found in [Cou50], pp. 9-10 (see also [Joh82], p. 125). These examples, together with the example below, cannot arise by conformal mapping to a bounded domain since then the Jacobian would equal $\frac{1}{2}|\nabla u|^2$.

That the Dirichlet principle in its 19th century formulation (see [BB98], pp. 82-83, 85) fails for a homeomorphism of the boundary, follows here by using later technology. By Gagliardo's Theorem for $1 < p < \infty$ (see [MP97], pp. 207-210 where $\mathbb{T}$ is replaced by a Lipschitz boundary; [Ada75], pp. 204-216 for a proof), the *trace* $F = Tu$ on the boundary of a function $u \in W^{1,p}(\mathbb{D})$ necessarily satisfies

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{F(e^{is})-F(e^{it})}{s-t}\right|^p dsdt < \infty. \tag{0.1}$$

Here the trace operator is the bounded linear operator

$$T : W^{1,p}(\mathbb{D}) \to L^p(\mathbb{T})\ ,\ 1 \le p < \infty,$$

such that $Tu = u$ on $\mathbb{T}$ whenever $u \in W^{1,p}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$. See p. 133 of [EG92] for a proof of its existence. Consequently in order to prove part (ii) of Theorem 0.2 it suffices to exhibit a homeomorphism $F$ for which (0.1) is infinite when $p = 2$.

In what follows, any homeomorphism $F$ from Kneser's Theorem is considered to be of the form $F(e^{it}) = r(t)e^{if(t)}$, $-\infty < t < \infty$, where $r(t)$ is a positive $2\pi$-periodic continuous function and $f(t)$ is a strictly increasing continuous function satisfying $f(t + 2n\pi) = f(t) + 2n\pi$. Thus the convex domain is situated (without loss of generality) to be starlike with respect to the origin.

*Proof.* (Part (ii) of Theorem 0.2) Consider the strictly increasing functions $f(t) = f_q(t) = \operatorname{sgn}(t)|\log|t|\,|^{1-q}$ on the interval $-\frac{1}{2} \le t \le \frac{1}{2}$ when $1 < q$. Then

$$\int_{-\frac{1}{2}}^{0}\int_0^{\frac{1}{2}}\left(\frac{f(s)-f(t)}{s-t}\right)^2 dsdt > \int_{-\frac{1}{2}}^{0}\int_0^{\frac{1}{2}}\left(\frac{f(s)}{s-t}\right)^2 dsdt = \int_0^{\frac{1}{2}}\frac{(f(s))^2}{s(2s+1)}ds.$$

By $2s + 1 \le 2$ and substituting $f_q$, the last integral is greater than

$$\frac{1}{2}\int_0^{\frac{1}{2}}\frac{|\log|s|\,|^{2-2q}}{s}ds,$$

which diverges for $q \le \frac{3}{2}$. For this restriction of $q$,

$$\left|\frac{e^{if(s)}-e^{if(t)}}{f(s)-f(t)}\right| \ge \frac{1}{\pi} \quad \text{for all } s,t \in [-\frac{1}{2},\frac{1}{2}].$$

Each $f(t)$ extends to a homeomorphism $[-\pi,\pi] \to [-\pi,\pi]$ and the above inequalities then show that (0.1) diverges for $p = 2$ and $F(e^{it}) = e^{if(t)}$. □

*Remark* 0.3. The traditional Dirichlet principle has been shown to fail for an absolutely continuous homeomorphism of the boundary.

It is also a classical result (see Proposition 7′ and Lemma 4′ [Ste70], pp. 151-152, but change the sign on $\alpha - 1$; see [Pee76], pp. 8-10, items a. and f.) that harmonic functions in $\mathbb{D}$ will have the $p$-norm of their gradients bounded by (0.1). It remains to show that every homeomorphism $F = re^{if}$ satisfies the inequality (0.1) when $p < 2$.

*Proof.* (Part (i) of Theorem 0.2) Every bounded convex domain is also a Lipschitz domain. Because $\Omega$ is also starlike with respect to the origin it follows for any two points on the boundary that the ratio formed by a change in the distance to the origin and a change in the argument is uniformly bounded. I.e. by the setup and definition of $F$ there is an $M < \infty$ so that $r \circ f^{-1}$ satisfies a Lipschitz condition $\left|\frac{r(s)-r(t)}{f(s)-f(t)}\right| \leq M$ for all $s \neq t$. Consequently

$$\left|\frac{r(s)e^{if(s)} - r(t)e^{if(t)}}{s-t}\right| \leq (M + \|r\|_\infty)\left(\frac{f(s)-f(t)}{s-t}\right).$$

Take $p > 1$. Assume for the moment that $f$ is absolutely continuous and write $g(s,t) = (\frac{f(s)-f(t)}{s-t})^p$. Integrating by parts as

$$\int g(s,t)ds = (s-t)g(s,t) - \int (s-t)\frac{\partial}{\partial s}g(s,t)ds$$

and evaluating, using the monotonicity of $f$ and $p > 1$, yields an intermediate step in Hardy's inequality (see p. 242 of [HLP52], for example)

$$\int_0^{2\pi}\left(\frac{f(s)-f(t)}{s-t}\right)^p ds \leq \frac{p}{p-1}\int_0^{2\pi}\left(\frac{f(s)-f(t)}{s-t}\right)^{p-1} f'(s)ds.$$

The last integrand is bounded by $(2\pi)^{p-1}|t-s|^{1-p}f'(s)$. Integrating the left and right-hand sides in $t$ from 0 to $2\pi$, applying Fubini with $p < 2$ and then the fundamental theorem of calculus in $s$ produces the bound

$$\int\int(\frac{f(s)-f(t)}{s-t})^p dsdt \leq \frac{p\,2^{p+1}\pi^2}{(p-1)(2-p)}$$

independent of $f$. When $f$ is not absolutely continuous it can be mollified by using a smooth and symmetric mollifier so that monotonicity and periodicity are retained. By the uniformity of the a priori bounds, positivity of the difference quotients and pointwise convergence of the mollifications of $f$ to $f$, Fatou's lemma yields the same bounds on the double integral for any such increasing continuous $f$. In general

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{F(e^{is})-F(e^{it})}{s-t}\right|^p dsdt \leq (M + \|r\|_\infty)^p\,\frac{p\,2^{p+1}\pi^2}{(p-1)(2-p)}\ ,\ 1 < p < 2,$$

which proves (i) for $1 \leq p < 2$. □

*Remark* 0.4. The above proof required $F$ to be a homeomorphism and $r \circ f^{-1}$ to satisfy a Lipschitz condition. Thus the conclusion remains true if $\Omega$ is replaced by a starlike Lipschitz domain. But then $u$ would not be a homeomorphism in general.

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