HARMONIC HOMEOMORPHISMS OF THE CLOSED DISC
to itself need be in $W^{1,p}$, $p < 2$, but not $W^{1,2}$

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Abstract. Harmonic maps $u$ from the closed disc onto bounded convex sets of the plane obey $u \in W^{1,p}$, $p < 2$.

The (complex-valued) Poisson extension $\mathcal{P} F$ of a homeomorphism $F$ of the circle $\mathbb{T}$ onto the boundary of a bounded convex domain $\Omega \subset \mathbb{C}$ extends $F$ to a homeomorphism of the closed disc $\mathbb{D}$ onto $\Omega$. This theorem was first proved by Hellmuth Kneser [Kne26]. See Chapter 3 of [Dur04].

Theorem 0.1 (H. Kneser). Let $\Omega \subset \mathbb{C}$ be a bounded convex domain. Let $F : \mathbb{T} \to \partial \Omega$ be a homeomorphism. Then

$$u(z) = \mathcal{P} F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} F(e^{it}) dt , \quad z \in \mathbb{D},$$

is a homeomorphism of $\mathbb{D}$ onto $\Omega$.

The question of Sobolev regularity in $\mathbb{D}$ for such harmonic homeomorphisms has recently arisen [AIMO]. There the authors study homeomorphisms of $\Omega$ onto $\mathbb{D}$ with integrable distortion function. The problem of determining when a given homeomorphism between boundaries has an extension to a homeomorphism with integrable distortion is shown to be equivalent to asking a question about the inverse mapping: when does a homeomorphism of the boundaries have an extension with finite Dirichlet integral? Here we prove

Theorem 0.2. With $\Omega, F$ and $u$ as in Kneser’s Theorem

(i) $u \in W^{1,p}(\mathbb{D})$ when $p < 2$

and

(ii) there exist $F : \mathbb{T} \to \mathbb{T}$ so that $u \notin W^{1,2}(\mathbb{D})$.

Here $W^{1,p}(\mathbb{D})$ is the Sobolev space of $L^p(\mathbb{D})$ functions $u$ that have distributional derivatives $\nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \in L^p(\mathbb{D})$ and norm $\|u\|_{1,p} = \left( \int_\mathbb{D} |u|^p dx dy \right)^{\frac{1}{p}} + \left( \int_\mathbb{D} |\nabla u|^p dx dy \right)^{\frac{1}{p}}$, $1 \leq p < \infty$. So (ii) implies that there exists a homeomorphism of $\partial \Omega$ onto $\mathbb{T}$ which does not admit an extension with integrable distortion [AIMO].

For $p = 2$ Friedrich Prym [Pry71] showed that the Poisson extension of a continuous function on $\mathbb{T}$ could fail to have finite Dirichlet integral in $\mathbb{D}$, i.e. fail to be
in \(W^{1,2}(\mathbb{D})\). By a recent account \cite{MP97}, p. 259 (see also \cite{Die81}, p. 38), Prym’s result was ignored. Hadamard \cite{Had06} later exhibited a lacunary series example which can be found in \cite{Con50}, pp. 9-10 (see also \cite{Ioh82}, p. 125). These examples, together with the example below, cannot arise by conformal mapping to a bounded domain since then the Jacobian would equal \(\frac{1}{2}|\nabla u|^2\).

That the Dirichlet principle in its 19th century formulation (see \cite{BB98}, pp. 82-83, 85) fails for a homeomorphism of the boundary, follows here by using later technology. By Gagliardo’s Theorem for \(1 < p < \infty\) follows here by using later technology. By Gagliardo’s Theorem for \(1 < p < \infty\) (see \cite{MP97}, pp. 207-210 where \(\mathcal{T}\) is replaced by a Lipschitz boundary; \cite{Ada75}, pp. 194-196 for a proof), the trace \(F = Tu\) on the boundary of a function \(u \in W^{1,p}(\mathbb{D})\) necessarily satisfies

\[
(0.1) \quad \int_0^{2\pi} \int_0^{2\pi} \frac{|F(e^{is}) - F(e^{it})|^p}{s-t} \, dsdt < \infty.
\]

Here the trace operator is the bounded linear operator

\[T : W^{1,p}(\mathbb{D}) \to L^p(\mathcal{T}), \ 1 \leq p < \infty,\]

such that \(Tu = u\) on \(\mathcal{T}\) whenever \(u \in W^{1,p}(\mathbb{D}) \cap C(\mathbb{D})\). See p. 133 of \cite{EG92} for a proof of its existence. Consequently in order to prove part (ii) of Theorem 0.2 it suffices to exhibit a homeomorphism \(F\) for which \((0.1)\) is infinite when \(p = 2\).

In what follows, any homeomorphism \(F\) from Kneser’s Theorem is considered to be of the form \(F(e^{it}) = r(t)e^{if(t)}, -\infty < t < \infty\), where \(r(t)\) is a positive \(2\pi\)-periodic continuous function and \(f(t)\) is a strictly increasing continuous function satisfying \(f(t + 2n\pi) = f(t) + 2n\pi\). Thus the convex domain is situated (without loss of generality) to be starlike with respect to the origin.

**Proof.** (Part (ii) of Theorem 0.2) Consider the strictly increasing functions \(f(t) = f_q(t) = \text{sgn}(t) |\log |t||^{1-q}\) on the interval \(-\frac{1}{2} \leq t \leq \frac{1}{2}\) when \(1 < q\). Then

\[
\int_{-\frac{1}{2}}^{0} \int_{0}^{\frac{1}{2}} \left(\frac{f(s) - f(t)}{s-t}\right)^2 \, dsdt > \int_{-\frac{1}{2}}^{0} \int_{0}^{\frac{1}{2}} \left(\frac{f(s)}{s-t}\right)^2 \, dsdt = \int_{0}^{\frac{1}{2}} \left(\frac{f(s)}{s(2s+1)}\right)^2 \, ds.
\]

By \(2s + 1 \leq 2\) and substituting \(f_q\), the last integral is greater than

\[
\frac{1}{2} \int_{0}^{\frac{1}{2}} \frac{|\log |s||^{2-2q}}{s} \, ds,
\]

which diverges for \(q \leq \frac{3}{2}\). For this restriction of \(q\),

\[
\left|\frac{e^{if(s)} - e^{if(t)}}{f(s) - f(t)}\right| \geq \frac{1}{\pi} \quad \text{for all } s, t \in [-\frac{1}{2}, \frac{1}{2}].
\]

Each \(f(t)\) extends to a homeomorphism \([-\pi, \pi] \to [-\pi, \pi]\) and the above inequalities then show that \((0.1)\) diverges for \(p = 2\) and \(F(e^{it}) = e^{if(t)}\). \(\square\)

**Remark 0.3.** The traditional Dirichlet principle has been shown to fail for an absolutely continuous homeomorphism of the boundary.

It is also a classical result (see Proposition 7’ and Lemma 4’ \cite{Ste70}, pp. 151-152, but change the sign on \(\alpha - 1\); see \cite{Pee76}, pp. 8-10, items a. and f.) that harmonic functions in \(\mathbb{D}\) will have the \(p\)-norm of their gradients bounded by \((0.1)\). It remains to show that every homeomorphism \(F = re^{if}\) satisfies the inequality \((0.1)\) when \(p < 2\).
Proof. (Part (i) of Theorem 0.2) Every bounded convex domain is also a Lipschitz domain. Because $\Omega$ is also starlike with respect to the origin it follows for any two points on the boundary that the ratio formed by a change in the distance to the origin and a change in the argument is uniformly bounded. I.e. by the setup and definition of $F$ there is an $M < \infty$ so that $r \circ f^{-1}$ satisfies a Lipschitz condition

$$\frac{|r(s) - r(t)|}{f(s) - f(t)} \leq M$$

for all $s \neq t$. Consequently

$$\left| \frac{r(s)e^{if(s)} - r(t)e^{if(t)}}{s-t} \right| \leq (M + \|r\|_{\infty}) \left( \frac{f(s) - f(t)}{s-t} \right).$$

Take $p > 1$. Assume for the moment that $f$ is absolutely continuous and write $g(s,t) = \left( \frac{f(s) - f(t)}{s-t} \right)^p$. Integrating by parts as

$$\int g(s,t)ds = (s-t)g(s,t) - \int (s-t)\frac{\partial}{\partial s}g(s,t)ds$$

and evaluating, using the monotonicity of $f$ and $p > 1$, yields an intermediate step in Hardy’s inequality (see p. 242 of [HLP52], for example)

$$\int_0^{2\pi} \left( \frac{f(s) - f(t)}{s-t} \right)^p ds \leq \frac{p}{p-1} \int_0^{2\pi} \left( \frac{f(s) - f(t)}{s-t} \right)^{p-1} f'(s) ds.$$

The last integrand is bounded by $(2\pi)^{p-1}|s-t|^{1-p}f'(s)$. Integrating the left and right-hand sides in $t$ from $0$ to $2\pi$, applying Fubini with $p < 2$ and then the fundamental theorem of calculus in $s$ produces the bound

$$\int \int \left( \frac{f(s) - f(t)}{s-t} \right)^p dsdt \leq \frac{p}{p-1}(2\pi)^{p+1}$$

independent of $f$. When $f$ is not absolutely continuous it can be mollified by using a smooth and symmetric mollifier so that monotonicity and periodicity are retained. By the uniformity of the a priori bounds, positivity of the difference quotients and pointwise convergence of the mollifications of $f$ to $f$, Fatou’s lemma yields the same bounds on the double integral for any such increasing continuous $f$. In general

$$\int_0^{2\pi} \int_0^{2\pi} \left| \frac{F(e^{it}) - F(e^{i\theta})}{s-t} \right|^p dsdt \leq (M + \|r\|_{\infty})^p \frac{p}{p-1}(2\pi)^{p+1}, 1 < p < 2,$$

which proves (i) for $1 \leq p < 2$. □

Remark 0.4. The above proof required $F$ to be a homeomorphism and $r \circ f^{-1}$ to satisfy a Lipschitz condition. Thus the conclusion remains true if $\Omega$ is replaced by a starlike Lipschitz domain. But then $\nu$ would not be a homeomorphism in general.

References


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