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ASYMPTOTICALLY HARMONIC SPACES IN DIMENSION 3

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Abstract. Let M be a Hadamard manifold of dimension 3 whose sectional curvature satisfies $-b^2 \leq K \leq -a^2 < 0$ and whose curvature tensor satisfies $\|\nabla R\| \leq C$ for suitable constants $0 < a \leq b$ and $C \geq 0$. We show that M is of constant sectional curvature provided M is asymptotically harmonic. This was previously only known if M admits a compact quotient.

1. Introduction

Let M be a Hadamard manifold, i.e., a simply connected, complete Riemannian manifold of nonpositive curvature. We will consider the case that the sectional curvature of M is bounded by negative constants, say, $-b^2 \leq K \leq -a^2$, where $0 < a \leq b < \infty$. Let SM denote the unit tangent bundle of M. For $v \in SM$, let γ_v be the geodesic with $\gamma_v'(0) = v$. If we let $b_{v,t}(x) = d(x, \gamma_v(t)) - t,\ x \in M$, then the *Busemann function* $b_v : M \to \mathbb{R}$ for γ_v is defined as $b_v(x) = \lim_{t\to\infty} b_{v,t}(x)$ (see [13] for details). It is known that on a Hadamard manifold, b_v is a C^2 function [10]. There are examples ([1]) where this result is sharp.

A Hadamard manifold is called *asymptotically harmonic* if the mean curvature of its horospheres (i.e., level sets of Busemann functions) is a universal constant (see section 2). Clearly, Euclidean spaces and rank one symmetric spaces of noncompact type are asymptotically harmonic since the isometry group acts transitively on the unit tangent bundle. Other examples are the nonsymmetric Damek-Ricci spaces, a class of homogeneous Hadamard manifolds which are in fact harmonic spaces [6]. It is an open problem if these examples exhaust all asymptotically harmonic Hadamard manifolds. See [9] for the homogeneous case.

Let M be an asymptotically harmonic manifold of bounded negative curvature. It was proved in [11] that if M is 3-dimensional and if M admits a compact quotient, then M is a symmetric space. In fact, the corresponding result holds in arbitrary dimension, as was proved in [4] (based also on [2] and [8]; see [12] for related discussion).

Our main result generalizes the result in dimension 3 as follows.

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Theorem. *Let M be a Hadamard manifold of dimension 3 whose sectional curvature satisfies $-b^2 \leq K \leq -a^2 < 0$ and whose curvature tensor satisfies $\|\nabla R\| \leq C$ for suitable constants $0 < a \leq b$ and $C \geq 0$. If M is asymptotically harmonic, then M is symmetric and hence of constant sectional curvature.*

2. Proof of the Theorem

Let (M, g) be a Hadamard manifold. We call M *asymptotically harmonic* if the mean curvature of its horospheres (see [13]) is a universal constant, that is, if its Busemann functions satisfy $\Delta b_v \equiv h$, $v \in SM$, where h is a nonnegative constant (resp. positive constant if $K < 0$).

Recall that this notion makes sense in the class of manifolds without conjugate points. Note that every *harmonic manifold* without conjugate points (see [3]) is asymptotically harmonic [14].

In this section we prove the Theorem as stated in the Introduction.

For $v \in SM$ and $x \in v^\perp$, define

$$u^+(v)(x) = \nabla_x \nabla b_{-v}\,, \quad u^-(v)(x) = -\nabla_x \nabla b_v\,.$$

Clearly,

$$u^\pm : SM \to \bigcup_{v \in SM} \mathrm{Hom}\,(v^\perp, v^\perp).$$

Note that $u^-(v) = -u^+(-v)$. Since $K \leq 0$, the endomorphisms $u^+(v)$ and $u^-(v)$, $v \in SM$, are positive semidefinite and negative semidefinite on $v^\perp$, respectively (and definite, if $K < 0$). We will frequently use that $u^\pm$ satisfy the Riccati equation along orbits of the geodesic flow $\phi^t : SM \to SM$.

If M is a *symmetric space*, then applying the geodesic symmetry based at $\pi(v) \in M$ yields that

$$u^+(v)(x) = -u^+(-v)(-x) = u^+(-v)(x) = -u^-(v)(x),$$

that is, $u^+ + u^- \equiv 0$. The converse also holds:

Lemma 2.1. *Let M be a Hadamard manifold. Let $X = \frac{-1}{2}(u^+ + u^-)$. If $X = 0$, then M is a symmetric space.*

Proof. Fix $v \in SM$ and consider the endomorphism fields $u^\pm(t) := u^\pm(\phi^t v)$ and $R(t) := R(\cdot, \gamma_v'(t))\gamma_v'(t) \in \mathrm{End}(\gamma_v'(t)^\perp)$ along γ_v. If $X = 0$, then the Riccati equation for $t \mapsto u^\pm(t)$ yields that

$$0 = (u^+)' + (u^+)^2 + R = 2(u^+)' + \big((u^-)' + (u^-)^2 + R\big) = 2(u^+)'\,.$$

Therefore, $(u^+)' = 0$, and hence, $R'(t) = 0$ holds for all $t \in \mathbb{R}$ (and along each geodesic in M). This in turn implies that $\nabla R = 0$ [15]. Since M is simply connected and complete, M is a symmetric space. □

Remark. The result obviously holds for simply connected M without conjugate points if we define $u^\pm$ to be the stable, resp. unstable, Riccati solution.

Lemma 2.2. *Let (M, g) be an asymptotically harmonic Hadamard manifold with $K < 0$. Define $V(v) = u^+(v) - u^-(v) \in \mathrm{End}(v^\perp), v \in SM$. Then $t \mapsto \det V(\phi^t v)$ is constant for every fixed $v \in SM$.*

Proof. Since (M, g) is an asymptotically harmonic manifold of strictly negative curvature, $\Delta b_v \equiv h$ holds for all $v \in SM$, where h is a positive constant. Since $K < 0$, it follows that $u^+(v)$ and $-u^-(v)$ are positive definite operators on $v^\perp$. In particular, $V(v)$ is invertible for all $v \in SM$.

Fix $v \in SM$ and consider the endomorphism field $V(t) := V(\phi^t v)$, likewise $X(t), u^+(t), u^-(t)$, along γ_v.

The Riccati equation for $u^\pm(t)$ yields that

$$XV + VX = (u^-)^2 - (u^+)^2 = (u^+)' - (u^-)' = V'.$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \log \det V(t) &= \operatorname{tr} V'(t) V^{-1}(t) \\ &= \operatorname{tr} (XV + VX) V^{-1}(t) \\ &= 2 \operatorname{tr} X. \end{aligned}$$

Since by definition,

$$\operatorname{tr} u^+ = \Delta b_{-v} = h, \ \ \operatorname{tr} u^- = -\Delta b_v = -h,$$

we conclude that $\operatorname{tr} X = 0$.

Hence, $\det V$ is constant along the geodesic flow. □

Lemma 2.3. *Let M be an asymptotically harmonic Hadamard manifold, such that $-b^2 \le K \le -a^2 < 0$ and $\|\nabla R\| \le C$ hold for constants $0 < a \le b$ and $C \ge 0$. Then $\det V$ is constant on stable leaves of SM.*

Proof. Let $\gamma = \gamma_v$ and $\tilde\gamma = \gamma_{\tilde v}$ denote two asymptotic geodesics, parametrized such that $\lim_{t\to\infty} d(\gamma(t), \tilde\gamma(t)) = 0$ (see [13]). There exist a constant $\rho > 0$ and parallel orthonormal frames $\{e_i(t)\}_i$ along $\gamma(t)$, resp. $\{\tilde e_i(t)\}_i$ along $\tilde\gamma(t)$, such that $e_1(t) = \gamma'(t), \tilde e_1(t) = \tilde\gamma'(t)$ and

$$d(e_i(t), \tilde e_i(t)) \le \rho e^{-ta} \quad \text{for all } t \ge 0\,, \tag{1}$$

where d denotes the Sasaki metric on TM (see the proof of Prop. 3.2 of [5]). In terms of these frames, we consider the Riccati equations $u'(t) + u^2(t) + R(t) = 0$ along γ and $\tilde u'(t) + \tilde u^2(t) + \tilde R(t) = 0$ along $\tilde\gamma(t)$ as matrix valued. Here, $R(t)$ denotes the matrix expression of $R(\cdot, \gamma'(t))\gamma'(t) \in \operatorname{End}(\gamma'(t)^\perp)$, similarly for $\tilde R(t)$ along $\tilde\gamma(t)$. Since $\|\nabla R\|$ is bounded on M, (1) and the mean value theorem yield that, for suitable $\rho' > 0$,

$$\|R(t) - \tilde R(t)\| \le \rho' e^{-ta} \quad \text{for all } t \ge 0\,. \tag{2}$$

Here and in the sequel, we use operator norms on $\operatorname{End}(\mathbb{R}^{n-1}), n = \dim M$.

We denote by $u^+(t)$ and $u^-(t)$ the stable and unstable solution of the Riccati equation along γ, and denote by $\tilde u^\pm(t)$ the corresponding quantities along $\tilde\gamma$. We will apply (2) and the Riccati equation in order to prove that $\|u^\pm(t) - \tilde u^\pm(t)\| \to 0$ as $t \to \infty$. Hence, writing $V(t) := V(\phi^t(v))$ and $\tilde V(t) := V(\phi^t(\tilde v))$ as matrices w.r.t. the orthonormal frames, it follows that $\|V(t) - \tilde V(t)\| \to 0$, as $t \to \infty$. If M is asymptotically harmonic, we conclude from Lemma 2.2 that $\det V$ is constant on stable leaves of SM.

It remains to prove that $\|u^{\pm}(t) - \tilde{u}^{\pm}(t)\| \to 0$ as $t \to \infty$:

(i) Let $y(t) := u^+(t) - \tilde{u}^+(t),\ \Delta(t) := 1/2(u^+(t) + \tilde{u}^+(t)),\ r(t) := R(t) - \tilde{R}(t)$. It follows that

$$y'(t) + \Delta(t)y(t) + y(t)\Delta(t) + r(t) = 0.$$

Note that $a\mathrm{Id} \leq \Delta(t) \leq b\mathrm{Id}$ (where $X \leq Y$ indicates that $Y - X$ is positive semidefinite). Hence, the solution $C(t)$ of the initial value problem $C'(t) = -C(t)\Delta(t),\ C(0) = \mathrm{Id}$ satisfies

$$\|C(s)^{-1}C(t)\| \leq e^{-a(t-s)} \quad \text{for all } s \leq t\,. \tag{3}$$

In terms of $C(t)$, we calculate that

$$y(t) = C(t)^T \left(y(0) - \int_0^t [C(s)^{-1}]^T r(s) C(s)^{-1}\, ds \right) C(t). \tag{4}$$

(This follows essentially from the variation of constant method, as used in [7].) Clearly, the first summand converges to 0 as $t \to \infty$. It follows from (2) and (3) that the operator norm of the second summand is bounded from above by

$$\rho' \int_0^t (e^{-a(t-s)})^2 e^{-as} ds,$$

which also converges to 0, as $t \to \infty$.

(ii) We may apply a similar argument to $y^-(t) := u^-(t) - \tilde{u}^-(t), \Delta^-(t) := 1/2(u^-(t) + \tilde{u}^-(t))$ and the corresponding differential equation for y^-. Note however that in this case, $-a\mathrm{Id} \geq \Delta^-(t) \geq -b\mathrm{Id}$. The corresponding solution $C^-(t)$ diverges; it satisfies $\|C^-(s)^{-1}C^-(t)\| \leq e^{-a(s-t)}$ for all $s \geq t$. Clearly, y^- satisfies (4) with C replaced by C^-. Since $y^-(t)$ is bounded, it follows that $y^-(0) = \int_0^\infty [C^-(s)^{-1}]^T r(s) C^-(s)^{-1}\, ds$ and hence, that

$$y^-(t) = C^-(t)^T \left(\int_t^\infty [C^-(s)^{-1}]^T r(s) C^-(s)^{-1}\, ds \right) C^-(t)\,,$$

which in turn yields that $\|y^-(t)\| \leq \rho' \int_t^\infty (e^{-a(s-t)})^2 e^{-as} ds \longrightarrow 0$, as $t \to \infty$. □

Corollary 2.1. $\det V$ *is a constant on* SM.

Proof. Let $p \in M$ and $v_1 \neq v_2 \in S_pM$. As $K \leq -a^2 < 0$, there exists a geodesic α such that α is asymptotic to γ_{v_1} and α^- is asymptotic to γ_{v_2}. Hence by Lemma 2.3, $\det V(v_1) = \det V(\alpha'(0))$, $\det V(v_2) = \det V(-\alpha'(0))$. On the other hand, for any $v \in SM$, note that $V(v) = u^+(v) - u^-(v) = -u^-(-v) + u^+(-v) = V(-v)$, and hence, $\det V(v_1) = \det V(v_2)$. Therefore, $\det V$ is constant on each fiber of the projection map $\pi : SM \to M$.

Let $p, q \in M$. Then, there exists a unique geodesic joining p to q. Hence, $\det V$ is constant on SM. □

Proof of the Theorem. Fix $p \in M$. Since $\dim M = 3$, we may identify the tangent sphere S_pM with the standard 2-sphere S^2, and $v^\perp$ with T_vS^2 for each $v \in S_pM = S^2$. Recall that all operators $V(v) \in \mathrm{End}(T_vS^2)$, $v \in S^2$, are positive definite. If none of the $V(v)$, $v \in S^2$, is a constant multiple of the identity, then the eigenspaces corresponding to, say, maximal eigenvalues of $V(v)$ define a line field (i.e., a one dimensional tangent distribution) on S^2 without singularities and hence, with index sum zero. This is a contradiction since $\chi(S^2) = 2 \neq 0$.

Since $\mathrm{tr}\, V \equiv 2h$, we conclude that for any $p \in M$ there exists a $v \in S_pM$, such that $V(v) = h\mathrm{Id}$. However, recall from Corollary 2.1 that $\det V$ is constant. Hence,

$\det V \equiv h^2$. This implies that $V(v) = h\mathrm{Id}$ holds for all $v \in SM$. Therefore, along the geodesic flow we obtain $0 = V' = XV + VX = 2hX$ (cf. Lemma 2.2) and hence, $X = 0$. Thus, M is a symmetric space, as follows from Lemma 2.1. Clearly, as $K < 0$, M is an irreducible symmetric space and therefore Einstein [13]. Since M is three dimensional, its sectional curvature is constant. □

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