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SPECTRUM OF THE $\overline{\partial}$-NEUMANN LAPLACIAN ON POLYDISCS

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Abstract. The spectrum of the $\overline{\partial}$-Neumann Laplacian on a polydisc in $\mathbb{C}^n$ is explicitly computed. The calculation exhibits that the spectrum consists of eigenvalues, some of which, in particular the smallest ones, are of infinite multiplicity.

1. Introduction

The $\overline{\partial}$-Neumann Laplacian $\Box_q$ on a bounded domain Ω in $\mathbb{C}^n$ is (a constant multiple of) the usual Laplacian acting diagonally on $(0,q)$-forms subject to the non-coercive $\overline{\partial}$-Neumann boundary conditions. It is a densely defined, non-negative, and self-adjoint operator. As such, its spectrum is a non-empty closed subset of the non-negative real axis. Unlike the usual Dirichlet Laplacian, its spectrum need not be purely discrete. (See [FS01] for a discussion on related subjects.) Spectral behavior of the $\overline{\partial}$-Neumann Laplacian is more sensitive to the boundary geometry of the domain than the Dirichlet/Neumann Laplacians. (See [Fu05a, Fu05b] and the references therein for related discussions.)

Spectral behavior of the $\overline{\partial}$-Neumann Laplacian on special domains often serves as a model for the general theory. One certainly cannot expect to explicitly calculate the spectrum for wide classes of domains. The spectrum for the ball and annulus was explicitly computed by Folland [Fo72]. In this note, we compute the spectrum for the polydiscs. Our computation exhibits that the spectrum of the $\overline{\partial}$-Neumann Laplacian on a polydisc consists of eigenvalues, some of which, in particular the smallest ones, are of infinite multiplicity. That the essential spectrum of the $\overline{\partial}$-Neumann Laplacian is non-empty is consistent with, in fact, equivalent to, the well-known fact that the $\overline{\partial}$-Neumann operator (the inverse of the $\overline{\partial}$-Neumann Laplacian) is non-compact (e.g., [K88]). It is noteworthy that for a polydisc, the bottom of the spectrum is always in the essential spectrum—a phenomenon not stipulated in general operator theory.

2. Preliminaries

We first recall the setup for the $\overline{\partial}$-Neumann Laplacian. We refer the reader to [FoK72, CS99] for an in-depth treatise of the $\overline{\partial}$-Neumann problem.

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Let Ω be a bounded domain in $\mathbb{C}^n$. For $0 \le q \le n$, let $L^2_{(0,q)}(\Omega)$ denote the space of $(0,q)$-forms with square integrable coefficients and with the standard Euclidean inner product whose norm is given by

$$\|\sum{}' a_J d\bar{z}_J\|^2 = \sum{}' \int_\Omega |a_J|^2 dV(z),$$

where the prime indicates the summation over strictly increasing q-tuples J. (We consider a_J to be defined on all q-tuples, antisymmetric with respect to J.) For $0 \le q \le n-1$, let $\overline{\partial}_q \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ be the $\overline{\partial}$-operator defined in the sense of distributions. This is a closed and densely defined operator. Let $\overline{\partial}^*_q$ be its adjoint. Then $\overline{\partial}^*_q$ is also a closed and densely defined operator with domain

$$\mathrm{Dom}(\overline{\partial}^*_q) = \{u \in L^2_{(0,q+1)}(\Omega) \mid \exists C > 0 \text{ such that } |\langle u, \overline{\partial}_q v\rangle| \le C\|v\|, \forall v \in \mathrm{Dom}(\overline{\partial}_q)\}.$$

For $1 \le q \le n-1$, let

$$Q_q(u,v) = (\overline{\partial}_q u, \overline{\partial}_q v) + (\overline{\partial}^*_{q-1} u, \overline{\partial}^*_{q-1} v)$$

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with $\mathrm{Dom}(Q_q) = \mathrm{Dom}(\overline{\partial}_q) \cap \mathrm{Dom}(\overline{\partial}^*_{q-1})$. It is evident that Q_q is non-negative, densely defined, and closed. The $\overline{\partial}$-Neumann Laplacian $\Box_q = \overline{\partial}^*_q \overline{\partial}_q + \overline{\partial}_{q-1}\overline{\partial}^*_{q-1} \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$ is the associated self-adjoint operator with domain

$$\mathrm{Dom}(\Box_q) = \{u \in L^2_{(0,q)}(\Omega) \mid u \in \mathrm{Dom}(\overline{\partial}_q) \cap \mathrm{Dom}(\overline{\partial}^*_{q-1}), \overline{\partial}_q u \in \mathrm{Dom}(\overline{\partial}^*_q), \overline{\partial}^*_{q-1} u \in \mathrm{Dom}(\overline{\partial}_{q-1})\}.$$

For the reader's convenience, we also briefly review relevant facts of the Bessel functions. Extensive treatment of the Bessel functions can be found, for example, in [W48]. The Bessel functions of integer orders are given via the following Laurent expansion:

$$\exp(\frac{z}{2}(t - \frac{1}{t})) = \sum_{m=-\infty}^{\infty} t^m J_m(z). \tag{2.1}$$

Evidently, $J_{-m}(z) = (-1)^m J_m(z)$, and when $m \ge 0$,

$$J_m(z) = \sum_{l=0}^{\infty} \frac{(-1)^l (z/2)^{2l+m}}{l!(l+m)!}.$$

Hence $J_m(z)$ is an entire function with a zero of order $|m|$ at the origin. By differentiating both sides of (2.1) with respect to t and with respect to z, we have the recurrence formulas

$$mJ_m(z) = \frac{z}{2}(J_{m+1}(z) + J_{m-1}(z)), \qquad J'_m(z) = \frac{1}{2}(J_{m-1}(z) - J_{m+1}(z)). \tag{2.2}$$

Therefore,

$$zJ_{m-1}(z) = zJ'_m(z) + mJ_m(z), \qquad zJ_{m+1}(z) = -zJ'_m(z) + mJ_m(z). \tag{2.3}$$

It follows that $J_m(z)$ satisfies the Bessel equation

$$J''_m(z) + \frac{1}{z}J'_m(z) + (1 - \frac{m^2}{z^2})J_m(z) = 0. \tag{2.4}$$

Thus $J_m(z)$ has only simple zeroes on $\mathbb{C} \setminus \{0\}$. Furthermore, it follows from (2.4) that the smallest positive zero of $J_m(z)$ cannot be less than $|m|$. On the other

hand, by multiplying both sides of (2.1) by t^{-m-1} and then integrating on $|t| = 1$, we obtain the following integral representation of the Bessel functions:

$$J_m(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(m\theta - z\sin\theta)\, d\theta.$$

From this integral representation, we know that $J_0(x)$ is positive on the interval $[k\pi, (k+1/2)\pi]$ when k is even and negative on the interval when k is odd. It follows that $J_0(x)$ has infinitely many zeroes on the positive real axis and all of these zeroes are on the intervals $((k+1/2)\pi, (k+1)\pi)$. From (2.3), we know that

$$J_{m-1}(z) = z^{-m}\frac{d}{dz}(z^m J_m(z)), \qquad J_{m+1}(z) = -z^m \frac{d}{dz}(z^{-m} J_m(z)). \tag{2.5}$$

It follows that $J_m(z)$ also has infinitely many zeroes on the positive real axis. Furthermore, the zeroes of $J_m(z)$ and those of $J_{m+1}(z)$ interlace. Let $\lambda_{m,j}$, $j = 1, 2, \ldots$, be the positive zeroes of $J_m(z)$, arranged in increasing order. Then it follows from (2.4) that

$$\int_0^1 r J_m(\lambda_{m,j} r) J_m(\lambda_{m,k} r)\, dr = \begin{cases} 0, & j \neq k, \\ \frac{1}{2} J_{m+1}^2(\lambda_{m,j}), & j = k. \end{cases}$$

Furthermore, for any given integer m, $\{\sqrt{r} J_m(\lambda_{m,j} r)\}_{j=1}^{\infty}$ forms a complete orthogonal basis for $L^2(0,1)$. Moreover, it follows from (2.5) that for $m \geq 0$, $\{r^{1/2+m}\} \cup \{\sqrt{r} J_m(\lambda_{m+1,j} r)\}_{j=1}^{\infty}$ forms a complete orthogonal basis for $L^2(0,1)$ and so does $\{\sqrt{r} J_m(\lambda_{m-1,j} r)\}_{j=1}^{\infty}$ for $m > 0$.

3. The computations

Let $P = P(a_1, \ldots, a_n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| < a_1, \ldots, |z_n| < a_n\}$. Write $\rho_j(z) = |z_j|^2 - a_j^2$. Then $P = \{z \in \mathbb{C}^n \mid \rho_j(z) < 0, j = 1, \ldots, n\}$. Suppose that

$$u = \sum_{|J|=q}{}' u_J d\bar{z}_J \in C^\infty(\overline{P}).$$

For any integer q between 1 and $n-1$, we now solve the $\overline{\partial}$-Neumann boundary value problem:

$$\Box_q u = \lambda u, \tag{3.1}$$

$$u \in \mathrm{Dom}(\overline{\partial}^*_{q-1}), \tag{3.2}$$

$$\overline{\partial}_q u \in \mathrm{Dom}(\overline{\partial}^*_q). \tag{3.3}$$

It follows from an easy integration by parts argument that $u \in \mathrm{Dom}(\overline{\partial}^*_{q-1})$, provided $u_{jK}(z) = 0$ when $|z_j| = a_j$ for any $(q-1)$-tuple K and $j \in \{1, \ldots, n\}$. Write $z_j = r_j e^{i\theta_j}$. To identify the eigenvalues of $\Box_q$, we use the method of separation of variables and look for eigenforms whose coefficients have the following product form:

$$u_J(z) = \prod_{k=1}^{n} u_J^k(z_k). \tag{3.4}$$

Then $u \in \mathrm{Dom}(\overline{\partial}^*_{q-1})$, provided

$$u_J^k(a_k e^{i\theta_k}) = 0, \qquad \text{when } k \in J. \tag{3.5}$$

For any $K = (k_1, \dots, k_{q+1})$, write

$$v_K = \sum_{l=1}^{q+1} (-1)^{l+1} \frac{\partial u_{K \setminus k_l}}{\partial \bar{z}_{k_l}},$$

where $K \setminus k_l$ means the deletion of the k_l entry from K. Then

$$\overline{\partial}_q u = \sum_{|K|=q+1}{}' \, v_K d\bar{z}_K.$$

Thus $\overline{\partial}_q u \in \mathrm{Dom}(\overline{\partial}_q^*)$ if $v_{jJ}(z) = 0$ whenever $|z_j| = a_j$ for any $j \in \{1, \dots, n\}$ and q-tuple J. Using the separation of variables (3.4), we have that $\overline{\partial}_q u \in \mathrm{Dom}(\overline{\partial}_q^*)$, provided, in addition to (3.5), u_J also satisfies

$$\frac{\partial u_J^k}{\partial \bar{z}_k}(a_k e^{i\theta_k}) = 0, \qquad \text{when } k \notin J. \tag{3.6}$$

Recall that $\Box_q = (-1/4)\Delta$, where Δ is the usual Laplacian acting diagonally. Denote by $\Delta_k = 4(\partial^2/\partial z_k \partial \bar{z}_k)$ the Laplacian in the z_k-variable. Then, with the separation of variables (3.4), the boundary value problem (3.1)-(3.3) is reduced to

$$\Delta_k u_J^k(z_k) = -\lambda_k u_J^k(z_k), \qquad u_J^k(a_k e^{i\theta_k}) = 0, \quad \text{for } k \in J, \tag{3.7}$$

and

$$\Delta_k u_J^k(z_k) = -\lambda_k u_J^k(z_k), \qquad \frac{\partial u_J^k}{\partial \bar{z}_k}(a_k e^{i\theta_k}) = 0, \quad \text{for } k \notin J, \tag{3.8}$$

with

$$\lambda = \frac{1}{4} \sum_{k=1}^{n} \lambda_k. \tag{3.9}$$

The boundary value problem (3.7) gives the eigenvalues for the Dirichlet Laplacian on the disc $|z_k| < a_k$. It is well known (and easy to see) that these eigenvalues are

$$\left(\frac{\lambda_{m_k, j_k}}{a_k} \right)^2 \tag{3.10}$$

and the associated eigenfunctions are

$$J_{m_k}(\lambda_{m_k, j_k} r_k / a_k) e^{i m_k \theta_k}, \tag{3.11}$$

for $m_k \in \mathbb{Z}$ and $j_k \in \mathbb{N}$.

To solve the boundary value problem (3.8), we separate the variables in polar coordinates: $u_J^k(z_k) = R(r_k)\Theta(\theta_k)$. Then (3.8) is reduced to

$$\frac{\Theta''}{\Theta} = -\mu, \qquad \Theta(\theta_k + 2\pi) = \Theta(\theta_k), \tag{3.12}$$

and

$$\frac{R''}{R} + \frac{1}{r_k}\frac{R'}{R} - \frac{\mu}{r_k^2} = -\lambda_k, \qquad \frac{R'}{R}(a_k) = -\frac{i}{a_k}\frac{\Theta'}{\Theta}. \tag{3.13}$$

From (3.12), we know that $\mu = m_k^2$, $m_k \in \mathbb{Z}$, with the associated eigenfunctions $e^{i m_k \theta_k}$. We first consider the case when $\lambda_k = 0$. In this case, we know from $\Theta = e^{i m_k \theta_k}$ and (3.13) that $R = r_k^{m_k}$. Since by interior elliptic regularity the eigenfunctions must be smooth at the origin, we know that 0 is an eigenvalue of the boundary value problem (3.8) with the associated eigenfunctions $z_k^{|m_k|}$.

Now we consider the case when $\lambda_k > 0$. Using the substitution $r = \sqrt{\lambda_k} r_k$, we reduce (3.13) to

$$R'' + \frac{1}{r}R' + (1 - \frac{m_k^2}{r^2})R = 0, \qquad \sqrt{\lambda_k} a_k R'(\sqrt{\lambda_k} a_k) - m_k R(\sqrt{\lambda_k} a_k) = 0. \tag{3.14}$$

From (2.4), we know that $R = J_{m_k}(r)$, and from (2.3), we know that $J_{m_k+1}(\sqrt{\lambda_k} a_k) = 0$. In summary, from the boundary value problem (3.8), we obtain the eigenvalues

$$\left(\frac{\lambda_{m_k+1,j_k}}{a_k}\right)^2 \tag{3.15}$$

with the associated eigenfunctions

$$J_{m_k}(\lambda_{m_k+1,j_k} r_k / a_k) e^{i m_k \theta_k}, \tag{3.16}$$

for $m_k \in \mathbb{Z}$ and $j_k \in \mathbb{N}$.

From the above computations, we now know that the spectrum of $\Box_q$ on the polydisc P contains the eigenvalues

$$\frac{1}{4} \sum_{k \in J} \left(\frac{\lambda_{m_k,j_k}}{a_k}\right)^2 \tag{3.17}$$

of infinite multiplicity with the associated eigenforms

$$\prod_{k \in J} \left(J_{m_k}(\lambda_{m_k,j_k} r_k / a_k) e^{i m_k \theta_k}\right) \prod_{k \notin J} z_k^{|m_k|} d\bar{z}_J, \tag{3.18}$$

and eigenvalues

$$\frac{1}{4} \sum_{k=1}^{n} \left(\frac{\lambda_{m_k,j_k}}{a_k}\right)^2 \tag{3.19}$$

with the associated eigenforms

$$\prod_{k \in J} \left(J_{m_k}(\lambda_{m_k,j_k} r_k / a_k) e^{i m_k \theta_k}\right) \prod_{k \notin J} \left(J_{m_k - 1}(\lambda_{m_k,j_k} r_k / a_k) e^{i(m_k - 1)\theta_k}\right) d\bar{z}_J, \tag{3.20}$$

for any strictly increase q-tuple J, $m_k \in \mathbb{Z}$, and $j_k \in \mathbb{N}$. It is interesting to note that the eigenvalues in the form of (3.19) are independent of the value of q, while those in the form of (3.17) detect the value of q.

It remains to show that the spectrum of $\Box_q$ consists of nothing else but the eigenvalues listed in (3.17) and (3.19). To do this, we use the following well-known fact from general operator theory (e.g., [Dav95], Lemma 1.2.2): Let T be a symmetric operator on a complex Hilbert space H. If there exists a complete orthonormal basis $\{f_j\}_{j=1}^{\infty}$ and $\lambda_j \in \mathbb{R}$ such that $Tf_j = \lambda_j f_j$, then T is essentially self-adjoint and the spectrum of $\overline{T}$ is the closure of $\{\lambda_j\}_{j=1}^{\infty}$ in $\mathbb{R}$. It follows from facts about the Bessel functions stated in the last paragraph of Section 2 that for each q-tuple J, the coefficients of $d\bar{z}_J$ in (3.18) and (3.20) form a complete orthogonal basis for $L^2(P)$. From Section 2 we also know that $\lambda_{m,j} \to \infty$ as $|m| \to \infty$ or $j \to \infty$. (Recall that $\lambda_{m,j} \geq |m|$.) Thus the spectrum of $\Box_q$ contains nothing else but eigenvalues listed in (3.17) and (3.19) with associated eigenforms listed in (3.18) and (3.19), respectively. The bottom of the spectrum is

$$\min_{|J|=q} \{\frac{\lambda_{0,1}^2}{4} \sum_{k \in J} \frac{1}{a_k^2}\},$$

which is always of infinite multiplicity. Note that in the above computations, we have restricted q to be between 1 and $n-1$. On the top degree $(0,n)$-forms, the $\overline{\partial}$-Neumann Laplacian is the negative one-fourth of the Dirichlet Laplacian, whose spectrum on a polydisc is well known and can be easily computed as above. In this case, the spectrum is purely discrete: It consists of eigenvalues of finite multiplicity in the form of (3.19).

Since we now know explicitly the spectrum and the associated eigenforms, it is then easy to explicitly express the $\overline{\partial}$-Neumann operator as an infinite sum of projections onto the eigenspaces. We leave this to the interested reader.

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