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MORSE–PALAIS LEMMA FOR NONSMOOTH FUNCTIONALS ON NORMED SPACES

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ABSTRACT. Using elementary differential calculus we get a version of the Morse-Palais lemma. Since we do not use powerful tools in functional analysis such as the implicit theorem or flows and deformations in Banach spaces, our result does not require the C^1-smoothness of functions nor the completeness of spaces. Therefore it is stronger than the classical one but its proof is very simple.

1. INTRODUCTION

In [9], Palais proved the Morse-Palais lemma for C^3 functions. This result was extended for C^2 functions by Kuiper in [6] (see also [8]). The Morse-Palais lemma gives us the similarity of the shapes of the graphs of $J(x)$ and $D^2J(0)(x,x)$, consequently the existence of the second derivatives is essential. Recently Li, Li, and Liu [7] obtained a version of the Morse-Palais lemma without the C^2-smoothness. Now we consider perturbed problems as follows: we study the multiplicity of solutions to the equation

$$(P) \qquad -\Delta u - \lambda u = f(x,u), \qquad x \in \Omega,$$

with the corresponding functional $J = J_1 - J_2$, where Ω is a bounded smooth open domain in $\mathbb{R}^n$, $J_1(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda |u|^2)dx$ and $J_2(u) = \int_\Omega \int_0^{u(x)} f(x,s)dsdx$. The problem (P) is the main application in [7].

The functional J_1 is smooth and we can investigate the shape of the graph of $D^2J_1(0)(x,x)$. But the perturbed part J_2 may not be C^1-smooth. We observe that $D^2J_1(0)$ is very nice: there exist a closed vector subspace W^+ and a finite-dimensional vector subspace W^- of the Sobolev space $W_0^{1,2}(\Omega)$ such that $W^+ \oplus W^-$ is a direct decomposition of $W_0^{1,2}(\Omega)$ and $D^2J_1(0)(y,y) \le 0 \le D^2J_1(0)(x,x)$ for any (x,y) in $W^+ \times W^-$.

By this observation we try to find a special version of the Morse-Palais lemma for problems similar to (P), in which we do not need any second derivatives. In the present paper we get this version, which does not require the C^1-smoothness of functions nor the completeness of spaces as follows.

Theorem 1.1. *Let $(H, ||.||)$ be a normed vector space and let J be a continuous and continuously directional differentiable real function on an open ball $B_H(0, 2\delta)$ in H,*

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that is, $DJ(x)(h) = \lim_{t\to 0} \dfrac{J(x+th)-J(x)}{t}$ *exists for any* (x,h) *in* $B_H(0,2\delta)\times H$ *and the map* $x \mapsto DJ(x)(h)$ *is continuous on* $B_H(0,2\delta)$ *for any fixed* h *of* H *and* $DJ(x)$ *is a linear mapping from* H *into* $\mathbb{R}$.

Assume that there exist a closed vector subspace H^+ *and a finite-dimensional vector subspace* H^- *of* H *such that* $H^+ \oplus H^-$ *is a direct decomposition of* H *and*

(i) $J(0)=0$ *and* $DJ(0)=0$.

(ii) $[DJ(x+y_2)-DJ(x+y_1)][-(y_2-y_1)] \;>0\;\forall\; x\in B'_{H^+}(0,\delta),\; y_1,y_2\in B'_{H^-}(0,\delta)$ *and* $y_1\neq y_2$.

(iii) $DJ(x+y)(x-y)\;>0 \qquad \forall\; x\in B'_{H^+}(0,\delta),\; y\in B'_{H^-}(0,\delta)$, *and* $(x,y)\neq(0,0)$.

(iv) $DJ(x)x > p(||x||) \qquad \forall\; x\in B'_{H^+}(0,\delta)\backslash\{0\}$, *where* $p:(0,\delta]\to(0,\infty)$ *is a non-decreasing function.*

Then there exist a positive real number ϵ, *an open neighborhood* U *of* 0 *in* H *and an isomorphism* ϕ *from* $B_{H^+}(0,\sqrt{\frac{p(\epsilon/2)}{2}})\times B_{H^-}(0,\sqrt{\frac{p(\epsilon/2)}{2}})$ *onto* U *such that:*

$$J(\phi(x+y)) = ||x||^2-||y||^2 \qquad \forall\;(x,y)\in B_{H^+}(0,\sqrt{\frac{p(\epsilon/2)}{2}})\times B_{H^-}(0,\sqrt{\frac{p(\epsilon/2)}{2}}).$$

Remark 1.1. Our Morse-Palais lemma is stronger than the classical one even in the case of finite-dimensional normed spaces and is applicable to the following function:

$$J(x,y)=x^2-y^2+|x|^{3/2}\;\forall\;(x,y)\in\mathbb{R}^2.$$

We note that the results in [6, 7] cannot be applied to this case.

Remark 1.2. Let H^+ and H^- be closed vector subspaces of a normed space $(H,||.||)$ such that the dimension of H^- is finite and $H^+\oplus H^-$ is a direct decomposition of H. Let $J_{1,1}$ and $J_{1,2}$ be the restrictions of continuous bilinear real mappings on $B'_{H^+}(0,2\delta)$ and $B'_{H^-}(0,2\delta)$ with a positive real number δ respectively. Assume that there is a positive real number η such that

$$J_{1,1}(x,x)\geq \eta||x||^2 \qquad \forall\; x\in B'_{H^+}(0,2\delta) \text{ and}$$
$$J_{1,2}(y,y)\leq -\eta||y||^2 \qquad \forall\; y\in B'_{H^-}(0,2\delta).$$

Let J_2 be a Fréchet differentiable real function on $B'_{H^+}(0,2\delta)\times B'_{H^-}(0,2\delta)$. Assume there is a positive real number M such that for any (x,y_1) and (x,y_2) in $B'_{H^+}(0,2\delta)\times B'_{H^-}(0,2\delta)$ we have

$$||DJ_2(x,y_1)||\leq M||(x,y_1)|| \text{ and}$$
$$||DJ_2(x,y_1)-DJ_2(x,y_2)||\leq M||y_1-y_2||.$$

Put $J_1(x,y)=J_{1,1}(x,x)+J_{1,2}(y,y)$ for any (x,y) in $B'_{H^+}(0,2\delta)\times B'_{H^-}(0,2\delta)$ and $J=J_1+\epsilon J_2$.

Then J satisfies the conditions of Theorem 1.1 for a sufficiently small positive real number ϵ. The J_1 associated with problem (P) belongs to the class of functionals considered in this remark.

Remark 1.3. In Theorem 1.1 we can replace the conditions (ii) and (iii) by the following:

(v) $[DJ(x_2+y_2)-DJ(x_1+y_1)][(x_2-x_1)-(y_2-y_1)]\;>0\;\forall\; x_1,x_2\in B'_{H^+}(0,\delta)$; y_1, $y_2\in B'_{H^-}(0,\delta)$ and $x_1+y_1\neq x_2+y_2$.

To get the Morse-Palais lemma one uses powerful tools in functional analysis such as the implicit theorem or flows and deformations in Banach spaces. In this paper we only apply elementary differential calculus to get results, hence we do not need the conditions essential for those theories. Therefore our result is stronger than the classical one but our proof is very simple. Of course we cannot get the differentiability of the map ϕ, but in some problems involving the topological degree we only need the continuity of ϕ (see [2, 5, 8]). The differentiability of J in our theorem is very weak but sufficient for many applications (see [1, 3, 4]). We shall prove our theorem in the second section.

2. Proof of the theorem

The proof of the theorem consists of the following lemmas.

Lemma 2.1. *There exists a positive real number $\epsilon_1 < \delta$ having the following property: for each $x \in B_{H^+}(0,\epsilon_1)$ there exists a unique $\varphi(x) \in B_{H^-}(0,\delta)$ such that $J(x+\varphi(x)) = \max\{J(x+y) : y \in B_{H^-}(0,\delta)\}$.*

Proof. First we show the uniqueness of $\varphi(x)$. Assume by contradiction that there exist z and z' in $B_{H^-}(0,\delta)$ and $z \neq z'$ such that

$$J(x+z) = J(x+z') = \max\{J(x+y) : y \in B_{H^-}(0,\delta)\}.$$

This implies $DJ(x+z)h = DJ(x+z')h = 0$ for any h in H^-. Choosing $h = z - z'$, we get $(DJ(x+z) - DJ(x+z'))(z-z') = 0$, which contradicts (ii).

Now, we prove the existence of ϵ_1. Note that $B'_{H^-}(0,\delta)$ is compact. Assume by contradiction that there exist a sequence $\{x_n\}$ converging to 0 in $B_{H^+}(0,\delta)$ and a sequence $\{y_n\}$ in $\partial B_{H^-}(0,\delta)$ such that

$$J(x_n+y_n) > J(x_n+y) \qquad \forall y \in B_{H^-}(0,\delta).$$

By the compactness of $\partial B_{H^-}(0,\delta)$, we may assume that $\{y_n\}$ converges to y_0 in $\partial B_{H^-}(0,\delta)$. By the continuity of J, we have

$$\lim_{n\to\infty} J(x_n+y_n) = J(y_0) \text{ and } \lim_{n\to\infty} J(x_n) = J(0).$$

This implies $J(y_0) \geq J(0)$. □

On the other hand, by the mean value theorem and (iii), there exists a real number $t_y \in (0,1)$ such that

$$J(y_0) - J(0) = DJ(t_y.y_0)(y_0) = -\frac{1}{t_y} DJ(t_y.y_0)(-t_y.y_0) < 0, \tag{2.1}$$

which is a contradiction.

Arguing as above, we may assume further that $\varphi(x) \in B_{H^-}(0,\delta/2)$ for any x in $B_{H^+}(0,\epsilon_1)$.

Lemma 2.2. *φ is continuous on $B_{H^+}(0,\varepsilon_1)$.*

Proof. Let $\{x_n\}$ be a sequence converging to x_0 in $B_{H^+}(0,\epsilon_1)$. By the compactness of $B'_{H^-}(0,\delta/2)$, without loss of generality we may assume that $\{\varphi(x_n)\}$ converges to $y_0 \in B'_{H^-}(0,\delta/2)$. We have

$$J(x_n+\varphi(x_n)) \geq J(x_n+y) \qquad \forall n \in \mathbb{N}, y \in B_{H^-}(0,\delta).$$

Taking the limits of both sides of the inequality, we see that $J(x_0+y_0) \geq J(x_0+y)$ for every y in $B_{H^-}(0,\delta)$. Thus by the uniqueness of $\varphi(x_0)$, y_0 should be $\varphi(x_0)$, which implies the continuity of φ. □

Lemma 2.3. *Put $j(x) = J(x + \varphi(x))$ for any x in $B_{H^+}(0, \varepsilon_1)$. Then j is a continuously directional differentiable real function on $B_{H^+}(0, \varepsilon_1)$ and*

$$Dj(x)h = DJ(x + \varphi(x))h \qquad \forall\ h \in H^+, x \in B(0, \varepsilon_1).$$

Proof. Fix an (x, h) in $B_{H^+}(0, \varepsilon_1) \times H^+$. By the continuously directional differentiability of J, the continuity of φ and the mean value theorem, there exists $\theta_t \in (0, 1)$ such that

$$\begin{aligned}\lim_{t\to 0} &\frac{J(x + th + \varphi(x + th)) - J(x + \varphi(x + th))}{t} \\ &= \lim_{t\to 0} DJ(x + \theta_t th + \varphi(x + th))h = DJ(x + \varphi(x))h.\end{aligned} \tag{2.2}$$

On the other hand we have

$$\lim_{t\to 0} \frac{J(x + th + \varphi(x)) - J(x + \varphi(x))}{t} = DJ(x + \varphi(x))h. \tag{2.3}$$

□

Furthermore, by the maximum property of φ,

$$\begin{aligned}J(x + th + \varphi(x)) - J(x + \varphi(x)) &\le J(x + th + \varphi(x + th)) - J(x + \varphi(x)) \\ &\le J(x + th + \varphi(x + th)) - J(x + \varphi(x + th)).\end{aligned} \tag{2.4}$$

Combining (2.2), (2.3), and (2.4) we have

$$\lim_{t\to 0} \frac{J(x + th + \varphi(x + th)) - J(x + \varphi(x))}{t} = DJ(x + \varphi(x))h.$$

Lemma 2.4. *We define*

$$\psi_1(x + y) = \begin{cases} \dfrac{\sqrt{J(x + \varphi(x))}}{||x||}x & \text{if } x \ne 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$\psi_2(x + y) = \begin{cases} \dfrac{\sqrt{J(x + \varphi(x)) - J(x + y)}}{||y - \varphi(x)||}(y - \varphi(x)) & \text{if } y \ne \varphi(x), \\ 0 & \text{if } y = \varphi(x), \end{cases}$$

$$\psi(x + y) = \psi_1(x + y) + \psi_2(x + y) \qquad \forall\ x \in B_{H^+}(0, \epsilon_1),\ y \in B_{H^-}(0, \delta).$$

Then ψ_1, ψ_2, and ψ are continuous on $B_{H^+}(0, \epsilon_1) \times B_{H^-}(0, \delta)$ and

$$J(x + y) = ||\psi_1(x + y)||^2 - ||\psi_2(x + y)||^2\ \forall\ x \in B_{H^+}(0, \epsilon_1),\ y \in B_{H^-}(0, \delta). \tag{2.5}$$

Proof. By the continuity of J and φ, we see that ψ_1, ψ_2, and ψ are continuous on $B_{H^+}(0, \epsilon_1) \times B_{H^-}(0, \delta)$. By a straightforward computation we get the last conclusion of the lemma. □

Lemma 2.5. *ψ is one-to-one on $B_{H^+}(0, \epsilon_1) \times B_{H^-}(0, \delta)$.*

Proof. Assume by contradiction that there exist x_1 and x_2 in $B_{H^+}(0, \epsilon_1)$ and y_1 and y_2 in $B_{H^-}(0, \delta)$ such that $x_1 + y_1 \ne x_2 + y_2$ and $\psi(x_1 + y_1) = \psi(x_2 + y_2)$. By the definition of ψ, we have

$$\begin{aligned}&\frac{x_1}{||x_1||} = \frac{x_2}{||x_2||}, \\ &\frac{y_1 - \varphi(x_1)}{||y_1 - \varphi(x_1)||} = \frac{y_2 - \varphi(x_2)}{||y_2 - \varphi(x_2)||}, \\ &j(x_1) = J(x_1 + \varphi(x_1)) = J(x_2 + \varphi(x_2)) = j(x_2), \\ &J(x_1 + y_1) = J(x_2 + y_2).\end{aligned}$$

Without loss of any generality, we may assume $x_2 = s.x_1$ with $s \geq 1$. Put $f(t) = j(tx_1) = J(tx_1 + \varphi(tx_1))$ for every $t \in (0, s)$. By Lemma 2.3 and condition (iii), we have

$$f'(t) = Dj(tx_1)x_1 = DJ(tx_1 + \varphi(tx_1))x_1 = \frac{1}{t}DJ(tx_1 + \varphi(tx_1))(tx_1 - \varphi(tx_1)) \ > 0.$$

Thus f is strictly increasing, s should be equal to 1 and $x_1 = x_2$.

Moreover, we have $J(x_1+y_1) = J(x_2+y_2)$ and we may suppose that $y_2-\varphi(x_2) = y_2 - \varphi(x_1) = r(y_1 - \varphi(x_1))$ with $r > 1$. Thus

$$\begin{aligned} J(x_1 + \varphi(x_1) + y_1 - \varphi(x_1)) &= J(x_1 + y_1) = J(x_2 + y_2) \\ &= J(x_1 + \varphi(x_1) + r(y_1 - \varphi(x_1))). \end{aligned}$$

By the mean value theorem, there exists $t_x \in (1, r)$ such that

$$\begin{aligned} &J(x_1 + \varphi(x_1) + r(y_1 - \varphi(x_1))) - J(x_1 + \varphi(x_1) + y_1 - \varphi(x_1)) \\ &\quad = DJ(x_1 + \varphi(x_1) + t_x(y_1 - \varphi(x_1)))((r-1).(y_1 - \varphi(x_1))) \\ &\quad = \frac{r-1}{t_x}DJ(x_1 + \varphi(x_1) + t_x(y_1 - \varphi(x_1)))(t_x.(y_1 - \varphi(x_1))) \\ &\quad = \frac{r-1}{t_x}[DJ(x_1 + \varphi(x_1) + t_x(y_1 - \varphi(x_1))) - DJ(x_1 + \varphi(x_1))][t_x(y_1 - \varphi(x_1))] \\ &\quad < 0, \end{aligned}$$

which is absurd and ψ should be one-to-one on $B_{H^+}(0, \epsilon_1) \times B_{H^-}(0, \delta)$. □

Lemma 2.6. *There is a positive real number $\varepsilon < \varepsilon_0$ such that*

$$B_{H^+}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}}) \times B_{H^-}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}}) \subset \psi(B_{H^+}(0, \epsilon) \times B_{H^-}(0, \delta)).$$

Proof. By the mean value theorem, for every $y \in B'_{H^-}(0, \delta) \setminus \{0\}$ there exists $t_y \in (0, 1)$ such that

$$J(y) = J(y) - J(0) = DJ(t_y.y)y = \frac{-1}{t_y}.DJ(t_y.y)(-t_y.y) < 0.$$

Since $\partial B'_{H^-}(0, \delta)$ is compact, it implies that there exists a positive real number C such that

$$J(y) < -C \qquad \forall\ y \in \partial B'_{H^-}(0, \delta). \tag{2.6}$$

We shall prove that there exists a positive real number $\epsilon \leq \frac{\epsilon_1}{2}$ such that

$$J(x + y) \leq 0 \qquad \forall\ (x, y) \in B'_{H^+}(0, \epsilon) \times \partial B'_{H^-}(0, \delta). \tag{2.7}$$

Assume by contradiction that there exists a sequence $\{(x_n, y_n)\}$ in $B'_{H^+}(0, \epsilon) \times \partial B'_{H^-}(0, \delta)$ such that x_n converges to 0 and $J(x_n + y_n) > 0$ for any integer n.

Because $\partial B'_{H^-}(0, \delta)$ is compact, we can assume that $\{y_n\}$ converges to y_0 in $\partial B'_{H^-}(0, \delta)$. Thus, by the continuity of J, we see that $J(y_0) \geq 0$, which contradicts (2.6). Therefore we get (2.7).

We choose ε in (2.7) such that

$$\varphi(B_{H^+}(0, \epsilon)) \ \subset \ B_{H^-}(0, \delta/2). \tag{2.8}$$

Now, fixing x in $B'_{H^+}(0,\epsilon)$, by the mean value theorem and condition (iv) there exists $s_x \in (1/2, 1)$ such that

$$\begin{aligned} J(x+\varphi(x)) \geq J(x) \;&>\; J(x) - J(x/2) \;=\; DJ(s_x x)(x/2) \\ &= \frac{1}{2s_x} DJ(s_x x)(s_x . x) > \;\frac{1}{2} p(||s_x x||) \;\geq\; \frac{1}{2} p(||x/2||). \end{aligned} \tag{2.9}$$

Combining (2.7) and (2.9) we get

$$J(x+\varphi(x)) - J(x+y) \geq J(x+\varphi(x)) \geq J(x) \;>\; \frac{1}{2}p(||x/2||) \;=\; \frac{p(\epsilon/2)}{2} \quad \forall\ (x,y) \in \partial B'_{H^+}(0,\epsilon) \times \partial B'_{H^-}(0,\delta).$$

By the continuity of J and the definition of ψ_1, it follows that

$$\{\frac{t}{||x||}x : 0 \leq t \leq \sqrt{\frac{p(\epsilon/2)}{2}}\} \subset \psi_1(B_{H^+}(0,\epsilon) \times B_{H^-}(0,\delta))\ \forall\ x \in \partial B_{H^+}(0,\epsilon). \tag{2.10}$$

Now let $z \in \partial B_{H^-}(0,\delta/2)$ and let $x \in B_{H^+}(0,\epsilon)$. By (2.8), there exists $k_z \geq 1$ such that $y = k_z.z + \varphi(x)$ belongs to $\partial B_{H^-}(0,\delta)$. Thus by the continuity of J, (2.9) and the definition of ψ_2, we have

$$\{\frac{t}{||z||}z : 0 \leq t \leq \sqrt{\frac{p(\epsilon/2)}{2}}\} \subset \psi_2(B_{H^+}(0,\epsilon) \times B_{H^-}(0,\delta)). \tag{2.11}$$

Combining (2.10) and (2.11) proves the lemma. □

Lemma 2.7. *Put*

$$U = (B_{H^+}(0,\epsilon) \times B_{H^-}(0,\delta)) \cap \psi^{-1}(B_{H^+}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}}) \times B_{H^-}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}}))$$

and let ϕ be the restriction of ψ^{-1} on $B_{H^+}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}}) \times B_{H^-}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}})$.

Then

(i) *ϕ is continuous on $B_{H^+}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}}) \times B_{H^-}(0, \frac{\sqrt{p(\epsilon/2)}}{\sqrt{2}})$ and*

(ii) $J(\phi(x+y)) = ||x||^2 - ||y||^2\ \forall\ (x,y) \in B_{H^+}(0, \sqrt{\frac{p(\epsilon/2)}{2}}) \times B_{H^-}(0, \sqrt{\frac{p(\epsilon/2)}{2}})$.

Proof. Let $(x_0, y_0) \in U$ and let $\{(x_n, y_n)\}$ be a sequence in U such that $\{\psi(x_n, y_n)\}$ converges to $\psi(x_0, y_0)$. We prove that $\{(x_n, y_n)\}$ converges to (x_0, y_0). By definition we see that $\{\psi_1(x_n, y_n)\}$ converges to $\psi_1(x_0, y_0)$.

We prove that $\{x_n\}$ converges to x_0. We consider two cases:

Case 1. $x_0 = 0$. We show that $\{x_n\}$ converges to 0. First note that $\{||\psi_1(x_n, y_n)||\}$ converges to 0. Thus, $\{j(x_n)\}$ converges to 0. Arguing as in Lemma 2.6, we have $j(x_n) > \frac{1}{2}p(||x_n/2||)$. Therefore $\{x_n\}$ converges to 0 because of the non-decreasing of p.

Case 2. $x_0 \neq 0$. We prove that $\{x_n\}$ converges to x_0. Note that $\{||\psi_1(x_n, y_n)||\}$ converges to $||\psi_1(x_0, y_0)||$. Thus, $\{j(x_n)\}$ converges to $j(x_0) > 0$ and $\{x_n\}$ does not converge to 0.

Since $\{\frac{\sqrt{j(x_n)}}{||x_n||}x_n\}$ converges to $\frac{\sqrt{j(x_0)}}{||x_0||}x_0$, it follows that $\{\sqrt{j(x_n)}\}$ converges to $\sqrt{j(x_0)}$ and then $\{\frac{x_n}{||x_n||}\}$ converges to $\frac{x_0}{||x_0||}$. On the other hand, $||x_n|| < \delta$

for every $n \in \mathbb{N}$. Thus we may assume that $\{||x_n||\}$ converges to $\alpha \in (0, \delta]$. This implies $\{x_n\}$ converges to $\dfrac{\alpha}{||x_0||}x_0$.

By the continuity of j, we get $\{j(x_n)\}$ converges to $j(\dfrac{\alpha}{||x_0||}x_0)$, which implies $j(x_0) = j(\dfrac{\alpha}{||x_0||}x_0)$. Since ψ is one-to-one on U, we get $\alpha = ||x_0||$. Therefore $\{x_n\}$ converges to x_0.

Furthermore, by the compactness of $B'_{H^-}(0, \delta)$, we can suppose that $\{y_n\}$ converges to y in $B'_{H^-}(0, \delta)$. By the continuity of φ and J, we see that $\{\varphi(x_n)\}$ and $\{\sqrt{J(x_n + \varphi(x_n)) - J(x_n + y_n)}\}$ converge to $\varphi(x_0)$ and $\sqrt{J(x_0 + \varphi(x_0)) - J(x_0 + y)}$ respectively. Thus we get $\psi_1(x_0, y_0) = \psi_1(x_0, y)$ and $\psi_2(x_0, y_0) = \psi_2(x_0, y)$. Therefore by Lemma 2.5, $(x_0, y_0) = (x_0, y)$ and ϕ is continuous on U.

Using (2.5), we get (i) and (ii). $\square$

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