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FIXED POINT INDICES AND INVARIANT PERIODIC SETS OF HOLOMORPHIC SYSTEMS

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Abstract. This note presents an intuitive method to study center families of periodic orbits of complex holomorphic differential equations near singularities, based on some iteration properties of fixed point indices. As an application of this method, we will prove Needham's theorem in a more general version.

1. Fixed point indices of holomorphic mappings

Let $\mathbb{C}^n$ be the complex vector space of dimension n, let U be an open set in $\mathbb{C}^n$ and let $f : U \to \mathbb{C}^n$ be a holomorphic mapping. If $p \in U$ is an isolated zero of f, say, there exists a bounded open set V with $p \in V \subset \overline{V} \subset U$ such that p is the unique solution of the equation $f(x) = 0$ in $\overline{V}$. Then we can define the zero index of f at p by

$$\pi_f(p) = \#\{x \in V; f(x) = q\},$$

where q is a regular value of f such that $|q|$ is small enough and $\#$ denotes the cardinality. $\pi_f(p)$ is well defined (see [9] or [17] for the details).

If $f : U \to \mathbb{C}^n$ is a holomorphic mapping and p is an isolated fixed point of f, then there is a ball B in U centered at p so that p is the unique fixed point of f in $\overline{B}$, in other words, p is the unique zero of the mapping

$$f - I : \overline{B} \to \mathbb{C}^n,$$

which puts each $x \in \overline{B}$ into $f(x) - x$, and then the *fixed point index* of f at p is well defined by

$$\mu_f(p) = \pi_{f-I}(p).$$

Fixed point indices of holomorphic mappings have the geometric properties stated in the following lemmas (see [16] and [17]). We will denote by Δ^n a ball in $\mathbb{C}^n$ centered at the origin.

Lemma 1. *Let $f : \Delta^n \to \mathbb{C}^n$ be a holomorphic mapping such that $0 \in \Delta^n$ is an isolated fixed point of f. Then $\mu_f(0) \geq 1$, and the equality holds if and only if the Jacobian matrix $f'(0)$ of f at 0 has no eigenvalue equal to 1.*

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Lemma 2. *Let U be an open and bounded set in $\mathbb{C}^n$ and let $f : \overline{U} \to \mathbb{C}^n$ be a holomorphic mapping such that f has no fixed point on the boundary ∂U. Then f has only finitely many fixed points in U.*

Lemma 3. *Let $f : \overline{\Delta^n} \to \mathbb{C}^n$ be a holomorphic mapping such that 0 is the unique fixed point of f in $\overline{\Delta^n}$. Then for any holomorphic mapping $g : \overline{\Delta^n} \to \mathbb{C}^n$ that is sufficiently close to f on the boundary $\partial\Delta^n$, g has only finitely many fixed points in Δ^n and*

$$\mu_f(0) = \sum_{\substack{g(x)=x\\ x\in\Delta^n}} \mu_g(x).$$

This lemma gives us an intuitive interpretation of the fixed point index: with the same condition of this lemma, for any holomorphic mapping $g : \overline{\Delta^n} \to \mathbb{C}^n$ such that g is sufficiently close to f on the boundary $\partial\Delta^n$ and all fixed points of g in Δ^n are simple, g has exactly $\mu_f(0)$ distinct fixed points in Δ^n. A fixed point x of g is called *simple* if $\det(g'(x) - I) \neq 0$, say, the Jacobian matrix $g'(x)$ has no eigenvalue 1, where I is the unit matrix.

Let $f : \Delta^n \to \mathbb{C}^n$ be a holomorphic mapping such that 0 is a fixed point of f. Then for any positive integer k, the k-th iteration f^k of f is well defined in some neighborhood V_k of 0, where $f^1 = f$, $f^2 = f \circ f$, $f^3 = f \circ f \circ f$, and so on. The following result can be found in a more general and more precise version in [16] when $n = 2$, and when $n > 2$ the proof is similar and will be given in the Appendix.

Proposition 1. *Let $m > 1$ be a prime number and let $f : \Delta^n \to \mathbb{C}^n$ be a holomorphic mapping such that $0 \in \Delta^n$ is an isolated fixed point of both f and f^m. If $f'(0)$ has an eigenvalue that is a primitive m-th root of 1, then*

$$\mu_{f^m}(0) > m. \tag{1.1}$$

A complex number λ is called a primitive m-th root of 1, if $\lambda^m = 1$ but $\lambda^k \neq 1$ for $k = 1, \dots, m-1$. The following lemma is due to M. Shub and D. Sullivan [14].

Lemma 4. *Let $m > 1$ be a positive integer and let $\Theta : \Delta^n \to \mathbb{C}^n$ be a holomorphic mapping with an isolated fixed point at the origin $0 \in \Delta^n$. Assume that for each eigenvalue λ of $\Theta'(0)$, either $\lambda = 1$ or $\lambda^m \neq 1$. Then 0 is still an isolated fixed point of Θ^m.*

In the rest of this section we will consider a holomorphic system

$$\dot{x} = \frac{dx}{dt} = F(x), x \in \Delta^n, \tag{1.2}$$

where $F : \Delta^n \to \mathbb{C}^n$ is a holomorphic mapping such that 0 is an isolated zero of F, say, 0 is an isolated singularity of the system. The following result is well known.

Lemma 5. *For any $\tau > 0$, there is a ball $B \subset \Delta^n$ centered at the origin 0 such that the local flow $\phi(t, x)$ of (1.2) is real analytic on $[0, \tau] \times B$, holomorphic with respect to x, $\phi([0, \tau] \times B) \subset \Delta^n$ and, putting $\Phi_\tau(x) = \phi(\tau, x)$, the Jacobian matrices $\Phi'_\tau(0)$ and $F'(0)$ of Φ_τ and F at 0, respectively, are related by*

$$\Phi'_\tau(0) = e^{\tau F'(0)}. \tag{1.3}$$

Now, we can prove the following result as a consequence of Proposition 1.

Corollary 1. *Consider the holomorphic system* (1.2) *and assume the Jacobian matrix* $F'(0)$ *has an eigenvalue* ωi, *with* $\omega \neq 0$ *being real. Let* $\Phi(x) = \phi(\frac{2\pi}{|\omega|}, x)$ *be the time* $\frac{2\pi}{|\omega|}$ *mapping of the local flow* $\phi(t,x)$ *of the system. Then* 0 *is an accumulated fixed point of* Φ.

Proof. By Lemma 5, Φ is well defined and holomorphic in a neighborhood of 0. To prove the conclusion by contradiction, we assume that 0 is an isolated fixed point of Φ. Then $\mu = \mu_\Phi(0)$ is well defined.

Let m be a prime number with $m > \mu_\Phi(0)$. Then 0 is an isolated fixed point of both $\Psi(x) = \phi(\frac{2\pi}{m|\omega|}, x)$ and the m-th iteration Ψ^m of Ψ, for

$$\Psi^m(x) = \phi(\frac{m2\pi}{m|\omega|}, x) = \Phi(x)$$

in a neighborhood of 0. The previous equality also implies that

$$\mu_{\Psi^m}(0) = \mu_\Phi(0) = \mu. \tag{1.4}$$

On the other hand, it follows from (1.3) that $\Psi'(0)$ has an eigenvalue equal to $e^{\frac{2\pi\omega}{m|\omega|}i}$, which is a primitive m-th root of 1, and then by Proposition 1,

$$\mu_{\Psi^m}(0) > m > \mu.$$

This contradicts (1.4). Therefore, 0 cannot be an isolated fixed point of Φ. $\square$

This corollary is the key ingredient of our method for studying invariant varieties of holomorphic systems. After some analysis about the fixed point set $Fix(\Phi)$ of Φ we will be able to prove that $Fix(\Phi)$ contains an analytic variety of complex dimension at least 1, consisting of 0 and periodic orbits of the same period of the system. Under certain conditions, we will also be able to prove that $Fix(\Phi)$ consists of periodic orbits of the same period $\frac{2\pi}{|\omega|}$. Here the period always means the least positive period.

2. Periodically invariant varieties of holomorphic systems

2.1. **Centers of planar holomorphic systems.** In the history of the qualitative theory of planar ODEs, one of the most interesting problems is to find the condition such that a singularity of an ODE is a *center* or *isochronous center*.

Let U be a domain in $\mathbb{R}^2$ and let $G \in C^k(U, \mathbb{R}^2), k \geq 1$. For the planar ODE

$$\dot{x} = G(x), x \in U, \tag{2.1}$$

an isolated singular point $p \in U$ is called a *center* if and only if there exists a punctured neighborhood $V \subset U$ of p, consisting of periodic orbits of (2.1) surrounding p, and p is called an *isochronous center* if and only if it is a center and all orbits of (2.1) near p have the same period.

Up to now, several classes of systems have been studied extensively, in relation to the existence of isochronous centers (see [2]). Among them, the equation

$$\dot{z} = P(z), z \in U \subset \mathbb{R}^2 \cong \mathbb{C},$$

where P is a complex holomorphic function defined on U, were considered in [1], [3], [5], [6], [7], [8], [12], [13] and [15], etc.

The following result about isochronous centers of complex holomorphic equations was given by Gregor [6] in 1958.

Theorem 1 (Gregor)**.** *Consider the system*

$$\dot{z} = P(z), z \in U \subset \mathbb{C}, \tag{2.2}$$

where P is a holomorphic function defined on U. A simple zero $0 \in U$ of P is a center of system (2.2) if and only if $P'(0)$ is pure imaginary. In this case 0 is an isochronous center and the common period of each cycle surrounding 0 is $T = \frac{2\pi}{|P'(0)|}$.

This result can be found summarized in [7], and several proofs of this theorem can be found in [1], [3], [8], [12] and [15].

2.2. **Invariant periodic sets of holomorphic systems in $\mathbb{C}^n$.** Now, consider an n-dimensional complex holomorphic system

$$\dot{x} = F(x), x \in \Delta^n, \tag{2.3}$$

where Δ^n is a ball centered at the origin 0 in $\mathbb{C}^n$ and $F : \Delta^n \to \mathbb{C}^n$ is a holomorphic mapping such that 0 is an isolated zero of F, say, 0 is an isolated singularity of (2.3).

When $n = 2$, as a generalization of Gregor's theorem, Needham and McAllister [11] proved that if

$$F'(0) = \begin{pmatrix} \mu i & 0 \\ 0 & \lambda \end{pmatrix},$$

where $\mu \neq 0$ is a real number and $\lambda \neq 0$ is a complex number, then there exists a one-dimensional complex manifold consisting of 0 and periodic orbits of (2.3) of the same period.

For arbitrary n, Needham [10] proved the following theorem.

Theorem 2 (Needham)**.** *If $F'(0)$ has a nonzero pure imaginary eigenvalue μi and if the other eigenvalues all have nonzero real parts, then there uniquely exists a one-dimensional complex submanifold of a neighborhood of 0 in Δ^n, consisting of 0 and periodic orbits of (2.3) of the same period $\frac{2\pi}{|\mu|}$.*

In this note, we shall use Corollary 1, together with some analysis of fixed point sets of certain moment mapping of the local flow of (2.3), to prove Needham's theorem in more general versions.

Theorem 3. *There is a complex analytic variety in Δ^n with pure complex dimension at least 1 consisting of 0 and periodic orbits of (2.3) of the same period, if and only if $F'(0)$ has a nonzero pure imaginary eigenvalue.*

Theorem 4. *If $F'(0)$ has a nonzero pure imaginary eigenvalue ωi and if for any other eigenvalue λ of $F'(0)$,*

$$\lambda / (\omega i) \neq \pm 2, \pm 3, \ldots,$$

then there exists a complex analytic variety in Δ^n of pure complex dimension at least 1, consisting of 0 and periodic orbits of (2.3) of the same period $\frac{2\pi}{|\omega|}$.

Theorem 5. *If $F'(0)$ has a nonzero pure imaginary eigenvalue ωi and if for any other eigenvalue λ of $F'(0)$,*

$$\lambda / (\omega i) \neq 0, \pm 1, \pm 2, \ldots,$$

then there uniquely exists a complex analytic disk D in Δ^n consisting of 0 and periodic orbits of (2.3) of the same period $\frac{2\pi}{|\omega|}$.

The term *pure dimension* means that the variety has the same dimension everywhere, the term *analytic disk* means that D is a holomorphic embedding of a disk in the complex plane and the uniqueness of D means that any analytic disk satisfying the condition in the theorem coincides with D in a neighborhood of 0 in $\mathbb{C}^n$.

Remark 1. The method in this note can also be applied to studying the analyticity of stable manifolds and unstable manifolds of holomorphic differential equations, by parameter transformation of the time t: we can use any line in the complex plane to replace the real line of t.

For example, if we replace the time t by $(a+ib)t$ with $b \neq 0$, then the invariant varieties in Theorems 3 to 5 become stable, or unstable, invariant sets in the new systems.

3. Proof of Theorems 3–5

Lemma 6. *There exist positive numbers δ and T_0, such that for each T in the interval $(0, T_0]$, the system (2.3) has no periodic orbit of period T intersecting $\overline{B}_\delta$, where $B_\delta = \{x \in \mathbb{C}^n; |x| < \delta\}$.*

Proof. Let $\phi = \phi(t,x)$ be the local flow of (2.3). By Lemma 5, for any $T_0 > 0$, there is a $\delta > 0$, such that $\phi(t,x)$ is real analytic on $[0,T_0] \times \overline{B}_\delta$ and complex holomorphic with respect to x. We shall show that the conclusion holds for sufficiently small δ and sufficiently small T_0.

Otherwise, for any fixed δ and T_0, there exist sequences $\{\eta_j\}$ in the interval $(0,\delta)$ and $\{t_j\}$ in the interval $(0,T_0)$ such that

$$\eta_j \to 0 \text{ and } t_j \to 0 \text{ as } j \to \infty, \tag{3.1}$$

and that for each j, (2.3) has a periodic orbit $\mathcal{O}_j$ of period t_j intersecting B_{η_j}.

By (3.1) and the fact that $\phi(0,x) \equiv x$ we have

$$\max_{x \in \overline{B}_\delta, 0 \le t \le t_j} |\phi(t,x) - x| \to 0 \text{ as } j \to \infty. \tag{3.2}$$

Considering that $\mathcal{O}_j \cap B_{\eta_j}$ contains infinitely many fixed points of the time t_j mapping $\Phi_{t_j}(x) = \phi(t_j, x)$ of the local flow ϕ and $\eta_j < \delta$, we conclude by Lemma 2 that the time t_j mapping Φ_{t_j} has a fixed point $x_j \in \partial B_\delta$ (the boundary of B_δ), for each j. Without loss of generality, assume

$$\lim_{j\to\infty} x_j = x_0 \tag{3.3}$$

for some $x_0 \in \partial B_\delta$.

We assume that δ is small enough such that 0 is the unique singularity of (2.3) in $\overline{B}_\delta$. Then there exist positive numbers t_0 and θ_0 with $t_0 < T_0$, such that

$$|\phi(t_0, x_0) - x_0| > \theta_0.$$

Thus, by (3.3), for sufficiently large j,

$$|\phi(t_0, x_j) - x_j| > \theta_0.$$

On the other hand, since x_j is a fixed point of the time t_j mapping Φ_{t_j}, we have that $\phi(t_j, x_j) = x_j$ for each j, and then, putting $t_0 = m_j t_j + r_j$ for each j, where

m_j is an integer and $0 \le r_j < t_j$, we have

$$\begin{aligned} & |\phi(r_j, x_j) - x_j| = |\phi(r_j, \phi(m_j t_j, x_j)) - x_j| \\ = \quad & |\phi(m_j t_j + r_j, x_j) - x_j| = |\phi(t_0, x_j) - x_j| > \theta_0. \end{aligned}$$

This contradicts (3.2), for $0 \le r_j < t_j$ and $x_j \in \partial B_\delta$. This completes the proof. $\square$

To prove Theorems 3–5, we need some known results about analytic sets. The reader is referred to [4], pages 14–57, for the details.

Lemma 7. *Let $\mathcal{A}$ be an analytic subset of an open subset of $\mathbb{C}^n$. Then for any $p \in \mathcal{A}$, the (complex) dimension $\dim_p \mathcal{A}$ of $\mathcal{A}$ at p is at least 1 if and only if p is an accumulated point of $\mathcal{A}$.*

Lemma 8. *If $\dim_p \mathcal{A} \ge 1$, then $\mathcal{A}$ has an irreducible component $\mathcal{B}$ containing p, with a pure complex dimension at least 1.*

Lemma 9. *Any irreducible component of $\mathcal{A}$ is the closure in $\mathcal{A}$ of some connected component of $\mathrm{reg}\mathcal{A}$, where $\mathrm{reg}\mathcal{A}$ is the set of all regular points of $\mathcal{A}$.*

A point $p \in \mathcal{A}$ is called regular, if there is a neighborhood V_p of p in $\mathbb{C}^n$, such that $V_p \cap \mathcal{A}$ is a complex submanifold of V_p.

Proof of Theorem 3. We first proof the necessity. Assume that there exists an analytic variety Σ of pure complex dimension at least 1, consisting of the singularity 0 and periodic orbits of (2.3) of the same period τ_0 (note that the period always indicates the least positive period).

By Lemma 6, there exist positive numbers δ and T_0 ($< \tau_0$) such that for each $T \in (0, T_0]$, the system (2.3) has no periodic orbit of period T intersecting $\overline{B}_\delta$.

Let M be a prime number such that $\tau_0/M < T_0$ and let $\Phi(x) = \phi(\tau_0/M, x)$ be the time τ_0/M mapping of the local flow $\phi(t,x)$ of (2.3). Then 0 is the unique fixed point of Φ located in $\overline{B}_\delta$. On the other hand, by the assumption about Σ, 0 is an accumulated point of Σ and all points of Σ are fixed points of $\Phi^M(x) = \phi(\tau_0, x)$, and then 0 is an accumulated point of fixed points of Φ^M.

Thus by Lemma 4, $\Phi'(0)$ has an eigenvalue Λ such that $\Lambda \ne 1$ but $\Lambda^M = 1$, which implies by Lemma 5 that $F'(0)$ has an eigenvalue λ with $\Lambda = e^{\frac{\tau_0 \lambda}{M}} \ne 1$ and $\Lambda^M = e^{\tau_0 \lambda} = 1$. Clearly, $\lambda = \omega i$ fore some real number $\omega \ne 0$. This completes the proof of the necessity.

The sufficiency follows from Theorem 4 directly. $\square$

Proof of Theorem 4. Assume $F'(0)$ has an eigenvalue ωi such that $\omega \ne 0$ is a real number and for any other eigenvalue λ

$$\lambda / (\omega i) \ne \pm 2, \pm 3, ... \tag{3.4}$$

Then by Corollary 1, the origin 0 is an accumulated point of fixed points of the time $\frac{2\pi}{|\omega|}$ mapping $\Phi = \phi(\frac{2\pi}{|\omega|}, \cdot)$ of the local flow $\phi(t,x)$ of (2.3) defined in a neighborhood of the origin.

By Lemma 5, for some ball B centered at the origin 0, $\phi(t,x)$ is real analytic on $[0, \frac{2\pi}{|\omega|}] \times \overline{B}$ and holomorphic with respect to $x \in \overline{B}$. Let $\mathcal{A}_B$ be the set of all fixed points of the mapping Φ in

$$V_B = \phi([0, \frac{2\pi}{|\omega|}] \times B) = \{\phi(t,x); t \in [0, \frac{2\pi}{|\omega|}], x \in B\}.$$

Then $\mathcal{A}_B$ is an analytic subset of V_B, and by Lemma 7 we have $\dim_0 \mathcal{A}_B \geq 1$, and then, by Lemma 8, there exists an irreducible component Σ_B of $\mathcal{A}_B$ containing p with pure complex dimension at least 1.

We first show that $\Sigma_B \backslash \{0\}$ consists of periodic orbits of (2.3). It suffices to prove that Σ_B is invariant by the flow $\phi(t,x)$. Note that by the definition of V_B, $\mathcal{A}_B$ is invariant by the local flow ϕ, say, $\phi(t,\mathcal{A}_B) = \mathcal{A}_B$ for all real t.

By Lemma 9, there exists a connected component S of $\mathrm{Reg}\mathcal{A}_B$ such that Σ_B is the closure of S in $\mathcal{A}_B$. Then each $p \in S$ has a neighborhood U_p in V_B, such that $S \cap U_p$ is a complex submanifold of U_p and S is the only connected component of $\mathcal{A}_B$ intersecting U_p, and then Σ_B is the only irreducible component of $\mathcal{A}_B$ intersecting U_p by Lemma 9. On the other hand, for sufficiently small $\varepsilon > 0, \Phi_\varepsilon(\Sigma_B) = \phi(\varepsilon, \Sigma_B)$ must intersect U_p, and then by the fact that $\Phi_\varepsilon = \phi(\varepsilon, \cdot)$ is a biholomorphic mapping from V_B onto $\Phi_\varepsilon(V_B)$, $\Phi_\varepsilon(\Sigma_B)$ is also an irreducible component of $\Phi_\varepsilon(\mathcal{A}_B) = \mathcal{A}_B$. (Recall that $\mathcal{A}_B$ is invariant by the flow $\phi(t,x)$.) Therefore, $\Phi_\varepsilon(\Sigma_B)$ and Σ_B are both irreducible components of $\mathcal{A}_B$ and both intersect U_p, and then $\Phi_\varepsilon(\Sigma_B) = \Sigma_B$, which implies that $\phi(t,\Sigma_B) = \Sigma_B$ for all real t. Thus Σ_B is invariant by ϕ and then $\Sigma_B \backslash \{0\}$ consists of periodic orbits.

It is clear that for each periodic orbit $\mathcal{O} \subset \Sigma_B \backslash \{0\}$, the period $T(\mathcal{O})$ of $\mathcal{O}$ equals to $\frac{2\pi}{|\omega| m_{\mathcal{O}}}$ for some positive integer $m_{\mathcal{O}}$. By Lemma 6, we may assume that B is small enough such that for some fixed $T_0 > 0$, the period of each orbit in $\Sigma_B \backslash \{0\}$ is larger than T_0. Then for each periodic orbit $\mathcal{O} \subset \Sigma_B$ we have $T_0 \leq T(\mathcal{O}) = \frac{2\pi}{|\omega| m_{\mathcal{O}}}$, with

$$m_{\mathcal{O}} \in \{1, 2, \ldots, m^*\}, \tag{3.5}$$

where m^* is the integral part of $2\pi \left(T_0|\omega|\right)^{-1}$.

We assert that if the ball B is small enough, then for each periodic orbit $\mathcal{O} \subset \Sigma_B \backslash \{0\}$, $m_{\mathcal{O}} = 1$. Otherwise, by (3.5) there is a fixed integer $m > 1$, and a sequence $\{x_k\} \subset \Sigma_B \backslash \{0\}$, such that $x_k \to 0$ as $k \to \infty$ and the periodic orbit passing through x_k has period $\frac{2\pi}{|\omega| m}$ for each k, say, $\phi(\frac{2\pi}{|\omega| m}, x_k) = x_k$ for each k.

Let M be any given prime number with $\frac{2\pi}{|\omega| mM} < T_0$. Then 0 is an isolated fixed point of $\Theta(x) = \phi(\frac{2\pi}{|\omega| mM}, x)$, but 0 is an accumulated point of fixed points of Θ^M, for

$$\Theta^M(x_k) = \phi(\frac{2\pi}{|\omega| m}, x_k) = x_k, k = 1, 2, \ldots.$$

Therefore, by Lemma 4, the Jacobian matrix $\Theta'(0)$ must have an eigenvalue Λ with $\Lambda \neq 1$ but $\Lambda^M = 1$. Hence, by (1.3), we have $\Lambda = e^{\frac{2\pi}{|\omega| mM}\lambda}$ for some eigenvalue λ of $F'(0)$ such that $\lambda \neq 0$ and $\frac{2\pi}{|\omega| m}\lambda$ is a multiple of $2\pi i$, and then $\lambda/(\omega i) = \pm km$ for some fixed positive integer k, which contradicts (3.4). Therefore, we have $m = 1$. The proof is complete. $\square$

Proof of Theorem 5. Assume that $\lambda_1, \ldots, \lambda_n$ are the n eigenvalues of the Jacobian matrix $F'(0)$ of F at 0, with $\lambda_1 = \omega i$ and

$$\lambda_l/\lambda_1 \neq 0, \pm 1, \pm 2, \pm 3, ..., l = 2, 3, \ldots, n. \tag{3.6}$$

Ignoring a linear transform of the phase space, we may assume that the Jacobian matrix $F'(0)$ is lower triangular and has $\lambda_1, \lambda_2, ..., \lambda_n$ down its main diagonal. We write this by

$$F'(0) = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n). \tag{3.7}$$

By Lemma 5, there exists a ball B centered at the origin 0 such that the local flow $\phi(t,x)$ of (2.3) is real analytic on $[0,\frac{2\pi}{|\omega|}]\times\overline{B}$ and complex holomorphic with respect to $x\in B$.

By (3.7) and Lemma 5,

$$\Lambda_l = e^{\frac{2\pi}{|\omega|}\lambda_l}, l=1,2,\dots,n,$$

are all the eigenvalues of the Jacobian matrix $\Phi'(0)$ of the time $\frac{2\pi}{|\omega|}$ mapping $\Phi(x)=\phi(\frac{2\pi}{|\omega|},x)$ at 0, and by (3.6) we have

$$\Lambda_l\neq 1, l=2,\dots,n. \tag{3.8}$$

It also follows from (3.7) and Lemma 5 that $\Phi'(0)=(1,\Lambda_2,\Lambda_3\dots,\Lambda_n)$ is a lower triangular matrix which has $1,\Lambda_2,\Lambda_3\dots,\Lambda_n$ down its main diagonal. Therefore, putting $x=(x_1,x_2,\dots,x_n)$ and $\Phi=(\varphi_1,\varphi_2,\dots,\varphi_n)$, by (3.8) we have

$$\det\left(\frac{\partial\left(\varphi_2,\varphi_3,\dots,\varphi_n\right)}{\partial\left(x_2,x_3,\dots,x_n\right)}-I_{n-1}\right)\Bigg|_{x=0}\neq 0,$$

where I_{n-1} is the $(n-1)\times(n-1)$ unit matrix. So, by the implicit function theorem, there uniquely exist one variable complex holomorphic function $x_l=x_l(x_1), l=2,\dots,n$, defined in a neighborhood of the origin in the complex plane $\mathbb{C}$, solving the system of the equations

$$\varphi_l(x_1,x_2,\dots,x_n)=x_l, l=2,\dots,n,$$

in a neighborhood of the origin 0, with

$$x_l(0)=0, l=2,3,\dots,n. \tag{3.9}$$

Therefore, in a neighborhood of the origin 0 in $\mathbb{C}^n$, the fixed point equation

$$\Phi(x_1,x_2,\dots,x_n)=(x_1,x_2,\dots,x_n) \tag{3.10}$$

is equivalent to the system of the n equations

$$\begin{cases} x_1=\varphi_1(x_1,x_2(x_1),\dots,x_n(x_1)),\\ x_l=x_l(x_1), l=2,\dots,n.\end{cases} \tag{3.11}$$

By Corollary 1, 0 is an accumulated point of fixed points of Φ, and then 0 is an accumulated point of zeros of the holomorphic function $x_1-\varphi_1(x_1,x_2(x_1),\dots,x_n(x_1))$ which is defined in a neighborhood of the origin 0 in $\mathbb{C}$. Therefore, we have

$$\varphi_1(x_1,x_2(x_1),\dots,x_n(x_1))\equiv x_1$$

in a neighborhood of $x_1=0$ in $\mathbb{C}$, and then (3.11) reads

$$\begin{cases} x_1=x_1,\\ x_l=x_l(x_1), l=2,\dots,n.\end{cases} \tag{3.12}$$

It is clear that there exists a positive number δ such that

$$\Sigma_\delta^*=\{(x_1,\varphi_2(x_1),\dots,\varphi_n(x_1));\ x_1\in\mathbb{C},|x_1|<\delta\}$$

is an analytic disk in Δ^n, which contains 0 by (3.9).

By the equivalence of (3.10) and (3.12), in a neighborhood of 0, Σ_δ^* coincides with the set $\mathcal{A}_B$ of all fixed points of Φ in

$$V_B=\phi([0,\frac{2\pi}{|\omega|}]\times B),$$

and then $0 \in \mathcal{A}_B$ has a neighborhood U_0 in $\mathcal{A}_B$ so that U_0 is an analytic disk in $\mathbb{C}^n$. On the other hand, by Theorem 4 there exists an analytic variety Σ of pure complex dimension at least 1, consisting of 0 and periodic orbits of the same period $\frac{2\pi}{|\omega|}$ of the system (2.3), and then by the equivalence of (3.10) and (3.12), $0 \in \Sigma$ has a neighborhood V_0 in Σ such that $V_0 \subset U_0$, which implies that the dimension of Σ is equal to 1, and then V_0 coincides with U_0 in a neighborhood of 0.

Now, we can conclude that Σ, Σ_δ^* and $\mathcal{A}_B$ coincide in a neighborhood of the origin. Therefore, when B is small enough, $\mathcal{A}_B$ is contained in $\Sigma \cap \Sigma_\delta^*$, and then $\mathcal{A}_B$ contains an analytic disk D consisting of 0 and periodic orbits of the same period $\frac{2\pi}{|\omega|}$.

The above argument also shows that any analytic disk D consisting of 0 and periodic orbits of (2.3) with the same period $\frac{2\pi}{|\omega|}$ coincides with Σ_δ^* in a neighborhood of 0, which implies the uniqueness, and the proof is complete. □

4. Appendix

Proof of Proposition 1. By the given condition, there exists a ball $B \subset \Delta^n$ with center 0 such that both f and f^m are well defined in a neighborhood of $\overline{B}$ and 0 is the unique fixed point of both f and f^m in $\overline{B}$.

Let λ be an eigenvalue of $f'(0)$ that is a primitive m-th root of 1. Then the Jacobian matrix $(f^m)'(0) = (f'(0))^m$ of f^m at 0 has the eigenvalue $\lambda^m = 1$, and then by Lemma 1 we have

$$\mu_{f^m}(0) \geq 2. \tag{4.1}$$

We first assume that none of eigenvalues of $f'(0)$ equals 1. Then applying Lemma 1 once more we have

$$\mu_f(0) = 1. \tag{4.2}$$

We can construct a sequence $f_k : \Delta^n \to \mathbb{C}^n$ of holomorphic mappings, converging to f uniformly, such that for each k, 0 is a simple fixed point of both f_k and f_k^m, say, $f_k(0) = f_k^m(0) = 0$, but neither $f_k'(0)$ nor $(f_k^m)'(0)$ has eigenvalue 1. This can be accomplished by small perturbations of the linear part of f at 0.

It is clear that for sufficiently large k, f_k^m is well defined on $\overline{B}$ and f_k^m converges to f^m uniformly on $\overline{B}$ as k tends to ∞. Therefore, by (4.1), Lemma 3 and the condition that 0 is a simple fixed point of f_k^m, we conclude that for sufficiently large k, f_k^m has another fixed point $x_k \neq 0$ in B. On the other hand, since 0 is the unique fixed point of f in $\overline{B}$, by Lemma 3 and (4.2), for sufficiently large k, 0 is the unique fixed point of f_k in B. Thus, we have $f_k(x_k) \neq x_k$.

Now that $f_k(x_k) \neq x_k, f_k^m(x_k) = x_k$ and m is prime, x_k is a periodic point of f_k with period m, say, the periodic orbit $\Gamma(x_k) = \{x_k, f_k(x_k), \dots, f_k^{m-1}(x_k)\}$ contains m distinct points. Since 0 is the unique fixed point of f^m in $\overline{B}$, by the convergence of f_k^m we have that $x_k \to 0$ as $k \to \infty$, and then by the condition $f(0) = 0$ and the convergence of f_k, the periodic orbit $\Gamma(x_k)$ is contained in B for sufficiently large k. Thus for sufficiently large k, $0, x_k, f_k(x_k), \dots, f_k^{m-1}(x_k)$ are $m+1$ distinct fixed points of f_k in B. By Lemma 3, we then have

$$\mu_{f^m}(0) \geq m + 1.$$

Therefore, (1.1) holds.

In general, we can consider the mapping

$$f_\varepsilon = (\varepsilon_1 z_1, \ldots, \varepsilon_n z_n) + f(z_1, \ldots, z_n)$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ is so chosen that $f'_\varepsilon(0)$ has eigenvalue λ but none other eigenvalue equals 1. It is easy to see that we can choose ε sufficiently small so that f_ε^m is sufficiently close to f^m, and then by Lemma 3, we have $\mu_{f^m}(0) \geq \mu_{f_\varepsilon^m}(0) \geq m+1$. This completes the proof. $\square$

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