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MORE ON PARTITIONING TRIPLES OF COUNTABLE ORDINALS

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Abstract. Consider an arbitrary partition of the triples of all countable ordinals into two classes. We show that either for each finite ordinal m the first class of the partition contains all triples from a set of type $\omega + m$, or for each finite ordinal n the second class of the partition contains all triples of an n-element set. That is, we prove that $\omega_1 \to (\omega + m, n)^3$ for each pair of finite ordinals m and n.

1. Background and motivation

If A is a set of ordinals and β is an ordinal, then $[A]^\beta$ denotes the set of all subsets of A which are order isomorphic to β. If α, ν, and $\{\beta_i \mid i < \nu\}$ are ordinals and $m < \omega$, then the *ordinary Ramsey relation* $\alpha \to (\beta_i)^m_{i<\nu}$ means that for each partition $f : [\alpha]^m \to \nu$ there are $i < \nu$ and $X \in [\alpha]^{\beta_i}$ with $f``[X]^m = \{i\}$. As usual, if $\nu < \omega$, then this relation might be written $\alpha \to (\beta_0, \ldots, \beta_{\nu-1})^m$; if $\beta_i = \beta$ for all $i < \nu$, then it might be written $\alpha \to (\beta)^m_\nu$; and the negation of any such relation is indicated by replacing the $\to$ with $\nrightarrow$. The study of these relations (as well as all of the notation defined above) was introduced by P. Erdős and R. Rado in [2].

The ordinary Ramsey theory of the countable ordinals is the theory of such partition relations with $\alpha \leq \omega_1$, those relations which describe the ordinary Ramsey-theoretic properties of individual countable ordinals or of the totality of all countable ordinals.

This theory has been studied thoroughly and quite successfully. Our understanding of it, as witnessed by the results below, is almost complete. In each of the relations listed below, α is an arbitrary countable ordinal, and m and n are arbitrary finite ordinals, unless otherwise indicated.

(1) $\omega \to (\omega)^m_n$ (F. P. Ramsey in [10]).
(2) (a) $\alpha \nrightarrow (\omega + 1, \omega)^2$ (P. Erdős and R. Rado in [2]).
  (b) If $m \geq 3$, then $\alpha \nrightarrow (\omega + 1, m + 1)^m$ (P. Erdős and R. Rado in [2]).
(3) (a) $\omega_1 \to (\omega_1)^1_\omega$.
  (b) If $m \geq 2$, then $\omega_1 \nrightarrow (m + 1)^m_\omega$ (P. Erdős and R. Rado in [2]).
(4) $\omega_1 \to (\alpha)^2_n$ (J. Baumgartner and A. Hajnal in [1]).
(5) (a) $\omega_1 \nrightarrow (\omega_1)^2_2$ (W. Sierpiński in [11]).

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(b) $\omega_1 \to (\omega_1, \omega+1)^2$ (P. Erdős and R. Rado in [2]).
(c) CH implies that $\omega_1 \nrightarrow (\omega_1, \omega+2)^2$ (A. Hajnal in [4]).
(d) It is consistent with Martin's Axiom that $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$ (S. Todorčević in [13]).

(6) (a) $\omega_1 \nrightarrow (\omega_1, 4)^3$ (P. Erdős and R. Rado in [2]).
(b) $\omega_1 \nrightarrow (\omega+2, \omega)^3$ (P. Erdős and R. Rado in [2]).
(c) $\omega_1 \to (\omega+1)^m_n$ (F. Galvin in [3]).
(d) If $m \geq 4$, then $\omega_1 \nrightarrow (\omega+2, m+1)^m$ (P. Erdős and R. Rado in [2]).

Taken together these results leave little of the story of the ordinary Ramsey theory of countable ordinals untold. Essentially only two open problems remain:

(1) For which $\alpha, \beta < \omega_1$ and $n < \omega$ does $\alpha \to (\beta, n)^2$?
(2) For which $\alpha < \omega_1$ and $n < \omega$ does $\omega_1 \to (\alpha, n)^3$?

Each of these questions seems very difficult to resolve, though some progress has been made on each of them.

Regarding the latter problem, E. C. Milner and K. Prikry (among others) have conjectured that the answer is "all of them". We definitely agree with their conjecture, though we have not been able to confirm it. In [9], they were able to prove the conjecture for $\alpha = \omega + \omega + 1$ and $n = 4$. Thus, the simplest unresolved cases of the question are whether $\omega_1 \to (\omega+2, 5)^3$ and whether $\omega_1 \to (\omega+\omega+2, 4)^3$. Below, we resolve the first case (and a little more) affirmatively. Unfortunately, we leave the second unresolved.

2. Our notation

Our basic notation and all terms which we leave undefined are explained in any standard text on set theory (for example, [5] or [7]).

We will code countable ordinals as finite sets of finite ordinals in various ways and will then use these codings to "lift" Ramsey properties of sets of finite ordinals to appropriate sets of countable ordinals. To that end, we will need to be able to quickly express certain relationships between sets of ordinals.

Suppose that x and y are sets of ordinals. If each element of x is less than each element of y, then we will say that x is *supremely less than* y and will write $x \ll y$. Note that if x is supremely less than y, then x and y are necessarily disjoint. Also, we will put $x \ll \emptyset$ for any nonempty set of ordinals x. We will write $x \sqsubseteq y$ and say that x is an *initial segment* of y, if $x \subseteq y$ and $x \ll y \smallsetminus x$. This is the proper analog of the usual notion of initial segments if we identify sets of ordinals with strictly increasing sequences of ordinals in the natural way.

We will call the maximal common initial segment of x and y the *stem* of x and y; we will denote it by $\sigma(x, y)$. What remains in x after the removal of its common stem with y we will call the *branch* of x from y; we will denote it by $\beta(x, y)$. We will denote the order-type of the stem of x and y by $\delta(x, y)$. In summary,

$$\begin{aligned} \sigma(x,y) &= \bigcup \{z \mid z \sqsubseteq x \wedge z \sqsubseteq y\}, \\ \beta(x,y) &= x \smallsetminus \sigma(x,y), \text{ and} \\ \delta(x,y) &= \operatorname{ot}(\sigma(x,y)). \end{aligned}$$

Note that $\sigma(x, y) = \sigma(y, x)$ always, but $\beta(x, y) \neq \beta(y, x)$ unless $x = y$.

The *lexicographic* ordering is defined in the usual way by putting $x <_{\text{lex}} y$ if $x \sqsubset y$ or $\min \beta(x, y) < \min \beta(y, x)$. We also define the *superlexicographic* ordering

by putting $x \ll_{\text{lex}} y$ if $x \sqsubset y$ or $\beta(x,y) \ll \beta(y,x)$. Note that $x \ll_{\text{lex}} y$ if and only if $x \neq y$ and there are sets of ordinals a, b, and c with $a \ll b \ll c$, and such that $x = a \cup b$ and $y = a \cup c$.

For each $k < \omega$ and $A \in [\omega_1]^{\omega^k}$ there is a unique order isomorphism $\pi_A : \langle [\omega]^k, <_{\text{lex}} \rangle \cong \langle A, < \rangle$. For $x \in [\omega]^k$, let $A(x) = \pi_A(x)$ be "the xth element of A".

A family of sets is a *filter base* if the intersection of any finitely many of its elements is infinite. In particular, any subfamily of a nonprincipal filter over ω is a filter base. An infinite set $X \in [\omega]^\omega$ is a *pseudo-intersection* of the filter base $\mathcal{F} \subseteq [\omega]^\omega$ if $X \subseteq^* F$ for every $F \in \mathcal{F}$. (Here, $X \subseteq^* F$ if $|X \smallsetminus F| < \omega$.) The cardinal $\mathfrak{p}$ is the *pseudo-intersection number*, the minimal cardinality of a filter base for which there is no pseudo-intersection. In particular, and most importantly, every filter base of cardinality less than $\mathfrak{p}$ must have a pseudo-intersection. As usual, the cardinal $\mathfrak{c}$ is the cardinality of the continuum. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

A filter $\mathcal{F} \subseteq [\omega]^\omega$ is a *Ramsey filter* if for every pair of finite ordinals m and n and every partition $f : [\omega]^m \to n$ there are $X \in \mathcal{F}$ and $i < n$ such that $f``[X]^m = \{i\}$. If we assume either CH or Martin's Axiom (or more generally that $\mathfrak{p} = \mathfrak{c}$), then such filters are easily constructed.

3. The proof

This section is devoted to a proof of Theorem 1 below. The proof is given in a series of lemmas.[1]

Theorem 1. *$\omega_1 \to (\omega + m, n)^3$ for all $m, n < \omega$.*

The general outline of the proof is based on an idea of J. Baumgartner and A. Hajnal from [1]. We will first prove the theorem with the aid of an additional combinatorial assumption, then demonstrate that we can find a suitable generic extension of the universe in which this combinatorial assumption (and hence the theorem) is true, and finally argue that because we can force the theorem to be true, the theorem must already be true. This method has also been used by S. Todorčević in [14] and E. C. Milner and K. Prikry in [8] and [9].

First, we prove Theorem 1 with the additional assumption that $\mathfrak{p} = \mathfrak{c} > \omega_1$.

Lemma 1. *If $\mathfrak{p} = \mathfrak{c} > \omega_1$, then $\omega_1 \to (\omega + m, n)^3$ for each $m, n < \omega$.*

Proof. Fix $m < \omega$. We will prove the lemma by induction on $n < \omega$. Since the lemma is trivial for $n \leq 3$, we assume that it holds for an arbitrary (but hereafter fixed) finite ordinal $n \geq 3$ and will deduce that it remains true with n replaced by $n + 1$.

Let $f : [\omega_1]^3 \to 2$ be an arbitrary (but hereafter fixed) partition of the triples of ω_1 into two classes. We need to show that either

(a) there is $A \in [\omega_1]^{\omega+m}$ with $f``[A]^3 = \{0\}$, or
(b) there is $B \in [\omega_1]^{n+1}$ with $f``[B]^3 = \{1\}$.

To do that, we will analyze the behavior of f on sets of type ω^k for $k < \omega$.

For each $k < \omega$, each $j < k$, each $A \in [\omega_1]^{\omega^k}$, and each $\beta \in \omega_1$ with $A \ll \{\beta\}$ we define a partition of $[\omega]^{2k-j}$ into two classes $f_{j,k,A,\beta} : [\omega]^{2k-j} \to \{0,1\}$ in the

[1] This result and the proof presented here first appeared in [6], albeit in a more terse form and with a few typographical errors.

following way: decompose each $x \in [\omega]^{2k-j}$ into three sets $x = a \cup b \cup c$ with $|a| = j$, $|b| = |c| = k - j$, and $a \ll b \ll c$, and let

$$f_{j,k,A,\beta}(x) = f_{j,k,A,\beta}(a \cup b \cup c) = f\{A(a \cup b), A(a \cup c), \beta\}.$$

Note that this is basically a partition of all pairs $x, y \in [\omega]^k$ with $x \ll_{\text{lex}} y$ and $\delta(x, y) = j$.

Fix a Ramsey filter $\mathcal{F}$. As $\mathcal{F}$ is Ramsey, for each such j, k, A, and β there are a set $X_{j,k,A,\beta} \in \mathcal{F}$ and an index $i_{j,k,A,\beta} \in \{0, 1\}$ with $f_{j,k,A,\beta}``[X_{j,k,A,\beta}]^{2k-j} = \{i_{j,k,A,\beta}\}$. Let $X_{k,A,\beta} = \bigcap\{X_{j,k,A,\beta} \mid j < k\}$. Note that $X_{k,A,\beta} \in \mathcal{F}$.

Consider the following statement for each $k < \omega$:

$$\begin{aligned}\Phi_k\colon &\text{ There are } j < k,\ A \in [\omega_1]^{\omega^k}, \text{ and } B \in [\omega_1]^{\omega_1} \text{ with } A \ll B \\ &\text{ such that } i_{j,k,A,\beta} = 0 \text{ for every } \beta \in B.\end{aligned}$$

The next lemma shows that any one of the Φ_k's is strong enough to establish Lemma 1 on its own.

Lemma 1.1. *If Φ_k is true for some $k < \omega$, then either* (a) *or* (b) *holds.*

Proof. Assume that Φ_k holds for some $k < \omega$. Let $j < k$, $A \in [\omega_1]^{\omega^k}$, and $B \in [\omega_1]^{\omega_1}$ witness this. Define a partition of $[B]^2$ into two pieces $g : [B]^2 \to \{0, 1\}$ by letting $g\{\beta, \gamma\} = i$ if and only if there is a set $X_{k,A,\beta,\gamma} \in \mathcal{F}$ with $f\{A(x), \beta, \gamma\} = i$ for every $x \in [X_{k,A,\beta,\gamma}]^k$. That g is a well-defined partition of $[B]^2$ follows from the fact that $\mathcal{F}$ is a Ramsey filter.

We know that $\omega_1 \to (\omega_1, \omega)^2$, so *either* there is $D \in [B]^{\omega_1}$ with $f\{A(x), \beta, \gamma\} = 1$ for each $\{\beta, \gamma\} \in [D]^2$ and $x \in [X_{k,A,\beta,\gamma}]^k$ *or* there is $C \in [B]^\omega$ with $f\{A(x), \beta, \gamma\} = 0$ for each $\{\beta, \gamma\} \in [C]^2$ and $x \in [X_{k,A,\beta,\gamma}]^k$.

In the first case, we apply our inductive hypothesis that $\omega_1 \to (\omega + m, n)^3$ to D to conclude that *either* there is $E \in [D]^{\omega+m}$ with $f``[E]^3 = \{0\}$ (in which case (a) holds, and we are done), *or* there is $F \in [D]^n$ with $f``[F]^3 = \{1\}$. In this latter case, let

$$X = \bigcap\{X_{k,A,\beta,\gamma} \mid \{\beta, \gamma\} \in [F]^2\},$$

and choose any $x \in [X]^k$. Then let $G = \{A(x)\} \cup F$, so that $G \in [\omega_1]^{n+1}$ and $f``[G]^3 = \{1\}$. Hence (b) holds, and we are done.

In the second case, we choose a finite ordinal r such that $r \to (m, n + 1)^3$, and then choose $C' \in [C]^r$. Let

$$\begin{aligned}X_0 &= \bigcap\{X_{j,k,A,\beta} \mid \beta \in C'\},\\ X_1 &= \bigcap\{X_{k,A,\beta,\gamma} \mid \{\beta, \gamma\} \in [C']^2\},\\ X &= X_0 \cap X_1.\end{aligned}$$

Fix $a \in [X]^j$. For each $i < \omega$, choose $b_i \in [X]^{k-j}$ with $a \ll b_0 \ll b_1 \ll b_2 \ll \dots$. Let $A' = \{A(a \cup b_i) \mid i < \omega\}$. Note that $f``[A', C']^{2,1} \cup [A', C']^{1,2} = \{0\}$. If (b) does not hold, then since $A' \to (\omega, n + 1)^3$ we can find $A'' \in [A']^\omega$ with $f``[A'']^3 = \{0\}$; and since $C' \to (m, n + 1)^3$ we can find $C'' \in [C']^m$ with $f``[C'']^3 = \{0\}$; hence $A'' \cup C'' \in [\omega_1]^{\omega+m}$ and $f``[A'' \cup C'']^3 = \{0\}$. Thus (a) holds, and the proof of the lemma is complete. $\square$

Let us set aside the previous lemma for a moment and consider the following statement for each $k < \omega$:

Ψ_k: There is $A \in [\omega_1]^{\omega^k}$ such that $f\{A(x), A(y), A(z)\} = 1$ for each trio of sets $x, y, z \in [\omega]^k$ with $x \ll_{\text{lex}} y \ll_{\text{lex}} z$ and $\delta(x, z) < \delta(x, y)$.

The next lemma demonstrates that Ψ_n is strong enough to establish Lemma 1 on its own. This is important because (as we will see below) the Φ_k's and the Ψ_k's are complementary: If enough of the Φ_k's fail, then many of the Ψ_k's will hold.

Lemma 1.2. *For each $k < \omega$, if Ψ_k is true, then there is $B \in [\omega_1]^{k+1}$ with $f``[B]^3 = \{1\}$. (In particular, if Ψ_n holds, then* (b) *holds.)*

Proof. Assume that $k < \omega$ and the set $A \in [\omega_1]^{\omega^k}$ witnesses that Ψ_k holds. Define $k+1$ sets of finite ordinals $\{x_{j,k} \mid j \leq k\} \subseteq [\omega]^k$ by the following formula:

$$\begin{aligned}
x_{0,k} &= \{0, 1, \ldots, k-1\} \\
x_{1,k} &= \{0, 1, \ldots, k-2\} \cup \{k\} \\
&\vdots \\
x_{j,k} &= \{0, 1, \ldots, k-j-1\} \cup \{k + t_j, k + t_j + 1, \ldots, k + t_{j+1} - 1\} \\
&\vdots \\
x_{k-1,k} &= \{0\} \cup \{k + t_{k-1}, k + t_{k-1} + 1, \ldots, k + t_{k-1} + (k-2)\} \\
x_{k,k} &= \{k + t_k, k + t_k + 2, \ldots, k + t_k + k\},
\end{aligned}$$

where $t_j = \sum_{i<j} j$ for each $j < \omega$ (so that $t_0 = t_1 = 0$, $t_2 = 1$, $t_3 = 3$, $t_4 = 6$, $t_5 = 10$, and so forth). For example, if $k = 6$, then the seven sets are:

$$\begin{array}{lcl}
x_{0,6} & = & \{\ 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5\ \} \\
x_{1,6} & = & \{\ 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 6\ \} \\
x_{2,6} & = & \{\ 0, \quad 1, \quad 2, \quad 3, \quad 7, \quad 8\ \} \\
x_{3,6} & = & \{\ 0, \quad 1, \quad 2, \quad 9, \quad 10, \quad 11\ \} \\
x_{4,6} & = & \{\ 0, \quad 1, \quad 12, \quad 13, \quad 14, \quad 15\ \} \\
x_{5,6} & = & \{\ 0, \quad 16, \quad 17, \quad 18, \quad 19, \quad 20\ \} \\
x_{6,6} & = & \{21, \quad 22, \quad 23, \quad 24, \quad 25, \quad 26\ \}
\end{array}$$

It is relatively easy to check for each triple $j_0 < j_1 < j_2 \leq k$ that $x_{j_0,k} \ll_{\text{lex}} x_{j_1,k} \ll_{\text{lex}} x_{j_2,k}$ and $\delta(x_{j_0,k}, x_{j_2,k}) < \delta(x_{j_0,k}, x_{j_1,k})$. It follows then from our assumption that Ψ_k holds for A with $f\{A(x_{j_0,k}), A(x_{j_1,k}), A(x_{j_2,k})\} = 1$ for each triple $j_0 < j_1 < j_2 \leq k$. In other words, f is identically one on the triples of the $(k+1)$-element set $\{A(x_{j,k}) \mid j \leq k\}$. □

With the two previous lemmas in hand, our proof of Lemma 1 is almost complete. We need only establish that either Φ_k is true for some $k < \omega$ or Ψ_n is true. We satisfy this need in Lemma 1.4 below. Before we can do that, however, we need to analyze the Ψ_k's a little more.

For each $k < \omega$ and each $D \in [\omega_1]^{\omega_1}$, let $W_k(D)$ be the set of all $A \in [D]^{\omega^k}$ such that $f\{A(x), A(y), A(z)\} = 1$ for each $x, y, z \in [\omega]^k$ with $x \ll_{\text{lex}} y \ll_{\text{lex}} z$ and

$\delta(x,z) < \delta(x,y)$, so that, for example, Ψ_k is simply the assertion that $W_k(\omega_1)$ is nonempty.

We will need the following facts in our proof of Lemma 1.4 below.

Lemma 1.3. *Assume that $k < \omega$ and $D \in [\omega_1]^{\omega_1}$.*

(1) *Suppose that $A \in W_k(D)$ and $X \in [\omega]^\omega$. Let $A \restriction X = \{A(x) \mid x \in [X]^k\}$. Then $A \restriction X$ is also in $W_k(D)$.*
(2) *Suppose $\{A_i \mid i < \omega\} \subseteq W_k(D)$ with $A_i \ll A_j$ and $f\{A_i(x), A_i(y), A_j(z)\} = 1$ for each pair $i < j < \omega$ and each triple $x, y, z \in [\omega]^k$ with $x \ll_{\text{lex}} y$. Let $A = \bigcup\{A_i \mid i < \omega\}$. Then $A \in W_{k+1}(D)$.*

Proof. The first follows from a trivial reindexing argument, which we leave to the reader. The second is a little more devious, but not too much so. To help us in its proof we define the "downshift" map $d : [\omega]^{k+1} \to [\omega]^k$ as follows. For each $x \in [\omega]^{k+1}$ let

$$d(x) = \{\ell - \min(x) - 1 \mid \ell \in x \wedge \ell \neq \min(x)\}.$$

Note that $A(x) = A_{\min(x)}(d(x))$ for each $x \in [\omega]^{k+1}$ and that if $\min(x) = \min(y)$, then $x \ll_{\text{lex}} y$ if and only if $d(x) \ll_{\text{lex}} d(y)$.

Clearly $A \in [D]^{\omega^{k+1}}$. As for the rest, let $x, y, z \in [\omega]^{k+1}$ with $x \ll_{\text{lex}} y \ll_{\text{lex}} z$ and $\delta(x,z) < \delta(x,y)$ be given. We need to show that $f\{A(x), A(y), A(z)\} = 1$. There are two cases to consider.

Case 1. Suppose $\delta(x,z) = 0$, so that $\min(x) = \min(y) < \min(z)$. Let $i = \min(x) = \min(y)$ and $j = \min(z)$. Then

$$f\{A(x), A(y), A(z)\} = f\{A_i(d(x)), A_i(d(y)), A_j(d(z))\} = 1,$$

by hypothesis since $i < j$ and $d(x) \ll_{\text{lex}} d(y)$.

Case 2. Suppose that $\delta(x,z) > 0$, so that $\min(x) = \min(y) = \min(z)$. Let $i = \min(x) = \min(y) = \min(z)$. Then

$$f\{A(x), A(y), A(z)\} = f\{A_i(d(x)), A_i(d(y)), A_i(d(z))\} = 1,$$

by hypothesis since $A_i \in W_k(D)$. □

This final lemma will complete our proof of Lemma 1 by demonstrating that either Ψ_n holds or Φ_k holds for some $k < \omega$. This, together with Lemmas 1.1 and 1.2, means that either (a) or (b) is obtained, and thus that Lemma 1 is true.

Lemma 1.4. *For each finite ordinal k, if Φ_j fails for every $j \leq k$, then Ψ_k holds. (In particular,* ***either*** *Φ_k holds for some $k < \omega$ and hence* (a) *or* (b) *holds by Lemma* 1.1 ***or*** *Ψ_n holds and hence* (b) *holds by Lemma* 1.2.*)*

Proof. Fix a finite ordinal $k \geq 1$. Assume that Φ_j fails for every $j \leq k$. We will prove that Ψ_k holds. Well, actually, we will prove by induction on $\ell \leq k$ that the stronger statement Ψ'_ℓ holds:

Ψ'_ℓ: For each $D \in [\omega_1]^{\omega_1}$ there are sets $A \in W_\ell(D)$ and $B \in [D]^{\omega_1}$ with $A \ll B$ such that $f\{A(x), A(y), \beta\} = 1$ for each $x, y \in [\omega]^\ell$ with $x \ll_{\text{lex}} y$ and each $\beta \in B$.

The sets A and B in the statement above are said to *witness Ψ'_ℓ for D*.

Note that Ψ'_0 is trivially true. Thus, we need only prove for each $\ell < k$ that $\Psi'_{\ell+1}$ follows from Ψ'_ℓ (and the failure of $\Phi_{\ell+1}$).

Suppose that $D \in [\omega_1]^{\omega_1}$. Recursively construct sequences $\{A_i \mid i < \omega\} \subseteq W_\ell(D)$ and $\{B_i \mid i < \omega\} \subseteq [D]^{\omega_1}$ as follows. First choose $A_0 \in W_\ell(D)$ and $B_0 \in [D]^{\omega_1}$ which witness Ψ'_ℓ for D. Next, for each $i < \omega$, assuming that A_i and B_i have been chosen, pick $A_{i+1} \in W_\ell(B_i)$ and $B_{i+1} \in [B_i]^{\omega_1}$ which witness Ψ'_ℓ for B_i. This completes the recursive construction.

Let $A = \bigcup\{A_i \mid i < \omega\}$. By Lemma 1.3, we know that $A \in W_{\ell+1}(D)$. As $\Phi_{\ell+1}$ fails, all but countably many $\beta \in D$ have $A \ll \{\beta\}$ and $i_{j,\ell+1,A,\beta} = 1$ for all $j \leq \ell$. Let

$$B = \{\beta \in D \mid A \ll \{\beta\} \wedge \forall j \leq \ell\, [i_{j,\ell+1,A,\beta} = 1]\}.$$

As $\mathfrak{p} > \omega_1$, we can find $Y \in [\omega]^\omega$ such that $Y \subseteq^* X_{\ell+1,A,\beta}$ for all $\beta \in B$. (Recall that $X_{\ell+1,A,\beta} = \bigcap\{X_{j,\ell+1,A,\beta} \mid j \leq \ell\}$.) For each $\beta \in B$, let N_β be the least $N < \omega$ such that $Y \smallsetminus N \subseteq X_{\ell+1,A,\beta}$. As B is uncountable, there must be $N < \omega$ and $B' \in [B]^{\omega_1}$ with $N_\beta = N$ for all $\beta \in B'$. Let $A' = A \restriction (Y \smallsetminus N) = \{A(x) \mid x \in [Y \smallsetminus N]^{\ell+1}\}$. By Lemma 1.3, $A' \in W_{\ell+1}(D)$. Moreover, these A' and B' satisfy $\Psi'_{\ell+1}$ for D. □

The following two lemmas finish our proof of Theorem 1, as outlined at the start of this section. Lemma 2 indicates that we can find a conservative generic extension of the universe in which for each $m, n < \omega$ the partition relation $\omega_1 \to (\omega + m, n)^3$ holds. Lemma 3 then shows that this relation is absolute. From this, Theorem 1 then follows.

Lemma 2. *There is a notion of forcing $\mathbb{P}$ such that*

$$\mathbb{P} \Vdash \ulcorner \dot{\omega}_1 = \check{\omega}_1 \text{ and } \dot{\omega}_1 \to (\omega + m, n)^3 \text{ for each } m, n < \omega \urcorner.$$

Proof. By Lemma 1, any notion of forcing $\mathbb{P}$ for which $\mathbb{P} \Vdash \ulcorner \dot{\omega}_1 = \check{\omega}_1$ and $\dot{\omega}_1 < \dot{\mathfrak{p}} = \dot{\mathfrak{c}} \urcorner$ will do. Constructions of such notions of forcing appear in [5], [7], [14], etc. □

Lemma 3 (J. Silver)**.** *Let m and n be finite ordinals, and $\{\alpha_i \mid i < n\}$ be countable ordinals. If there is a notion of forcing $\mathbb{P}$ such that*

$$\mathbb{P} \Vdash \ulcorner \dot{\omega}_1 = \check{\omega}_1 \text{ and } \dot{\omega}_1 \to (\check{\alpha}_i)^m_{i<n} \urcorner,$$

then, indeed, $\omega_1 \to (\alpha_i)^m_{i<n}$. Thus if one can conservatively force such a partition relation to be true, then one did not need to force in the first place.

Proof. See the presentation in either [1] or [12]. □

4. Conclusion

The following question still remains largely unanswered:

For which $\alpha < \omega_1$ and $n < \omega$ does $\omega_1 \to (\alpha, n)^3$?

By the main result of [9] and Theorem 1 above, the simplest unresolved cases of this question are now:

Problem 1. Does $\omega_1 \to (\omega + \omega + 2, 4)^3$?

Problem 2. Does $\omega_1 \to (\omega + \omega, 5)^3$?

References

1. J. Baumgartner and A. Hajnal, *A proof (involving Martin's axiom) of a partition relation*, Fund. Math. **78** (1973), no. 3, 193–203. MR0319768 (47:8310)
2. P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489. MR0081864 (18:458a)
3. F. Galvin, *On a partition theorem of Baumgartner and Hajnal*, Infinite and Finite Sets, Colloq. Math. Soc., János Bolyai, vol. 10, North-Holland, 1975, pp. 711–729. MR0376355 (51:12531)
4. A. Hajnal, *Some results and problems on set theory*, Acta Math. Acad. Sci. Hungar. **11** (1960), 277–298. MR0150044 (27:47)
5. T. Jech, *Set theory*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978, Pure and Applied Mathematics. MR0506523 (80a:03062)
6. A. Jones, *Some results in the partition calculus*, Ph.D. thesis, Dartmouth College, June 1999.
7. K. Kunen, *Set theory*, North-Holland Publishing Co., Amsterdam, 1983, An introduction to independence proofs, Reprint of the 1980 original. MR0756630 (85e:03003)
8. E. C. Milner and K. Prikry, *A partition theorem for triples*, Proc. Amer. Math. Soc. **97** (1986), no. 3, 488–494. MR0840635 (87f:04006)
9. ———, *A partition relation for triples using a model of Todorčević*, Discrete Math. **95** (1991), no. 1-3, 183–191, Directions in infinite graph theory and combinatorics (Cambridge, 1989). MR1141938 (93i:03065)
10. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1930), 264–286.
11. W. Sierpiński, *Sur un problème de la théorie des relations*, Annali E. Scuola Normale Superiore de Pisa Ser. 2 **2** (1933), 285–287.
12. J. Silver, *A large cardinal in the constructible universe*, Fund. Math. **69** (1970), 93–100. MR0274278 (43:43)
13. S. Todorčević, *Forcing positive partition relations*, Trans. Amer. Math. Soc. **280** (1983), no. 2, 703–720. MR0716846 (85d:03102)
14. ———, *Partition relations for partially ordered sets*, Acta Math. **155** (1985), no. 1-2, 1–25. MR0793235 (87d:03126)

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