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ON CHARACTERIZATION AND PERTURBATION OF LOCAL C-SEMIGROUPS

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Abstract. Let $S(\cdot)$ be a (C_0)-group with generator $-B$, and let $\{T(t); 0 \leq t < \tau\}$ be a local C-semigroup commuting with $S(\cdot)$. Then the operators $V(t) := S(-t)T(t)$, $0 \leq t < \tau$, form a local C-semigroup. It is proved that if C is injective and A is the generator of $T(\cdot)$, then $A+B$ is closable and $\overline{A+B}$ is the generator of $V(\cdot)$. Also proved are a characterization theorem for local C-semigroups with C not necessarily injective and a theorem about solvability of the abstract inhomogeneous Cauchy problem: $u'(t) = Au(t) + Cf(t), 0 < t < \tau;\ u(0) = Cx$.

1. Introduction and result

The aim of this paper is to present some theorems about characterization and perturbation of local C-semigroups and the solvability of the associated inhomogeneous Cauchy problem.

Let X be a complex Banach space and let $B(X)$ be the Banach algebra of all bounded (linear) operators on X. When $0 < \tau < \infty$ (resp. $\tau = \infty$), a family $\{T(t); 0 \leq t < \tau\}$ in $B(X)$ is called a *local C-semigroup* (resp. *(global) C-semigroup*) on X if

(a) $T(\cdot)x : [0, \tau) \to X$ is continuous for each $x \in X$.

(b) $T(s+t)C = T(s)T(t)$ for all $0 \leq s, t, t+s < \tau$ and $T(0) = C$.

C-semigroups have been studied in many papers, among them are [1], [2], [3], [7], and [10]. Local C-semigroups have been studied in [4], [6], [8], [9], [11], [12], [13].

When $\tau = \infty$ and $C = I$, the identity operator, $T(\cdot)$ is a classical (C_0)-semigroup. Thus (C_0)-semigroups form a subclass of the class of C-semigroups, and clearly, every C-semigroup can be viewed as a local C-semigroup defined on $[0, \tau)$ for any $0 < \tau < \infty$. In general, a local C-semigroup on $[0, \tau)$ for some $\tau < \infty$ is not necessarily extendable to the whole half line $[0, \infty)$. Results concerning extension of local C-semigroups can be found in [4] and [12].

We first state the following general characterization theorem for local C-semigroups. Its proof will be given in Section 2.

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Theorem 1. *A strongly continuous family $\{T(t); 0 \le t < \tau\}$ is a local C-semigroup if and only if $T(\cdot)$ commutes with C and satisfies $T(0) = C$ and*

$$(1.1) \qquad (1 * T)(s)[T(t) - C] = [T(s) - C](1 * T)(t) \text{ for all } 0 \le s, t, s + t < \tau.$$

We are mainly interested in those local C-semigroups which are nondegenerate. A local C-semigroup $\{T(t); 0 \le t < \tau\}$ will be said to be *nondegenerate* if one of the equivalent conditions in the next lemma is satisfied.

Lemma 2. *The following statements are equivalent:*
(c1) *C is injective.*
(c2) $\lim_{t \to 0^+} T(t)x = 0$ *implies* $x = 0$.
(c3) $T(t)x \equiv 0$ *on* $(0, s_1)$ *for some* $s_1 \in (0, \tau/2)$ *implies* $x = 0$.
(c4) $T(t)x \equiv 0$ *on* $(0, \tau/2)$ *implies* $x = 0$.

Proof. (c1) $\Rightarrow$ (c2). If $T(t)x \to 0$ as $t \to 0^+$, by (a) and (b) we have $Cx = \lim_{t \to 0^+} T(t)x = 0$, and (c1) implies $x = 0$.

(c2) $\Rightarrow$ (c3) $\Rightarrow$ (c4) are obvious.

(c4) $\Rightarrow$ (c1). If $Cx = 0$, then from (b) we see that $T(s)T(t)x = 0$ for all $0 < s, t < \tau/2$, which implies $x = 0$ by (c4). Hence C is injective. □

When $\tau = \infty$, the above definition of nondegeneracy coincides with the usual definition of nondegeneracy for C-semigroups, i.e., $T(t)x = 0$ for all $t > 0$ implies $x = 0$. But, when $\tau < \infty$, (c4) is even strictly stronger than the following condition:
(c′) $T(t)x \equiv 0$ on $(0, s_2)$ for some $s_2 \in (\tau/2, \tau)$ implies $x = 0$,
because, unlike the case $\tau = \infty$, (c′) is not equivalent to the injectivity of C when $\tau < \infty$. Also, unlike C-semigroups, a local C-semigroup $\{T(t); 0 \le t < \tau\}$ with $\tau < \infty$ need not be commutative, although C commutes with each $T(t), t \in [0, \tau)$, and $\{T(t); 0 \le t < \tau/2\}$ is a commutative subfamily, by (b). These interesting phenomena are illustrated by the following example.

Let $U : [\tau/2, \tau) \to B(X)$ be a strongly continuous function such that $U(\tau/2) = 0$ and $U(t)$ is injective for all $t \in (\tau/2, \tau)$. The operator $C := 0 \oplus I \in B(X \times X)$, with I the identity operator on X, is not injective. We define $T : [0, \tau) \to B(X \times X)$ by

$$T(t) := \begin{cases} 0 \oplus I & \text{if } 0 \le t < \tau/2, \\ U(t) \oplus I & \text{if } \tau/2 \le t < \tau. \end{cases}$$

Then $T(\cdot)$ is strongly continuous and satisfies

$$T(s)T(t) = T(t)T(s) = 0 \oplus I = T(s + t)C \text{ for all } 0 \le s, t, s + t < \tau.$$

Hence this $T(\cdot)$ is a local C-semigroup with C not injective. But, it satisfies condition (c′). Indeed, if there is a $t \in (\tau/2, \tau)$ such that $T(t)x = 0$, then since

$$C(t)x = (U(t) \oplus I)(x_1, x_2) = (U(t)x_1, x_2)$$

and $U(t)$ is injective, we have $x_1 = x_2 = 0$ and hence $x = 0$. Moreover, since the family $\{U(t); \tau/2 \le t < \tau\}$ need not be commutative, the above local C-semigroup $T(\cdot)$ may not be a commutative family.

When C is injective, $T(\cdot)$ is indeed commutative (see [8, Lemma 1.1]).

Now for a nondegenerate local C-semigroup $T(\cdot)$, by either (c2) or (c4) one can define the *generator* A by

$$\text{(d)} \qquad x \in D(A) \text{ and } Ax = y \Leftrightarrow \int_0^t T(s)y\,ds = T(t)x - Cx \text{ for all } t \in [0, \tau).$$

It is known [8] that this definition is equivalent to the one in the sense of Da Prato [1]:

$$\text{(d}'\text{)} \qquad \begin{cases} D(A) = \{x \in X;\ \lim\limits_{h\to 0^+} (T(h)x - Cx)/h \in R(C)\}, \\ Ax = C^{-1} \lim\limits_{h\to 0^+} (T(h)x - Cx)/h \text{ for } x \in D(A). \end{cases}$$

The following characterization of a nondegenerate local C-semigroup is proved in [8, Lemma 2.1 and Proposition 2.2].

Proposition 3. *Let $C \in B(X)$ be an injection and let $\{T(t); 0 \le t < \tau\}$ be a strongly continuous family of operators on X.*

(i) *If $T(\cdot)$ is a local C-semigroup with generator A, then A is closed and satisfies $C^{-1}AC = A$, $R(\int_0^t T(s)ds) \subset D(A)$ and*

$$T(t)x - Cx = \begin{cases} A\int_0^t T(s)xds & \text{for } x \in X, \\ \int_0^t T(s)Axds & \text{for } x \in D(A) \end{cases} \tag{1.2}$$

for all $t \in [0, \tau)$.

(ii) *If A is a closed operator satisfying $R(\int_0^t T(s)ds) \subset D(A)$ and (1.2), then $T(\cdot)$ is a local C-semigroup with generator $C^{-1}AC$.*

Consequently, a strongly continuous family $\{T(t); 0 \le t < \tau\}$ is a local C-semigroup with generator A if and only if A is closed and satisfies $C^{-1}AC = A$, $R(\int_0^t T(s)ds) \subset D(A)$ and (1.2).

The following characterization of generator in terms of solvability of the abstract Cauchy problem is proved in [8, Corollary 2.6].

Proposition 4. *An operator A is the generator of a local C-semigroup $\{T(t); t \in [0, \tau)\}$ if and only if the abstract Cauchy problem*

$$\mathrm{ACP}(A; Cx + C(1 * g), 0) \qquad \begin{cases} u'(t) = Au(t) + Cx + \int_0^t Cg(s)ds, \quad 0 < t < \tau, \\ u(0) = 0 \end{cases}$$

has a unique solution for every $x \in X$ and $g \in L^1_{loc}([0, \tau), X)$. The solution is $u(t) = \int_0^t T(s)xds + \int_0^t \int_0^s T(s - r)g(r)drds$.

This proposition can be used to deduce assertion (ii) of the next theorem, which is concerned with the solvability of the abstract Cauchy problem

$$\mathrm{ACP}(A; Cf, Cx) \qquad \begin{cases} u'(t) = Au(t) + Cf(t), \quad 0 < t < \tau, \\ u(0) = Cx \end{cases}$$

for suitable vector x and vector function f.

Theorem 5. *Let A be the generator of a nondegenerate local C-semigroup $\{T(t); 0 \le t < \tau\}$ on X, $x \in X$, and $f \in C([0, \tau), X)$.*

(i) *If $\mathrm{ACP}(A; Cf, Cx)$ has a strong solution u, then $u \equiv T(\cdot)x + T * f$.*

(ii) *If $u := T(\cdot)x + T * f \in C([0, \tau), [D(A)])$, where $[D(A)]$ is the Banach space $D(A)$ equipped with the graph norm, then u is a strong solution of $\mathrm{ACP}(A; Cf, Cx)$.*

(iii) *If either $f \in C^1([0, \tau), X)$ or $f \in C([0, \tau), [D(A)])$, then for every $x \in D(A)$ $\mathrm{ACP}(A; Cf, Cx)$ has the unique strong solution $u(t) := T(t)x + (T * f)(t)$, $0 \le t < \tau$.*

Theorem 5 is well known for the case $\tau = \infty$ (cf. [7, Theorem 7.1 and Corollary 7.5]) and the case $C = I$ (cf. [5]).

As an application of Proposition 4, it is deduced in [9, Theorem 2] that if A is the generator of a local C-semigroup and if $B \in B(X)$ satisfies $R(B) \subset R(C)$ and $BCx = CBx$ for $x \in D(A)$, then $A + B$ also generates a local C-semigroup. In the following theorem we extend this result to those unbounded perturbation operators B that generate some commuting (C_0)-groups.

Theorem 6. *Let $T(\cdot)$ be a local C-semigroup on a Banach space X and let $S(\cdot)$ be a (C_0)-group with generator $-B$. Suppose $S(t)T(s) = T(s)S(t)$ for all $0 \le s, t < \tau$. Let $V(t) := S(-t)T(t)$ for $t \in [0, \tau)$. Then*

(i) *$\{V(t); t \in [0, \tau)\}$ is the unique local C-semigroup commuting with $S(\cdot)$ and $T(\cdot)$ and satisfying*

$$\int_0^t S(u)(1 * V)(u)du = \int_0^t S(u)(1 * T)(t - u)du. \tag{1.3}$$

(ii) *If C is injective and A is the generator of $T(\cdot)$, then $A + B$ is closable and $\overline{A + B}$ is the generator of $V(\cdot)$. In particular, if $B \in B(X)$ commutes with $T(\cdot)$, then $A + B$ generates the local C-semigroup $\{e^{-tB}T(t); 0 \le t < \tau\}$.*

We remark that a multiplicative perturbation theorem for local C-semigroups has been proved in [13].

The proofs of Theorems 1, 5, and 6 will be given in Sections 2 and 3, and a simple example of an application of Proposition 4 and Theorems 5 and 6 will be given in Section 4.

2. Proofs of Theorems 1 and 5

Proof of Theorem 1. Suppose $T(\cdot)$ is a local C-semigroup on X. Then, by the commutativity of $(1 * T)(t)$ with $T(s)$ and C for $0 \le s, t, s + t < \tau$, we have

$$\begin{aligned} T(s)(1 * T)(t)x &= \int_0^t T(s)T(u)xdu = \int_0^t T(s + u)Cxdu = \int_s^{s+t} T(u)Cxdu \\ &= \int_t^{s+t} T(u)Cxdu + \int_0^t T(u)Cxdu - \int_0^s T(u)Cxdu \\ &= (1 * T)(s)T(t)x + C(1 * T)(t)x - (1 * T)(s)Cx \end{aligned}$$

for $x \in X$ and $0 \le s, t, s + t < \tau$. This shows (1.1).

Conversely, suppose that $T(\cdot)$ satisfies (1.1). For fixed $s, t \in [0, \tau)$ with $s + t < \tau$, we replace s and t in (1.1) by $s + t - r$ and r, respectively, to obtain

$$(1*T)(s+t-r)T(r)x - T(s+t-r)(1*T)(r)x = (1*T)(s+t-r)Cx - C(1*T)(r)x$$

for all $x \in X$ and $r \in [0, s + t)$. By integrating the right-hand side with respect to r from 0 to t, we obtain from $CT(\cdot) = T(\cdot)C$ that

$$\begin{aligned} &\int_0^t (1 * T)(s + t - r)Cxdr - \int_0^t C(1 * T)(r)xdr \\ &\quad = \int_s^{s+t} (1 * T)(r)Cxdr - \int_0^t C(1 * T)(r)xdr \\ &\quad = (\int_0^{s+t} - \int_0^s - \int_0^t)(1 * T)(r)Cxdr. \end{aligned}$$

On the other hand, from the left-hand side we have

$$\begin{aligned}
&\int_0^t (1*T)(s+t-r)T(r)xdr - \int_0^t T(s+t-r)(1*T)(r)xdr \\
&\quad = (1*T)(s+t-r)(1*T)(r)x|_0^t + \int_0^t T(s+t-r)(1*T)(r)xdr \\
&\qquad - \int_0^t T(s+t-r)(1*T)(r)xdr \\
&\quad = (1*T)(s)(1*T)(t)x - (1*T)(s+t)(1*T)(0)x \\
&\quad = (1*T)(s)(1*T)(t)x.
\end{aligned}$$

Therefore $(1*T)(\cdot)$ satisfies

$$(1*T)(s)(1*T)(t)x = (\int_0^{s+t} - \int_0^s - \int_0^t)(1*T)(r)Cxdr$$

for all $x \in X$ and $s, t, s+t \in [0,\tau)$. By differentiation, it is clear that $T(\cdot)$ is a local C-semigroup. □

Proof of Theorem 5. (i) Suppose u is a strong solution of ACP$(A; Cf, Cx)$. Then $u(0) = Cx$ and $u' = Au + Cf$, so that $u - Cx = 1 * u' = 1 * (Au + Cf)$. Therefore we have

$$\begin{aligned}
T*u - C(1*T)(\cdot)x = T*(u-Cx) &= (1*T)A*u + 1*(T*Cf) \\
&= (T-C)*u + C(1*T)*f \\
&= T*u - C(1*u) + C1*(T*f).
\end{aligned}$$

Since C is injective, we have $1*u = (1*T)(\cdot)x + 1*(T*f)$. By differentiation, we have $u = T(\cdot)x + T*f$. This also proves that the strong solution of ACP$(A; Cf, Cx)$ is unique.

(ii) Suppose $u := T(\cdot)x + T*f \in C([0,\tau),[D(A)])$. Then $u(0) = Cx$, $Au \in C([0,\tau),X)$ and $1*u = (1*T)(\cdot)x + (1*T*f)$, so that, by the closedness of A, we have

$$1*(Au) = A(1*u) = T(\cdot)x - Cx + (T-C)*f = u - Cx - C(1*f).$$

By differentiation, we have $u' = Au + Cf$.

Derivation of (ii) from Proposition 4. By Proposition 4, ACP$(A; Cx+C(1*f), 0)$ has a unique solution v given by $v(t) = \int_0^t u(s)ds$ for $t \in [0,\tau)$, where $u = T(\cdot)x + T*f$. If $u \in C([0,\tau),[D(A)])$, then by the closedness of A, we have $v(t) \in D(A)$ and $Av(t) = \int_0^t Au(s)ds$ is continuously differentiable in X on $[0,\tau)$. Hence $u = v' = Av + Cx + C(1*f)$ is continuously differentiable and satisfies $u'(t) = Au(t) + Cf(t)$, $0 < t < \tau$, and $u(0) = T(0)x = Cx$, i.e., u is a strong solution of ACP$(A; Cf, Cx)$.

(iii) Consider the following two cases.

If $f \in C^1([0,\tau),X)$, then $T*f \in C^1([0,\tau),X)$ so that

$$\begin{aligned}
A[1*(T*f)] &= A(1*T)*f = (T-C)*f \\
&= T*f - C(1*f) \in C^1([0,\tau),X).
\end{aligned}$$

Since A is closed, it follows from differentiation that $(T*f)(t) \in D(A)$ for all $t \in [0,\tau)$ and $A(T*f) = (T*f)' - Cf \in C([0,\tau),X)$, i.e., $T*f \in C([0,\tau),[D(A)])$.

If $f \in C([0,\tau),[D(A)])$, then $A(T*f) = T*Af \in C([0,\tau),X)$, and hence we also have $T*f \in C([0,\tau),[D(A)])$.

Since $T(\cdot)x \in C([0,\tau),[D(A)])$ for $x \in D(A)$, in both the above cases, we have $u := T(\cdot)x + T*f \in C([0,\tau),[D(A)])$ for every $x \in D(A)$. Therefore u is a strong solution of ACP$(A; Cf, Cx)$ for every $x \in D(A)$, by (ii). □

3. Proof of Theorem 6

We first prove the following proposition.

Proposition 7. *Let $T(\cdot)$ and $S(\cdot)$ be two commuting local C-semigroups with generators A and B, respectively. Then the following hold.*

(i) *$A+B$ is closable and satisfies:*

$$(A+B) \subset C^{-1}(A+B)C \text{ and } \overline{A+B} \subset C^{-1}\overline{A+B}C.$$

(ii) *If, in addition, either one of $T(\cdot)$ and $S(\cdot)$ is even a (C_0)-semigroup, then*

$$C^{-1}\overline{A+B}C = \overline{A+B}.$$

Proof. (i) First, we show that $A+B$ is closable. Let $\{x_n\}$ be a null sequence in $D(A+B)$ such that $(A+B)x_n$ converges to a vector $y \in X$. We need to show $y=0$. Observe that $S(t)T(s) = T(s)S(t)$ implies that $S(t)Ax = AS(t)x$ for $x \in D(A)$. Hence we have

$$\begin{aligned}(1*T)(t)(1*S)(s)y &= \lim_{n\to\infty}(1*T)(t)(1*S)(s)(A+B)x_n \\ &= \lim_{n\to\infty}\{[T(t)-C](1*S)(s)x_n + (1*T)(t)[S(s)-C]x_n\} \\ &= [T(t)-C](1*S)(s)0 + (1*T)(t)[S(s)-C]0 = 0\end{aligned}$$

and then $T(t)S(s)y = 0$ for all $s,t \in (0,\tau)$, by differentiation. Then the strong continuity of $T(\cdot)$ and $S(\cdot)$ at 0 implies $C^2y = 0$ and hence $y=0$. Therefore $A+B$ is closable.

Let $x \in D(A+B) = D(A)\cap D(B)$. Since A and B are generators, by Proposition 3(i), we have $C^{-1}ACx = Ax$ and $C^{-1}BCx = Bx$, so that $Cx \in D(A)\cap D(B) = D(A+B)$ and $ACx = CAx$ and $BCx = CBx$. Hence $(A+B)Cx = ACx + BCx = C(A+B)x$ and so $x \in D(C^{-1}(A+B)C)$ and $(A+B)x = C^{-1}(A+B)Cx$. Hence $(A+B) \subset C^{-1}(A+B)C$. Next, we show $\overline{A+B} \subset C^{-1}\overline{A+B}C$. If $x \in D(\overline{A+B})$, then there is a sequence $\{x_n\}$ in $D(A+B)$ such that $(x_n,(A+B)x_n) \to (x,\overline{A+B}x)$. As above, we have $(A+B)Cx_n = C(A+B)x_n \to C\overline{A+B}x$. This with the fact that $Cx_n \to Cx$ implies that $Cx \in D(\overline{A+B})$ and $\overline{A+B}Cx = C\overline{A+B}x$, or $\overline{A+B}x = C^{-1}\overline{A+B}Cx$. Therefore $\overline{A+B} \subset C^{-1}\overline{A+B}C$.

(ii) Assume $S(\cdot)$ is a (C_0)-semigroup. It remains to show the inclusion:

$$C^{-1}\overline{A+B}C \subset \overline{A+B}.$$

Let $x \in D(C^{-1}\overline{A+B}C)$ and $y := C^{-1}\overline{A+B}Cx$. Then $Cy = \overline{A+B}Cx$. So, there is a sequence $\{z_n\}$ in $D(A+B)$ such that $(z_n,(A+B)z_n) \to (Cx,Cy)$ strongly as $n\to\infty$. Therefore we have for every $s,t \in [0,\tau)$

$$\begin{aligned}&(1*T)(s)(1*S)(t)Cy \\ &= \lim_{n\to\infty}(1*T)(s)(1*S)(t)(A+B)z_n \\ &= \lim_{n\to\infty}[(1*S)(t)[T(s)-C]z_n + (1*T)(s)(S(t)-I)z_n] \\ &= (1*S)(t)[T(s)-C]Cx + (1*T)(s)(S(t)-I)Cx.\end{aligned}$$

Since $T(\cdot)$, $S(\cdot)$, and C commute, it follows from the injectivity of C that

$$(1*T)(s)[(1*S)(t)y-(S(t)-I)x]=[T(s)-C](1*S)(t)x$$

for every $s,t\in[0,\tau)$. By the definition of generator, this implies that $(1*S)(t)x\in D(A)$ and

$$A(1*S)(t)x=(1*S)(t)y-(S(t)-I)x=(1*S)(t)y-B(1*S)(t)x.$$

Hence we have for every $t\in[0,\tau)$

$$(1*S)(t)y=(A+B)(1*S)(t)x=\overline{A+B}(1*S)(t)x.$$

By differentiation, we have $S(t)x\in D(\overline{A+B})$ and $S(t)y=\overline{A+B}S(t)x$. Since $S(0)=I$, this implies that $x\in D(\overline{A+B})$ and $y=\overline{A+B}x$. Therefore $C^{-1}\overline{A+B}C\subset\overline{A+B}$.

Proof of Theorem 6. (i) Since $S(\cdot)$ commutes with $T(\cdot)$, clearly $V(\cdot)$ is a local C-semigroup commuting with $S(\cdot)$ and $T(\cdot)$. $V(\cdot)$ satisfies (1.3):

$$\begin{aligned}\int_0^t S(u)(1*V)(u)du&=\int_0^t S(u)\int_0^u S(-s)T(s)dsdu=\int_0^t\int_s^t S(u-s)T(s)duds\\&=\int_0^t\int_0^{t-s}S(u)T(s)duds=\int_0^t S(u)\int_0^{t-u}T(s)dsdu\\&=\int_0^t S(u)(1*T)(t-u)du.\end{aligned}$$

Suppose $V_1(\cdot)$ and $V_2(\cdot)$ are two functions satisfying (1.3). Then the function $V(\cdot):=V_1(\cdot)-V_2(\cdot)$ satisfies $\int_0^t S(u)(1*V)(u)du=0$ for all $t\in[0,\tau)$. Hence $S(t)(1*V)(t)=0$ for all $t\in[0,\tau)$. Since $S(t)$ is injective, we must have $V(\cdot)\equiv 0$.

(ii) Since $S(-t)$ is a (C_0)-semigroup with generator B, an application of Proposition 7 (with $S(t)$ therein replaced by $S(-t)$) yields that $A+B$ is closable and $C^{-1}\overline{A+B}C=\overline{A+B}$.

Since $(1*T)(t)A\subset A(1*T)(t)=T(t)-C$ for $t\in[0,\tau)$, and since A is closed and $S(t)Ay=AS(t)y$ for $y\in D(A)$ we have $R(\int_0^t S(u)(1*T)(t-u)du)\subset D(A)$ and

$$\begin{aligned}\int_0^t S(u)(1*T)(t-u)duA&\subset A\int_0^t S(u)(1*T)(t-u)du\\&=\int_0^t S(u)A(1*T)(t-u)du\\&=\int_0^t S(u)[T(t-u)-C]du\\&=\frac{d}{dt}\int_0^t S(u)(1*T)(t-u)du-\int_0^t S(u)Cdu\\&=\frac{d}{dt}\int_0^t S(u)(1*V)(u)du-\int_0^t S(u)Cdu\\&=S(t)(1*V)(t)-\int_0^t S(u)Cdu\end{aligned}$$

for all $t \in [0, \tau)$. This and (1.3) imply that

$$\int_0^t S(u)(1*V)(u)Adu \subset A\int_0^t S(u)(1*V)(u)du = S(t)(1*V)(t) - \int_0^t S(u)Cdu. \tag{3.1}$$

Differentiating (1.3), we obtain

$$S(t)(1*V)(t) = \int_0^t S(u)T(t-u)du \text{ for all } t \in [0, \tau). \tag{3.2}$$

Since $1 * V$ commutes with $S(\cdot)$, it commutes with the generator $-B$, i.e., $(1*V)(u)x \in D(B)$ and $B(1*V)(u)x = (1*V)(u)Bx$ for $x \in D(B)$, so that

$$S'(u)(1*V)(u)x = -BS(u)(1*V)(u)x = -S(u)B(1*V)(u)x = -S(u)(1*V)(u)Bx$$

for all $u \in [0, \tau)$. Then, using integration by parts, the closedness of B, and (1.3), we obtain for $x \in D(B)$ and $t \in [0, \tau)$,

$$\begin{aligned} S(t)(1*V)(t)x &= -\int_0^t S(u)(1*V)(u)Bxdu + \int_0^t S(u)V(u)xdu \\ &= -\int_0^t BS(u)(1*V)(u)xdu + \int_0^t S(u)V(u)xdu \\ &= -B\int_0^t S(u)(1*T)(t-u)xdu + \int_0^t S(u)V(u)xdu. \end{aligned}$$

Combining this and (3.2), and by the closedness of B again, we obtain

$$\int_0^t S(u)(1*T)(t-u)duB \subset B\int_0^t S(u)(1*T)(t-u)du = -\int_0^t S(u)T(t-u)du + \int_0^t S(u)V(u)du \tag{3.3}$$

for every $t \in [0, \tau)$.

Now we obtain from (1.3), (3.2) and (3.3) that

$$\int_0^t S(u)(1*V)(u)Bdu \subset B\int_0^t S(u)(1*V)(u)du = \int_0^t S(u)V(u)du - S(t)(1*V)(t). \tag{3.4}$$

Hence, summing (3.1) and (3.4) and then taking the closure of $A + B$ we have for every $t \in [0, \tau)$

$$\int_0^t S(u)(1*V)(u)(\overline{A+B})du \subset (\overline{A+B})\int_0^t S(u)(1*V)(u)du = \int_0^t S(u)[V(u) - C]du. \tag{3.5}$$

Since $\overline{A+B}$ is closed, differentiation with respect to t yields

$$R(S(t)(1*V)(t)) \subset D(\overline{A+B})$$

and

$$S(t)(1*V)(t)\overline{A+B} \subset \overline{A+B}S(t)(1*V)(t) = S(t)[V(t) - C] \tag{3.6}$$

for all $t \in [0, \tau)$. Since $S(t)$ is injective, $(1 * V)(t)\overline{A+B} \subset V(t) - C$. On the other hand, since $S(\cdot)$ commutes with $V(\cdot)$, we have $R((1 * V)(t)S(t)) \subset D(\overline{A+B})$ and $\overline{A+B}(1 * V)(t)S(t) = [V(t) - C]S(t)$. Then, by the surjectivity of $S(t)$, we obtain that $R((1 * V)(t)) \subset D(\overline{A+B})$ and

$$\overline{A+B}(1 * V)(t) = V(t) - C.$$

Thus we have shown

$$(1 * V)(t)\overline{A+B} \subset \overline{A+B}(1 * V)(t) = V(t) - C.$$

The conclusion of the theorem now follows from Proposition 3.

4. An illustrative example

Consider the following initial value problems in c_0:

$$\begin{cases} v_n'(t) = n(1+in)u_n(t) + e^{-n}q_n + \int_0^t e^{-n}g_n(s)ds,\ 0 < t < 1, \\ v_n(0) = 0, \end{cases} \tag{4.1}$$

$$\begin{cases} u_n'(t) = n(1+in)u_n(t) + e^{-n}f_n(t),\ 0 < t < 1, \\ u_n(0) = e^{-n}q_n. \end{cases} \tag{4.2}$$

The family $\{T(t); 0 \le t \le 1\}$, defined by $T(t)x := (e^{-n}e^{nt}x_n),\ x = (x_n) \in c_0$, $0 \le t \le 1$, is a local C-semigroup with $C := \bigoplus_{n=1}^{\infty} e^{-n} \in B(c_0)$ and with generator $A := \bigoplus_{n=1}^{\infty} n$. The diagonal matrix $-B := -iA^2 = \bigoplus_{n=1}^{\infty}(-in^2)$ generates the C_0-group $S(\cdot)$, defined by $S(t)x := (e^{-in^2t}x_n),\ x = (x_n) \in c_0$, $t \ge 0$, which commutes with $T(\cdot)$. By Theorem 6, $\overline{A+B}$ generates the local C-semigroup $\{V(t); 0 \le t < 1\}$, defined by $V(t)x := (e^{-n}e^{n(1+in)t}x_n),\ x = (x_n) \in c_0$.

If, for instance, $g(t) = (g_n(t)) \in c_0$ for all $t \in [0,1)$ and the functions $\{g_n\}$ are uniformly continuous on $[0,1)$, then $g \in C([0,1); c_0)$. Now it follows from Proposition 4 that, for any $q \in c_0$, (4.1) has a unique solution $v \in C([0,1); c_0)$, which is given by

$$\begin{aligned} v(t) &= \int_0^t V(s)qds + \int_0^t \int_0^s V(s-r)g(r)drds \\ &= \left(e^{-n}\Big[\frac{1}{n(1+in)}(e^{n(1+in)t} - 1)q_n + \int_0^t \int_0^s e^{n(1+in)(s-r)}g_n(r)drds\Big]\right), \end{aligned}$$

$0 \le t < 1$.

If $(n^2 f_n(t)) \in c_0$ for all $t \in [0,1)$ and the functions $\{n^2 f_n\}$ are uniformly continuous on $[0,1)$, then $f \in C([0,1); [D(\overline{A+B})])$. It follows from Theorem 5 that, for any $q \in c_0$ with $\lim_{n\to\infty} n^2 q_n = 0$, (4.2) has a unique solution $u \in C([0,1); c_0)$, which is given by

$$\begin{aligned} u(t) &= V(t)q + \int_0^t V(t-s)g(s)ds \\ &= \left(e^{-n}\Big[e^{n(1+in)t}q_n + \int_0^t e^{n(1+in)(t-s)}f_n(s)ds\Big]\right),\ 0 \le t < 1. \end{aligned}$$

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