

DIMENSION DISTORTION OF HYPERBOLICALLY CONVEX MAPS

S. ROHDE

(Communicated by Juha M. Heinonen)

ABSTRACT. In this note, we provide an answer to a question of D. Mejia and Chr. Pommerenke, by constructing a hyperbolically convex subdomain G of the unit disc \mathbb{D} so that the conformal map from \mathbb{D} to G maps a set of dimension 0 on $\partial\mathbb{D}$ to a set of dimension 1.

1. INTRODUCTION AND STATEMENT OF THE RESULT

A subdomain G of the unit disc \mathbb{D} in \mathbb{C} is *hyperbolically convex* if it is convex with respect to the hyperbolic metric of \mathbb{D} . These domains are just intersections of hyperbolic halfplanes. Conformal maps from \mathbb{D} onto hyperbolically convex domains have been systematically studied beginning with Ma and Minda ([2] and [3]); see the papers [5], [6] and [7] by Mejia and Pommerenke for further references. Hyperbolically convex domains appear naturally in various situations such as the Hayman-Wu problem, where the Brown-Flynn's theorem (the boundary of hyperbolically convex domains has length $< \pi^2$) has been used by Fernández, Heinonen and Martio [1] to obtain the bound $4\pi^2$ for the Hayman-Wu constant (see [9] for the best bound to date and for references).

In [5], Mejia and Pommerenke proved that, for conformal maps f of \mathbb{D} onto hyperbolically convex domains,

$$\dim f(A) < 1$$

if $A \subset \mathbb{T}$ has $\dim A < 1$ and if $f(A)$ is uniformly perfect. They asked (Conjecture 3 in [6]) if the assumption of uniform perfectness can be omitted. We will answer this question in the negative by proving

Theorem 1. *There is a hyperbolically convex domain $G \subset \mathbb{D}$ and a set $A \subset \mathbb{T}$ with $\dim A = 0$ and $\dim f(A) = 1$, where f is the conformal map from \mathbb{D} onto G .*

Our domain is of Schottky type (meaning that the complement of \overline{G} is a union of *disjoint* hyperbolic halfplanes; see [5] and Theorem 2 therein). The inductive construction of the domain G as a decreasing sequence of hyperbolically convex domains G_n is slightly easier to describe in the upper halfplane \mathbb{H} : Fix a sequence $\varepsilon_1, \varepsilon_2, \dots$ of small positive numbers and positive integers n_1, n_2, \dots (to be determined later). Next, construct a Cantor-like set $S \subset [0, 1] \subset \mathbb{R}$ in the following way.

Received by the editors November 9, 2004 and, in revised form, November 16, 2005.

2000 *Mathematics Subject Classification.* Primary 30C35.

The author was partially supported by NSF Grants DMS-0201435 and DMS-0244408.

©2006 American Mathematical Society
Reverts to public domain 28 years from publication

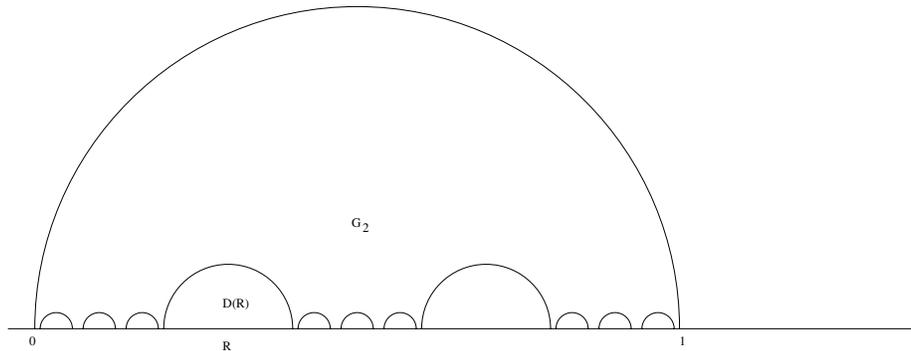


FIGURE 1.1. The second approximation to the hyperbolically convex domain G

Set $S_0 = [0, 1]$. Suppose that S_k has already been constructed and consists of $N_k = n_1 n_2 \cdots n_k$ intervals $I_j^{(k)}$ of size

$$s_k = (\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k) / (n_1 n_2 \cdots n_k).$$

Form S_{k+1} from S_k by replacing each of the $I_j^{(k)}$ by n_{k+1} equally spaced subintervals $I_\ell^{(k+1)}$ of size $s_{k+1} = (\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{k+1}) / (n_1 n_2 \cdots n_{k+1})$. Set

$$S = \bigcap_k \bigcup_j I_j^{(k)}.$$

For instance, the middle-third Cantor set is the special case $n_k = 2$ and $\varepsilon_k = 2/3$. Denote the “removed” intervals by $R_\ell^{(k+1)}$, $1 \leq \ell \leq (n_{k+1} - 1)n_1 \cdots n_k$. For an interval $I \subset [0, 1]$, let $D(I)$ be the open halfdisc in \mathbb{H} with diameter I . Set

$$G_k = D([0, 1]) \setminus \bigcup_{1 \leq m \leq k} \bigcup_j \overline{D(R_j^{(m)})}$$

and

$$G = \bigcap_k G_k;$$

see Figure 1.1.

Fix a point $z_0 \in G$. Let g_k , respectively g , denote the conformal map of G_k , respectively G , onto \mathbb{D} normalized by $g_k(z_0) = 0$ and $g'_k(z_0) > 0$. Because ∂G is a simple curve, g extends continuously to \overline{G} . We will show that an appropriate choice of the parameters ε_k and n_k yields a set S of dimension 1 such that $g(S)$ has dimension 0, proving the theorem (with $f = g^{-1}$ and $A = g(S)$). More precisely, we will prove

Theorem 2. *If*

$$\log n_k = o\left(\frac{1}{\sqrt{\varepsilon_k}}\right) \quad \text{as } k \rightarrow \infty,$$

then

$$\dim g(S) = 0.$$

On the other hand, if

$$\log \frac{1}{\varepsilon_k} = o(\log n_k) \quad \text{as } k \rightarrow \infty,$$

then

$$\dim S = 1.$$

Theorem 1 follows immediately by taking any pair of sequences ε_k, n_k that satisfies both conditions, for instance $\varepsilon_k = 1/k^2$ and $n_k = \exp(\sqrt{k})$.

It was realized in [5] that domains of Schottky type are the “critical domains” for the validity of the dimension distortion property: Mejia and Pommerenke proved (Theorem 2 of [5]) that $\dim A < 1$ would imply $\dim f(A) < 1$ for *all* hyperbolicly convex f if $\dim A < 1$ implies $\dim f(A) < 1$ for *those* hyperbolicly convex f that are of *Schottky type*. Notice that our domain G is of Schottky type, so that the assumption of their Theorem 2 does not hold.

The phenomenon responsible for the strong dimension-distortion is the following: if ε_k is very small and I is one of the $I_j^{(k)}$, then near I the domain G_k looks like a half-strip, whose sides are the two geodesics ending in the endpoints of I . Therefore the harmonic measure of I seen from z_0 is exponentially smaller than the size s_k of I . For our set S to have the properties $\dim S = 1$ and $\dim g(S) < 1$ we will need $\varepsilon_k \rightarrow 0$. Thus S cannot be uniformly perfect, in accordance with Theorem 3 of [5].

2. PROOFS

The letter C denotes various constants whose value may change even within a line. Recall the definition of S from Section 1 and assume that $\varepsilon_j < \frac{1}{2}$ for all j . Denote by $\hat{I}_j^{(k)}$ the (smaller) arc of ∂G that has the same endpoints as $I_j^{(k)}$. Fix $z_0 \in G$.

Lemma 2.1. *For all j and k we have*

$$\omega(z_0, \hat{I}_j^{(k)}) \leq K_1 \exp\left(-K_2 \sum_{\ell=1}^k \frac{1}{\sqrt{\varepsilon_\ell}}\right),$$

where K_2 is a universal constant and K_1 depends on z_0 and G only, but not on j or k .

Proof. Fix an interval $I = I_{j_0}^{(k_0)}$. Denote by r_k the length of the intervals $R_j^{(k)}$. Then $(n_{k+1} - 1)r_{k+1} + n_{k+1}s_{k+1} = s_k$ so that we have

$$r_{k+1} = \frac{1 - \varepsilon_{k+1}}{n_{k+1} - 1} s_k > \frac{\varepsilon_{k+1}}{n_{k+1}} s_k = s_{k+1}.$$

It follows that

$$(2.1) \quad s_{k+1} < h_{k+1} := \sqrt{r_{k+1}s_{k+1}} < r_{k+1} < s_k.$$

Let Γ denote the family of all curves γ in G that join a small (fixed) circle centered at z_0 with $\hat{I}_j^{(k)}$. By conformal invariance of the modulus $M(\Gamma)$ and by Pfluger’s theorem (see for instance [8], Chapter 9) we have

$$\omega(z_0, \hat{I}_j^{(k)}) \leq C \exp\left(-\frac{\pi}{M(\Gamma)}\right),$$

where C depends on z_0 and the circle but not on j or k . Consider the strips $S_k = \{x + iy : s_k < y < h_k\}$. It follows from (2.1) that they are pairwise disjoint. For $1 \leq k \leq k_0$, denote by Q_k the component of $G \cap S_k$ that separates I from z_0 . (We will assume without much loss of generality that $\Im z_0 > h_1$.) Each Q_k is, or is contained in, a topological quadrilateral bounded by two horizontal line segments

(top and bottom) and two circular arcs (sides) of circles of diameter at least r_k . The length ℓ_k of the top is maximal if both sides come from circles of diameter r_k . Because both circles are separated by an interval $I_j^{(k)}$ of size s_k , it follows that

$$\ell_k \leq s_k + 2(r_k/2 - \sqrt{(r_k/2)^2 - h_k^2})$$

and it thus follows from (2.1) using $\sqrt{1-x} \geq 1-x$ that

$$\ell_k < 5s_k.$$

Denote by Q'_k the smallest rectangle containing Q_k : it has horizontal sides of length ℓ_k and vertical sides of length $h_k - s_k$. Every curve $\gamma \in \Gamma$ has to pass through each of the Q'_k ($1 \leq k \leq k_0$) and therefore (e.g. [8], Proposition 9.3)

$$\frac{1}{M(\Gamma)} \geq \sum_{k=1}^{k_0} \frac{1}{M(Q'_k)} = \sum_{k=1}^{k_0} \frac{h_k - s_k}{\ell_k} \geq \sum_{k=1}^{k_0} \frac{h_k}{5s_k} \geq \frac{1}{5\sqrt{2}} \sum_{k=1}^{k_0} \frac{1}{\sqrt{\varepsilon_k}},$$

where the last inequality follows using $r_k/s_k = (n_k/\varepsilon_k)(1 - \varepsilon_k)/(n_k - 1)$ and $s_k < 1/2$. The lemma follows from Pfluger's theorem, with $K_2 = \pi/(5\sqrt{2})$. \square

Proof of Theorem 2. For every k , we have

$$g(S) \subset \bigcup_j g(\hat{I}_j^{(k)}).$$

By Lemma 2.1, the arc $g(\hat{I}_j^{(k)})$ has length

$$\ell_{j,k} \leq 2\pi K_1 \exp(-K_2 \sum_{\ell=1}^k \frac{1}{\sqrt{\varepsilon_\ell}}).$$

For every $\alpha > 0$,

$$\sum_j \ell_{j,k}^\alpha \leq C n_1 n_2 \cdots n_k \exp(-K_2 \alpha \sum_{\ell=1}^k \frac{1}{\sqrt{\varepsilon_\ell}}) = C \prod_{\ell=1}^k \left(n_\ell \exp(-K_2 \alpha \frac{1}{\sqrt{\varepsilon_\ell}}) \right)$$

and the assumption $\log n_j = o(\frac{1}{\sqrt{\varepsilon_j}})$ implies that the factors go to zero as $j \rightarrow \infty$. Thus $\dim g(S) \leq \alpha$.

To prove that $\dim S = 1$ if $\log(1/\varepsilon_j) = o(\log n_j)$, fix $\alpha < 1$ and consider the measure μ on S defined by

$$\mu(I_j^{(k)}) = \frac{1}{n_1 n_2 \cdots n_k}$$

for all j and k . The theorem follows at once from the ‘‘easy direction’’ of Frostman’s Lemma ([8], Exercise 10.2.5, or [4], Theorem 8.8) if we can prove that, for every interval I , we have

$$(2.2) \quad \mu(I) \leq C_\alpha |I|^\alpha.$$

To this end, let k be such that $s_{k+1} \leq |I| < s_k$. Because $r_k > s_k$, there is only one index j such that $I_j^{(k)} \cap I \neq \emptyset$. Let $t = |I|/s_k < 1$. Then I can intersect at most $tn_{k+1} + 2$ of the n_{k+1} subintervals $I_\ell^{(k+1)}$ of $I_j^{(k)}$. Therefore

$$(2.3) \quad \mu(I) \leq (tn_{k+1} + 2) \frac{1}{n_1 n_2 \cdots n_{k+1}}.$$

By assumption,

$$\log \frac{1}{\varepsilon_p} \leq \frac{1-\alpha}{\alpha} \log n_p$$

for p large enough, so that

$$\frac{1}{n_1 n_2 \cdots n_k} \leq C \left(\frac{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}{n_1 n_2 \cdots n_k} \right)^\alpha = C s_k^\alpha.$$

Now (2.3) implies that

$$\mu(I) \leq C(t s_k^\alpha + 2 s_{k+1}^\alpha) \leq C(t^{1-\alpha} |I|^\alpha + 2 |I|^\alpha)$$

and (2.2) is proved. \square

ACKNOWLEDGEMENT

I would like to thank the referee for his suggestions, resulting in improved readability of this paper.

REFERENCES

- [1] J. L. Fernández, J. Heinonen, O. Martio *Quasilines and conformal mappings*, J. Analyse Math. 52 (1989), 117–132. MR0981499 (90a:30017)
- [2] W. C. Ma, D. Minda, *Hyperbolically convex functions*, Ann. Polon. Math. 60 (1994), no. 1, 81–100. MR1295110 (95k:30037)
- [3] W. C. Ma, D. Minda, *Hyperbolically convex functions II*, Ann. Polon. Math. 71 (1999), no. 3, 273–285. MR1704303 (2000j:30020)
- [4] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge University Press (1995). MR1333890 (96h:28006)
- [5] D. Mejia, Chr. Pommerenke, *Hyperbolically convex functions, dimension and capacity*, Complex Var. Theory Appl. 47 (2002), no. 9, 803–814. MR1925176 (2003f:30017)
- [6] D. Mejia, Chr. Pommerenke, *Hyperbolically convex functions*, Analysis and applications—ISAAC 2001 (Berlin), 89–95, Int. Soc. Anal. Appl. Comput., 10, Kluwer Acad. Publ., Dordrecht, 2003. MR2022741 (2004k:30108)
- [7] D. Mejia, Chr. Pommerenke, *On the derivative of hyperbolically convex functions*, Ann. Acad. Sci. Fenn. Math. 27 (2002), no. 1, 47–56. MR1884348 (2003b:30016)
- [8] Chr. Pommerenke, *Boundary behaviour of conformal maps*, Springer (1992). MR1217706 (95b:30008)
- [9] S. Rohde, *On the theorem of Hayman and Wu*, Proc. Amer. Math. Soc. 130 (2002), no. 2, 387–394. MR1862117 (2002i:30010)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195