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INVERSE SPECTRAL THEORY FOR STURM-LIOUVILLE PROBLEMS WITH FINITE SPECTRUM

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Dedicated to the memory of F.V. Atkinson

Abstract. For any positive integer n and any given n distinct real numbers we construct a Sturm-Liouville problem whose spectrum is precisely the given set of n numbers. Such problems are of Atkinson type in the sense that the weight function or the reciprocal of the leading coefficient is identically zero on at least one subinterval.

1. Introduction

In this paper we show that, for any $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ and any given real numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \tag{1.1}$$

there exists a regular self-adjoint Sturm-Liouville problem consisting of an equation

$$-(py')' + qy = \lambda wy \text{ on } J = (a, b),\ -\infty < a < b < \infty, \tag{1.2}$$

with coefficients satisfying

$$\frac{1}{p}, q, w \in L^1(J, \mathbb{R}) \tag{1.3}$$

and separated boundary conditions

$$\begin{aligned} \cos\alpha y(a) - \sin\alpha(py')(a) = 0,\ 0 \le \alpha < \pi, \\ \cos\beta y(b) - \sin\beta(py')(b) = 0,\ 0 < \beta \le \pi, \end{aligned} \tag{1.4}$$

whose spectrum σ is given by

$$\sigma = \{\lambda_j : j = 1, 2, \dots, n\}. \tag{1.5}$$

If, in addition to (1.3), the coefficients satisfy

$$p > 0,\ w > 0 \text{ a.e. on } J, \tag{1.6}$$

then this is a classical Sturm-Liouville problem which is well known to have a spectrum consisting of an *infinite but countable* number of real eigenvalues $\{\lambda_n : n \in \mathbb{N} = \{1, 2, 3, ...\}\}$ which can be ordered to satisfy

$$-\infty < \lambda_1 < \lambda_2 < \lambda_3 < .... \tag{1.7}$$

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The classical theory of Sturm-Liouville problems was extended by Atkinson in [1]. Let

$$r = \frac{1}{p}. \tag{1.8}$$

Atkinson observed that the basic theory for equation (1.2) holds if the conditions (1.6) are relaxed to

$$r \geq 0,\ w \geq 0 \text{ on } J, \tag{1.9}$$

thus allowing $r = 1/p$ and w to be identically zero on one or more subintervals of J. Theorem 8.4.6, p. 215 in [1] contains the following result.

Proposition 1. *Let* (1.3), (1.9) *hold. Assume that for some* $c, d \in J$,

$$\int_a^c w > 0,\ \int_d^b w > 0,\ \int_a^b r > 0. \tag{1.10}$$

If r *and* w *are positive on a common subinterval of* J, *then the Sturm-Liouville problem* (1.2), (1.4) *has an infinite number of eigenvalues satisfying* (1.7).

The statement on page 212 of [1], 'The eigenvalues ... form a sequence $\lambda_0 < \lambda_1 < ...$, possibly finite ...', strongly suggests that Atkinson realized that without the hypotheses of Proposition 1, there may be only a finite number of eigenvalues, but he gave no proof or example. This was confirmed by Kong, Wu and Zettl [4] who showed that for any positive integer n there exist such problems with exactly n eigenvalues. It is also shown in [4] that, given any n disjoint open intervals of the real line there exists a problem of Atkinson type which has exactly one eigenvalue in each of the given n open intervals. But the construction in [4] does not directly yield a problem with pre-assigned eigenvalues.

Definition 1. If r or w is identically zero on one or more subintervals of J, we say that the equation (1.2), or the boundary value problem (1.2), (1.4) is of 'Atkinson type'. It is convenient to think of such problems in terms of their system formulation:

$$u' = rv,\ v' = (q - \lambda w)u;\ u = y,\ v = pu'.$$

Remark 1. When r is identically zero on a subinterval then, of course, this means that p is infinite on this subinterval. Notation can be a psychological barrier. We suspect that the universally used notation for equation (1.1) has been an obstacle preventing Atkinson's ideas and results from being more widely used. Atkinson suggests the notation

$$-(\frac{1}{p}y')' + qy = \lambda wy.$$

Although this notation is more suggestive since the solutions depend on $1/p$, not p, it has not been widely adopted.

2. An inverse spectral construction

Our main result is

Theorem 1. *Let* $n \in \mathbb{N}$. *Given any* n *real numbers* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ *there exists a Sturm-Liouville problem of Atkinson type and with Dirichlet boundary conditions*

$$y(a) = 0 = y(b) \tag{2.1}$$

whose spectrum is precisely the set $\{\lambda_j : j = 1, 2, ..., n\}$.

Proof. First we observe that we may assume that $\lambda_1 > 0$ since the eigenvalues can be shifted by adding kwy to both sides of equation (1.2). The proof consists of two parts: (i) we construct two matrix eigenvalue problems, one of which has $\{\lambda_j : j = 1, 2, ..., n\}$ as its eigenvalues, and (ii) using a method of Atkinson we construct a Sturm-Liouville problem which is equivalent to this matrix eigenvalue problem.

Choose numbers $\{\lambda_j^{'} : j = 1, 2, ..., n\}$ such that

$$0 < \lambda_1' < \lambda_1 < \lambda_2' < \lambda_2 < \cdots < \lambda_n^{'} < \lambda_n. \tag{2.2}$$

By Gantmacher and Krein [3, Supplement II], there exist unique numbers

$$m_1, m_2, \ldots, m_n > 0$$

and

$$l_0, l_1, l_2, \ldots, l_n > 0, \ \sum_{j=0}^{n} l_j = 1$$

such that $\{\lambda_j : j = 1, 2, ..., n\}$ are the eigenvalues of the matrix eigenvalue problem

$$Az = \lambda Mz, \tag{2.3}$$

and $\{\lambda_j^{'} : j = 1, 2, ..., n\}$ are the eigenvalues of the matrix eigenvalue problem

$$A^{'}z = \lambda Mz,$$

where M is the diagonal matrix

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n)$$

A is the tridiagonal matrix

$$A = \begin{bmatrix} l_0^{-1} + l_1^{-1} & -l_1^{-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -l_1^{-1} & l_1^{-1} + l_2^{-1} & -l_2^{-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -l_{n-2}^{-1} & l_{n-2}^{-1} + l_{n-1}^{-1} & -l_{n-1}^{-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -l_{n-1}^{-1} & l_{n-1}^{-1} + l_n^{-1} \end{bmatrix}$$

and A' agrees with A except for the entry in the last row and last column which is replaced by l_{n-1}^{-1}.

We now construct a Sturm-Liouville problem whose spectrum is the set of eigenvalues of the matrix problem (2.3). Let

$$b = 1 + \sum_{j=1}^{n} m_j$$

and consider the partition

$$0 = b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < a_{n+1} = b, \tag{2.4}$$

where

$$a_1 = l_0, \ b_k = a_k + m_k, \ a_{k+1} = b_k + l_k \quad \text{for } k = 1, 2, \ldots, n.$$

Let K be the union of the intervals $[a_k, b_k]$ for $k = 1, 2, \ldots, n$. For example if $n = 2$,

$$K = [l_0, \, l_0 + m_1] \cup [l_0 + m_1 + l_1, \, l_0 + m_1 + l_1 + m_2].$$

The lengths of these two intervals are m_1 and m_2 and the complementary intervals of K have lengths l_0, l_1, l_2.

Let $\varkappa_K$ denote the characteristic function of K, i.e.,

$$\varkappa_K(t) = \begin{cases} 1, & t \in K, \\ 0, & t \in [0,b]\backslash K. \end{cases}$$

Now we consider the following problem of Atkinson type:

$$u' = (1-\varkappa_K)\, v,\ v' = -\lambda\, \varkappa_K\, u,\ u(0) = 0 = u(b). \tag{2.5}$$

Such problems have been studied by Volkmer in [5], [6].

We claim that the spectrum of (2.5) is precisely the set $\{\lambda_j : j = 1, 2, ..., n\}$. To prove the claim we write (2.5) as a difference equation following the same method Atkinson used on pp. 203, 204 of [1]. Let u, v be a solution of (2.5). We note that u and v are alternately constant on the intervals of the partition (2.4), say $u(x) = u_k$ on $[a_k, b_k]$ and $v(x) = v_k$ on $[b_k, a_{k+1}]$. We add the definitions $u_0 = u(a) = 0$ and $u_{n+1} = u(b) = 0$. By integrating the equations in (2.5) over suitable subintervals of the partition we obtain

$$\begin{aligned} v_k - v_{k-1} &= -\lambda u_k m_k \quad \text{for } k = 1, 2, \ldots, n, \\ u_{k+1} - u_k &= v_k l_k \quad \text{for } k = 0, 1, 2, \ldots, n. \end{aligned}$$

It follows that, for $k = 1, 2, \ldots, n$,

$$l_k^{-1}(u_{k+1} - u_k) - l_{k-1}^{-1}(u_k - u_{k-1}) = v_k - v_{k-1} = -\lambda u_k m_k.$$

This shows that λ is an eigenvalue of (2.3) with eigenvector $(u_1, u_2, \ldots, u_n)$. Retracing the steps one sees that eigenvalues of (2.3) are also eigenvalues of (2.5). □

Remark 2. The eigenvalues λ'_k of the matrix eigenvalue problem $A'z = \lambda M z$ are eigenvalues of (2.5) but with the boundary conditions $u(0) = v(b) = 0$. So we can prescribe the spectra (2.2) for the boundary conditions $u(0) = u(b) = 0$ and $u(0) = v(b) = 0$ simultaneously.

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