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# $\mathcal{K}$-BI-LIPSCHITZ EQUIVALENCE OF REAL FUNCTION-GERMS

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ABSTRACT. In this paper we prove that the set of equivalence classes of germs of real polynomials of degree less than or equal to $k$, with respect to $\mathcal{K}$-bi-Lipschitz equivalence, is finite.

## 1. INTRODUCTION

Finiteness theorems of different kinds appear in the modern development of Singularity Theory. When one considers a classification problem, it is important to know if the problem is "tame" or not. In other words, how difficult the problem is and if there is any hope to develop a complete classification. For the problem of topological classification of polynomial function-germs, a finiteness result was conjectured by R. Thom [13] and proved by Fukuda [5]. He proved that the number of equivalence classes of polynomial function-germs of degree less than or equal to any fixed $k$, with respect to topological equivalence, is finite. Note that R. Thom also discovered that this finiteness result does not hold for polynomial map-germs $P\colon \mathbb{R}^3 \to \mathbb{R}^3$ [13]. Finiteness theorems for polynomial map-germs, in the real and in the complex case, with respect to topological equivalence were the subject of investigation of various authors (see, for example, [12], [10], [3], [4]) and many interesting results were obtained in this direction. Mostowski [9] and Parusinski [11] proved that the set of equivalence classes of semialgebraic sets with a complexity bounded from below by any fixed $k$ is finite. A finiteness result does not hold for polynomial function-germs with respect to bi-Lipschitz equivalence. Henry and Parusinski [6] showed that this problem is not tame, i.e., it has "moduli".

Here we consider the problem of the $\mathcal{K}$-bi-Lipschitz classification of polynomial function-germs ($\mathcal{K}$-equivalence is the contact equivalence defined by Mather [7]). We show that this problem is still tame. The main idea of the proof is the following. First, we consider Lipschitz functions "of the same contact". Namely, $f$ and $g$ are of the same contact if $\dfrac{f}{g}$ is positive and bounded away from zero and infinity. We show that two functions of the same contact are $\mathcal{K}$-bi-Lipschitz equivalent. The next step is related to the geometry of contact equivalence. Recall that two function-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are $\mathcal{C}^\infty$-contact equivalent if there exists a

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$\mathcal{C}^\infty$-diffeomorphism in the product space $(\mathbb{R}^n \times \mathbb{R}, 0)$ which leaves $\mathbb{R}^n$ invariant and maps the graph $(f)$ to the graph $(g)$. This definition is due to Mather [7] for map-germs $f, g\colon \mathbb{R}^n \to \mathbb{R}^p$. Montaldi extended this notion by introducing a purely geometrical definition of contact: two pairs of submanifolds of $\mathbb{R}^n$ have the same contact type if there is a diffeomorphism of $\mathbb{R}^n$ taking one pair to the other, and relating this with the $\mathcal{K}$-equivalence of convenient map-germs [8]. This geometrical interpretation also exists for a topological version of $\mathcal{K}$-equivalence (cf. [1]).

In this paper, we give a definition of Montaldi's construction for the bi-Lipschitz case and show that the existence of a bi-Lipschitz analogue to Montaldi's construction, for two germs $f$ and $g$, implies that $f \circ h$ and $g$ are of the same contact, for some bi-Lipschitz map-germ $h$. Finally, the finiteness result follows from Montaldi's construction and the Mostowski-Parusinski [11] theorem on Lipschitz stratifications.

## 2. Basic definitions and results

**Definition 2.1.** Two function-germs $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ are called *$\mathcal{K}$-bi-Lipschitz equivalent* (or *contact bi-Lipschitz equivalent*) if there exist two germs of bi-Lipschitz homeomorphisms $h : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{R}^n, 0) & \xrightarrow{(id, f)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\ h \downarrow & & H \downarrow & & h \downarrow \\ (\mathbb{R}^n, 0) & \xrightarrow{(id, g)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \end{array}$$

where $id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map and $\pi_n : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is the canonical projection.

The function-germs $f$ and $g$ are called *$\mathcal{C}$-bi-Lipschitz equivalent* if $h = id$.

In other words, two function-germs $f$ and $g$ are $\mathcal{K}$-bi-Lipschitz equivalent if there exists a germ of a bi-Lipschitz map $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(x, y)$ can be written in the form $H(x, y) = (h(x), \tilde{H}(x, y))$, where $h$ is a bi-Lipschitz map-germ, $\tilde{H}(x, 0) = 0$ and $H$ maps the germ of the graph $(f)$ onto the graph $(g)$.

Recall that graph $(f)$ is the set defined as follows:

$$\operatorname{graph}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = f(x)\}.$$

**Definition 2.2.** Two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are called *of the same contact at a point* $x_0 \in \mathbb{R}^n$ if there exist a neighborhood $U_{x_0}$ of $x_0$ in $\mathbb{R}^n$ and two positive numbers $c_1$ and $c_2$ such that, for all $x \in U_{x_0}$, we have

$$c_1 f(x) \leq g(x) \leq c_2 f(x).$$

We use the notation: $f \approx g$.

*Remark* 2.3. It is clear that if two function-germs $f$ and $g$ are of the same contact, then the germs of their zero-sets are equal.

The main results of the paper are the following.

**Theorem 2.4.** *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two germs of Lipschitz functions. Then $f$ and $g$ are $\mathcal{C}$-bi-Lipschitz equivalent if and only if one of the following two conditions is true:*

i) $f \approx g$,
ii) $f \approx -g$.

**Theorem 2.5.** *Let $\mathcal{P}_k(\mathbb{R}^n)$ be the set of all polynomials of $n$ variables with degree less than or equal to $k$. Then the set of equivalence classes of the germs at $0$ of the polynomials in $\mathcal{P}_k(\mathbb{R}^n)$, with respect to $\mathcal{K}$-bi-Lipschitz equivalence, is finite.*

## 3. Functions of the same contact

*Proof of Theorem* 2.4. Suppose that the germs of the Lipschitz functions $f$ and $g$ are $\mathcal{C}$-bi-Lipschitz equivalent. Let $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ be the germ of a bi-Lipschitz homeomorphism satisfying the conditions of Definition 2.1. Let $V_+$ be the subset of $\mathbb{R}^n \times \mathbb{R}$ of points $(x, y)$ where $y > 0$, and let $V_-$ be the subset of $\mathbb{R}^n \times \mathbb{R}$ where $y < 0$. Clearly, we have one of the following possibilities:

1) $H(V_+) = V_+$ and $H(V_-) = V_-$, or
2) $H(V_+) = V_-$ and $H(V_-) = V_+$.

Let us consider the first possibility. In this case, the functions $f$ and $g$ have the same sign on each connected component of the set $f(x) \neq 0$. Moreover,

$$\begin{aligned} |g(x)| &= \| (x,0) - (x, g(x)) \| = \| H(x,0) - H(x, f(x)) \| \\ &\leq c_2 \ \| (x,0) - (x, f(x)) \| = c_2 |f(x)|, \end{aligned}$$

where $c_2$ is a positive real number. Using the same argument we can show

$$c_1 |f(x)| \leq |g(x)|, \qquad c_1 > 0.$$

Hence, $f \approx g$.

Let us consider the second possibility. Let $\xi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ be a map-germ defined as follows:

$$\xi(x, y) = (x, -y).$$

Applying the same arguments to a map $\xi \circ H$, we will conclude that $f \approx -g$.

Reciprocally, suppose that $f \approx g$. Let us construct a map-germ

$$H : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0).$$

Then the following may occur:

$$H(x,y) = \begin{cases} (x, 0) & \text{if } \ y = 0, \\ (x, \dfrac{g(x)}{f(x)} y) & \text{if } \ 0 \leq |y| \leq |f(x)|, \\ (x, y - f(x) + g(x)) & \text{if } \ 0 \leq |f(x)| \leq |y|, \\ (x, y) & \text{otherwise.} \end{cases} \tag{1}$$

The map $H(x, y) = (x, \tilde{H}(x, y))$ defined above is bi-Lipschitz. In fact, $H$ is injective because, for any fixed $x^\star$, we can show that $\tilde{H}(x^\star, y)$ is a continuous and monotone function. Moreover, $H$ is Lipschitz if $0 \leq |f(x)| \leq |y|$. Let us show that $H$ is Lipschitz if $0 \leq |y| \leq |f(x)|$. By Rademacher's theorem, in almost every $x$ near $0 \in \mathbb{R}^n$, all the partial derivatives $\dfrac{\partial f}{\partial x_i}, \dfrac{\partial g}{\partial x_i}$ exist; hence the derivatives $\dfrac{\partial \tilde{H}}{\partial x_i}$

exist in almost every $x$ near $0 \in \mathbb{R}^n$. By the Mean Value Theorem and continuity of $\tilde{H}$, it is enough to show that the derivatives $\dfrac{\partial \tilde{H}}{\partial x_i}$ are bounded on the domain $0 \le |y| \le |f(x)|$, for all $i = 1, \ldots, n$. We have

$$\frac{\partial \tilde{H}}{\partial x_i} = \frac{(\frac{\partial g}{\partial x_i} f(x) - \frac{\partial f}{\partial x_i} g(x))\, y}{(f(x))^2} = \frac{\partial g}{\partial x_i} \frac{y}{f(x)} - \frac{\partial f}{\partial x_i} \frac{g(x)}{f(x)} \frac{y}{f(x)}.$$

Since $|y| \le |f(x)|$, then $\dfrac{y}{f(x)}$ is bounded. The expression $\dfrac{g(x)}{f(x)}$ is bounded since $f \approx g$. Moreover, $\dfrac{\partial g}{\partial x_i}$ and $\dfrac{\partial f}{\partial x_i}$ are bounded because $f$ and $g$ are Lipschitz functions.

Since $H^{-1}$ can be constructed in the same form as (1), we conclude that $H^{-1}$ is also Lipschitz and, thus, $H$ is a bi-Lipschitz map. □

## 4. Montaldi's construction

**Definition 4.1.** Two germs of Lipschitz functions are called *$\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent* (or *contact equivalent in the sense of Montaldi*) if there exists a germ of a bi-Lipschitz map $M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and $M(\text{graph}\,(f)) = \text{graph}\,(g)$. The map $M$ is called a *Montaldi map.*

**Theorem 4.2.** *Two germs of Lipschitz functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent if and only if they are $\mathcal{K}$-bi-Lipschitz equivalent.*

*Proof.* It is clear that the $\mathcal{K}$-bi-Lipschitz equivalence implies the $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence. To prove the converse, let $f$ and $g$ be $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent. Then

$$M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\} \quad \text{and} \quad M(\text{graph}\,(f)) = \text{graph}\,(g).$$

Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be defined by $h(x) = \pi_n(M(x, f(x)))$.

*Claim* 1. $h$ is a bi-Lipschitz map-germ.

*Proof of Claim* 1. Since $g$ is a Lipschitz function, the projection ${\pi_n}_{|_{\text{graph}\,(g)}}$ is a bi-Lipschitz map. By the same argument, the map $x \mapsto (x, f(x))$ is bi-Lipschitz. The map $M$ is bi-Lipschitz by Definition 4.1.

*Claim* 2. One of the following assertions is true:

i) $f \approx g \circ h$,
ii) $f \approx -(g \circ h)$.

*Proof of Claim* 2. Since $M$ is a bi-Lipschitz map, it follows that there exist two positive numbers $c_1$ and $c_2$ such that

$$c_1\, |f(x)| \le \| M(x, f(x)) - M(x, 0) \| \le c_2 |f(x)|.$$

By the above construction,

$$\| M(x, f(x)) - M(x, 0) \| = \| (h(x), g(h(x))) - M(x, 0) \| \ge |g(h(x))|.$$

Therefore, $|g(h(x))| \le c_2 |f(x)|$.

Using the same procedure for the map $M^{-1}$, we obtain that

$$c_1|f(x)| \leq |g(h(x))|.$$

Since $M$ is a homeomorphism and $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$, the same argument as in Theorem 2.4 implies that, for all $x \in \mathbb{R}^n$,

$$\operatorname{sign} f(x) = \operatorname{sign} g(h(x)),$$

or, for all $x \in \mathbb{R}^n$,

$$\operatorname{sign} f(x) = -\operatorname{sign} g(h(x)).$$

Hence, Claim 2 is proved. □

*End of the proof of Theorem* 4.2. Using Claim 2 we obtain, by Theorem 2.4 , that $f$ and $g \circ h$ are $\mathcal{C}$-bi-Lipschitz equivalent. By Claim 1, $f$ and $g$ are $\mathcal{K}$-bi-Lipschitz equivalent. □

## 5. Finiteness theorem

This section is devoted to a proof of the finiteness theorem (Theorem 2.5). Actually, this result follows from Theorem 4.2 and the equisingularity statement of the Mostowski-Parusinski theorem, but we are going to present an independent proof, using just the finiteness statement of the same theorem.

Let $\mathcal{P}_k(\mathbb{R}^n)$ be the set of all polynomials of $n$ variables with degree less than or equal to $k$. Let $\mathcal{S}_k(\mathbb{R}^n)$ be the set of all continuous semialgebraic functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such that there exists a polynomial $P \in \mathcal{P}_k(\mathbb{R}^n)$; $|f(x)| = |P(x)|$, for all $x \in \mathbb{R}^n$. Clearly, there exists a positive integer number $K$ such that, for all $f \in \mathcal{S}_k(\mathbb{R}^n)$, the algebraic complexity of $f$ (see [2] for a definition) is less than or equal to $K$. For any function $f \in \mathcal{S}_k(\mathbb{R}^n)$, we associate the germ at $0 \in \mathbb{R}^n$ of the semialgebraic set $X_f \subset \mathbb{R}^n \times \mathbb{R}$, defined by $X_f = \mathbb{R}^n \times \{0\} \cup \operatorname{graph}(f)$. We say that $X_f$ and $X_g$ are *strongly bi-Lipschitz equivalent* if there exists a germ of a bi-Lipschitz homeomorphism $h\colon (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $h(X_f) = X_g$. Note that if $f$ and $g$ are $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent, the corresponding sets $X_f$ and $X_g$ are strongly bi-Lipschitz equivalent, but not vice versa. By the Mostowski-Parusinski Theorem ([9], [11]) the set of equivalence classes of the sets $X_f$ for $f \in \mathcal{S}_k(\mathbb{R}^n)$, with respect to strong bi-Lipschitz equivalence, is finite. We are going to show that the set of equivalence classes with respect to $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence is also finite.

Let $f \in \mathcal{S}_k(\mathbb{R}^n)$. Let $Y_1, \ldots, Y_p$ be the set of connected components of the set $f(x) \neq 0$. Let us define a transformation $F_i\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$. Let $(x, y)$ be a coordinate system in $\mathbb{R}^n \times \mathbb{R}$ such that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let

$$H(x,y) = \begin{cases} (x,y) & \text{if } x \notin Y_i, \\ (x, y - f(x)) & \text{if } x \in Y_i. \end{cases} \tag{2}$$

Since $f(x)$ is a Lipschitz function, $F_i$ is a bi-Lipschitz map. The transformation $F_i$ maps the set $X_f$ to the set $X_{f_i}$ where $f_i \in \mathcal{S}_k(\mathbb{R}^n)$. In fact,

$$f_i(x) = \begin{cases} f(x) & \text{if } x \notin Y_i, \\ -f(x) & \text{if } x \in Y_i. \end{cases} \tag{3}$$

We say that $f_i$ is obtained from $f$ by an elementary transformation. Let $\{f_{\alpha_1}, \ldots, f_{\alpha_p}\}$ be the set of all the functions which can be obtained from $f$ by a sequence of elementary transformations. Clearly, this set is finite.

By the Mostowski-Parusinski Theorem there exists a finite set of functions

$$f^1, \ldots, f^m \in \mathcal{S}_k(\mathbb{R}^n)$$

such that, for any $g \in \mathcal{S}_k(\mathbb{R}^n)$, there exists a number $i$ such that $X_g$ is strongly bi-Lipschitz equivalent to $X_{f^i}$. Let us prove that there exists a function $f^i_{\alpha_j}$ obtained from $f^i$ by a sequence of elementary transformations such that $g$ is $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent to $f^i_{\alpha_j}$, i.e., there exists a Montaldi map between $X_g$ and $X_{f^i_{\alpha_j}}$.

Suppose that there exists a germ of a bi-Lipschitz map $h\colon (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $h(X_{f^i}) = X_g$. Let $Y_1, \ldots, Y_p$ be the connected component of the $f^i(x) \neq 0$. If $h$ is not a Montaldi map, then there exists a family of the components $Y_1, \ldots, Y_r$, $r \leq p$ such that the image of the set $Y_j \times \{0\}$ belongs to a part of graph $(g)$, and the image of the part of graph $(f^i)$ above $Y_j$ belongs to $\mathbb{R}^n \times \{0\}$. Let $f^i_j$ be the function obtained from $f^i$ by an elementary transformation $F^i_j$. Then the map $h \circ F^i_j$ is a Montaldi map on the set $Y_j \times \mathbb{R}$ and is equal to $h$ on the complement of $Y_j \times \mathbb{R}$. When we apply this construction for all the sets $Y_1, \ldots, Y_r$ we obtain a Montaldi map $M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $M(X_{f^i_{\alpha_j}}) = X_g$, for some $f^i_{\alpha_j}$. Since $\mathcal{P}_k(\mathbb{R}^n) \subset \mathcal{S}_k(\mathbb{R}^n)$, the set of equivalence classes in $\mathcal{P}_k(\mathbb{R}^n)$, with respect to $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence, is finite.

By Theorem 4.2, the set of equivalence classes with respect to $\mathcal{K}$-equivalence is also finite. The theorem is proved. □

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