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SHARP BEREZIN LIPSCHITZ ESTIMATES

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ABSTRACT. F.A. Berezin introduced a general "symbol calculus" for linear operators on reproducing kernel Hilbert spaces. For the Segal-Bargmann space $H^2(\mathbf{C}^n, d\mu)$ of Gaussian square-integrable entire functions on complex n-space, $\mathbf{C}^n$, or for the Bergman spaces $A^2(\Omega)$ of Euclidean volume square-integrable holomorphic functions on bounded domains Ω in $\mathbf{C}^n$, we show here that earlier Lipschitz estimates for Berezin symbols of arbitrary bounded operators are sharp.

1. Introduction

The Segal-Bargmann Hilbert space $H^2(\mathbf{C}^n, d\mu)$ of Gaussian square-integrable entire functions on complex n-space [2] has the Bergman "reproducing kernel" property that

$$f(a) = \int_{\mathbf{C}^n} K(a,z)f(z)d\mu(z) = \langle f(\cdot), K(\cdot,a)\rangle$$

for all f in $H^2(\mathbf{C}^n, d\mu)$ and a in $\mathbf{C}^n$ where $a \cdot z = a_1\bar{z}_1 + \cdots + a_n\bar{z}_n$, $|a|^2 = a \cdot a$, $K(a,z) = e^{a\cdot z/2}$ is the Bergman kernel function and

$$d\mu(z) = (2\pi)^{-n}\exp\{-|z|^2/2\}dv(z)$$

(dv is Lebesgue volume measure). It is easy to check that

$$k_a(z) = K(z,a)\{K(a,a)\}^{-\frac{1}{2}}$$

is a unit vector in the Hilbert space structure that $H^2(\mathbf{C}^n, d\mu)$ inherits as a subspace of $L^2(\mathbf{C}^n, d\mu)$.

Although we use the space $H^2(\mathbf{C}^n, d\mu)$ as a model, we will also consider the analogous Bergman spaces $A^2(\Omega)$ for Ω a bounded domain in $\mathbf{C}^n$ and the Gaussian $d\mu$ replaced by dv. Their kernel functions will also be denoted by $K(a,z)$ as in [15, pp. 39-54]. Note that $K(a,z)$ is always analytic in a and conjugate-analytic in z with

$$\overline{K(a,z)} = K(z,a)$$

and that $K(a,a)$ is positive. The "bounded symmetric domains" Ω give interesting special cases which are "closest" to the model $H^2(\mathbf{C}^n, d\mu)$.

For bounded linear operators X on $H^2(\mathbf{C}^n, d\mu)$ or $A^2(\Omega)$, Berezin [3], [4] considered the mapping $Ber : X \to \tilde{X}$, where $\tilde{X}(a) = \langle Xk_a, k_a\rangle$. The function $\tilde{X}(\cdot)$ is real-analytic on $\mathbf{C}^n$ or Ω and is uniquely determined by X [3]; [11, pp. 43, 139].

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It is clear that $|\tilde{X}(\cdot)| \leq \|X\|$, where $\|\cdot\|$ is the usual operator norm. Because *Ber* is linear and one-to-one, it "encodes" operator-theoretic information into function theory in a striking but somewhat impenetrable way.

In fact, since $k_a \to 0$ weakly as $|a| \to \infty$ for $H^2(\mathbf{C}^n, d\mu)$, it is also clear that *Ber* maps compact operators on these Hilbert spaces into functions which vanish at infinity (there is an analogous result for $A^2(\Omega)$ with "nice" Ω). Because of these properties, the mapping *Ber* has found useful applications in dealing with operators "of function-theoretic significance" such as Toeplitz and Hankel operators [1], [5], [6], [7], [12], [9]. The functions $\tilde{X}$ also arise in Berezin's well-known quantization program [4]. It is, therefore, of interest to determine the range of *Ber*.

In this note, we consider the infinitesimal Bergman metric on Ω (or on $\mathbf{C}^n$) defined by writing

$$g_{j,k}(z) = 2\frac{\partial}{\partial \overline{z}_j}\frac{\partial}{\partial z_k} \ln K(z,z) \tag{$*$}$$

(the constant "2" conforms with [16]). It is well known [14] that $(*)$ gives a Riemannian metric on Ω (it is just the usual Euclidean metric on $\mathbf{C}^n$). Following [16], we define the Bergman length of a C^1–curve $\gamma : [0,1] \to \Omega$ by

$$l(\gamma) = \int_0^1 \{G(s)\}^{\frac{1}{2}} ds$$

where

$$G(s) = \sum_{j,k=1}^{n} g_{jk}(\gamma(s))\overline{\gamma_j'(s)}\gamma_k'(s).$$

The Bergman metric distance function $\beta(a,b)$ is defined by

$$\beta(a,b) = \inf\{l(\gamma) : \gamma(0) = b, \gamma(1) = a\}.$$

In [8], the following estimates for all bounded domains Ω and for $\mathbf{C}^n$ were obtained:

$$\begin{aligned} |\tilde{X}(a) - \tilde{X}(b)| &\leq 2\|X\|\{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}} \\ &\leq \sqrt{2}\,\|X\|\beta(a,b). \end{aligned} \tag{$**$}$$

In the next two sections, we show, by varying X, a, b, that these estimates are sharp.

2. Preliminary results

For the remainder of this note, we pick γ to be a geodesic curve for the Bergman metric, emanating from b in Ω with $\gamma(0) = b$ and with $\gamma'(0) = v$, a fixed unit vector in $\mathbf{C}^n$. It is then clear from Theorem 9.9 of [14, p. 53] that, for some $t_0 > 0$ and $0 < t < t_0$,

$$\beta(\gamma(t), b) = \int_0^t \{G(s)\}^{\frac{1}{2}} ds$$

and, by the fundamental theorem of the integral calculus,

$$\lim_{t\to 0} \frac{\beta(\gamma(t), b)}{t} = \{G(0)\}^{\frac{1}{2}} > 0. \tag{$\dagger$}$$

As in [8], consider the rank-one projection $P_a(f) = \langle f, k_a\rangle k_a$ onto the span of k_a. We will use some standard results from [13] about trace-class operators. Recall that

$$\tilde{X}(a) = trace(XP_a).$$

A slight elaboration of Theorem 1 of [8] yields

Proposition 1. *The rank-two self-adjoint operator $P_a - P_b$ has real eigenvalues $\{\lambda, -\lambda\}$ where $\lambda \geq 0$ with $\lambda = \{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}}$. It follows that the operator norm of $P_a - P_b$ is given by*

$$\|P_a - P_b\| = \{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}},$$

while the trace norm of $P_a - P_b$ is given by

$$\|P_a - P_b\|_{trace} = 2\{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}}.$$

Proof. Since P_a, P_b are both rank-one self-adjoint operators with

$$trace(P_a) = trace(P_b) = 1,$$

it follows that there is an orthonormal basis in which $P_a - P_b$ is diagonal with two (possibly) nonzero real eigenvalues$\{\lambda, -\lambda\}$ and $\lambda \geq 0$. It follows that $(P_a - P_b)^2$ has nonzero eigenvalues $\{\lambda^2, \lambda^2\}$. On the other hand, it is easy to calculate

$$\begin{aligned} trace(P_a - P_b)^2 &= trace(P_a - P_aP_b - P_bP_a + P_b) \\ &= 2 - 2\ trace(P_aP_b) \\ &= 2 - 2\langle P_aP_bk_b, k_b\rangle \\ &= 2 - 2\langle P_ak_b, k_b\rangle \\ &= 2 - 2\langle k_b, k_a\rangle\langle k_a, k_b\rangle \\ &= 2\{1 - |\langle k_a, k_b\rangle|^2\}. \end{aligned}$$

It follows that $\lambda^2 = \{1 - |\langle k_a, k_b\rangle|^2\}$ and the proposition follows easily. □

For the Skwarczynski distance function

$$\rho(a, b) = \{1 - |\langle k_a, k_b\rangle|\}^{\frac{1}{2}},$$

[16] shows that

$$(\dagger\dagger) \qquad \lim_{t\to 0} \frac{\rho(\gamma(t), b)}{t} = \frac{1}{2}\{G(0)\}^{\frac{1}{2}}.$$

We now have the key technical result:

Proposition 2. *For γ a Bergman metric geodesic, as above, we have*

$$\begin{aligned} \lim_{t\to 0} \frac{\|P_{\gamma(t)} - P_b\|_{trace}}{t} &= \sqrt{2}\{G(0)\}^{\frac{1}{2}} \\ &= \sqrt{2} \quad \lim_{t\to 0} \frac{\beta(\gamma(t), b)}{t}. \end{aligned}$$

Proof. From Proposition 1, we have

$$\|P_a - P_b\|_{trace} = 2\{1 + |\langle k_a, k_b\rangle|\}^{\frac{1}{2}}\rho(a, b).$$

It follows that

$$\lim_{t\to 0} \frac{\|P_{\gamma(t)} - P_b\|_{trace}}{t} = 2 \lim_{t\to 0}\{1 + |\langle k_{\gamma(t)}, k_b\rangle|\}^{\frac{1}{2}} \lim_{t\to 0} \frac{\rho(\gamma(t), b)}{t}.$$

It is easy to check that

$$\lim_{t\to 0}\{1+|\langle k_{\gamma(t)},k_b\rangle|\}^{\frac{1}{2}}=\sqrt{2},$$

and the rest follows from $(\dagger),(\dagger\dagger)$. □

3. Sharpness of the estimates

To obtain the promised sharpness results, we need to consider a family of "optimal" bounded operators X_t. This has been detected by a process of trial and error which is best left unexamined. For $\gamma(t)$ a Bergman metric geodesic as above, with $\gamma(0)=b$ and $\gamma'(0)=v$, a unit vector, we consider the family of operators

$$X_t=P_{\gamma(t)}-P_b.$$

Note that, by Proposition 1,

$$\|X_t\|=\{1-|\langle k_{\gamma(t)},k_b\rangle|^2\}^{\frac{1}{2}}.$$

We can also calculate easily that

$$\tilde{X}_t(\gamma(t))-\tilde{X}_t(b)=2\,\{1-|\langle k_{\gamma(t)},k_b\rangle|^2\}.$$

Again using Proposition 1, we have

$$\|P_{\gamma(t)}-P_b\|_{trace}=2\,\{1-|\langle k_{\gamma(t)},k_b\rangle|^2\}^{\frac{1}{2}}.$$

We now obtain

Sharpness Theorem 1. *For X_t as above,*

(a) $$\frac{|\tilde{X}_t(\gamma(t))-\tilde{X}_t(b)|}{\|X_t\|\|P_{\gamma(t)}-P_b\|_{trace}}\equiv 1$$

while

(b) $$\lim_{t\to 0}\frac{|\tilde{X}_t(\gamma(t))-\tilde{X}_t(b)|}{\|X_t\|\,\beta(\gamma(t),b)}=\sqrt{2}.$$

Proof. (a) is a direct calculation. Proposition 2 yields

$$\lim_{t\to 0}\frac{\|P_{\gamma(t)}-P_b\|_{trace}}{\beta(\gamma(t),b)}=\frac{\lim_{t\to 0}\frac{\|P_{\gamma(t)}-P_b\|_{trace}}{t}}{\lim_{t\to 0}\frac{\beta(\gamma(t),b)}{t}}=\sqrt{2}.$$

This, with (a), yields (b). □

Remark. The sharpness of $(**)$ provides striking evidence of the efficiency of the trace estimates of [13, pp. 47, 48, 92].

4. Open problems and remarks

For bounded domains Ω in $\mathbf{C}^n$, and the special case of operators X of "multiplication by a bounded analytic function φ", [8] provided a Bloch-type estimate for all a,b in Ω:

$(\dagger\dagger\dagger)$ $$|\varphi(a)-\varphi(b)|\le\sqrt{2}\|\varphi\|_\infty\beta(a,b).$$

Problem 1. Determine the sharp constants B_Ω for which

$$|\varphi(a) - \varphi(b)| \le B_\Omega \|\varphi\|_\infty \beta(a, b)$$

for all bounded analytic φ on Ω and all a, b in Ω.

From (†††), by choosing φ in the set of coordinate functions $\{z_j : j = 1, 2, 3, ..., n\}$, we see that for each bounded Ω there is a constant C_Ω with

$$|a - b| \le C_\Omega \beta(a, b)$$

for all a, b in Ω.

Problem 2. Determine sharp values of C_Ω.

Remark. Jingbo Xia has pointed out that the proof of Theorem 4 of [8] can be used, along with a direct calculation, to provide a strengthened version of Theorem 2 of [8]: for any X in the full algebra of bounded operators $Op\{H^2(\mathbf{C}^n, d\mu)\}$, $\tilde{X}$ and its partial derivatives of all orders are **bounded**. Subsequently, Miroslav Engliš and Genkai Zhang [10] have provided the appropriate generalization of this result for **invariant** differential operators applied to $\tilde{X}$ on bounded symmetric Ω.

5. Dedication

I dedicate this note to the memory of my late wife Charlaine Ackerman Coburn (1939-2005). She was sharp and constant.

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